

ON A HYPERGEOMETRIC SUMMATION THEOREM DUE TO QURESHI ET AL.

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ABSTRACT. We first aim at proving an interesting easily derivable summation formula. Then it is easily seen that this formula immediately yields a hypergeometric summation theorem recently added to the literature by Qureshi et al. Moreover we apply the main formulas to present some interesting summation formulas, whose special cases are also seen to yield the earlier known results.

1. Introduction and preliminaries

It is well known that the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ of the first and second kind are the following special cases ($\alpha = \beta = -\frac{1}{2}$ and $\alpha = \beta = \frac{1}{2}$, respectively) of the classical Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ defined by (see, for example, [3, pp. 169–170] and [5])

$$(1.1) \quad P_n^{(\alpha, \beta)}(x) := \sum_{k=0}^n \binom{n+\alpha}{k} \binom{n+\beta}{n-k} \left(\frac{x-1}{2}\right)^{n-k} \left(\frac{x+1}{2}\right)^k.$$

Also, in terms of the *generalized hypergeometric series* ${}_pF_q$ defined by (see [6, p. 73]):

$$(1.2) \quad {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n z^n}{(\beta_1)_n \cdots (\beta_q)_n n!} \\ = {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \quad (p, q \in \mathbb{N}_0),$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol or the *shifted factorial*, since

$$(1)_n = n! \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \mathbb{N} := \{1, 2, 3, \dots\}),$$

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which is defined (for $\lambda, \nu \in \mathbb{C}$), in terms of the familiar gamma function Γ , by

$$(1.3) \quad (\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \cdots (\lambda + \nu - 1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases}$$

it being *understood conventionally* that $(0)_0 := 1$ and \mathbb{C} the set of complex numbers, (1.1) can be written as

$$(1.4) \quad P_n^{(\alpha, \beta)}(x) = \binom{n + \alpha}{n} {}_2F_1 \left[\begin{matrix} -n, n + \alpha + \beta + 1; \\ \alpha + 1; \end{matrix} \frac{1 - x}{2} \right].$$

In fact, we have the following relationships (see, for example, [8, p. 125] and [5]):

$$(1.5) \quad T_n(x) = \binom{n - \frac{1}{2}}{n}^{-1} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x)$$

and

$$(1.6) \quad U_n(x) = \frac{1}{2} \binom{n + \frac{1}{2}}{n + 1}^{-1} P_n^{(\frac{1}{2}, \frac{1}{2})}(x).$$

Moreover, upon setting $x = \cos \theta$ in Equations (1.5) and (1.6), it is easily seen that (see, for example, [8, p. 71, Equation 1.8(10)] and [5])

$$(1.7) \quad T_n(\cos \theta) = \cos(n\theta)$$

and

$$(1.8) \quad U_n(\cos \theta) = \frac{\sin[(n + 1)\theta]}{\sin \theta}.$$

Also, for the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ defined by Equations (1.5) and (1.6), it is known that (see, for example, [4, p. 468, Entries 7.3.1(209) and 7.3.1(211)])

$$(1.9) \quad {}_2F_1 \left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}; \\ \frac{1}{2}; \end{matrix} x \right] = (1 - x)^{\frac{n}{2}} T_n \left(\frac{1}{\sqrt{1 - x}} \right)$$

and

$$(1.10) \quad {}_2F_1 \left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}; \\ \frac{3}{2}; \end{matrix} x \right] = \frac{(1 - x)^{\frac{n}{2}}}{n + 1} U_n \left(\frac{1}{\sqrt{1 - x}} \right).$$

Upon setting $x = -\frac{b^2}{a^2}$ in Equations (1.9) and (1.10), recently Qureshi et al. [5, Equations (18) and (19)] obtained two interesting hypergeometric summation formulae given in the following theorem.

Theorem 1. For $n \in \mathbb{N}_0$, we have

$$(1.11) \quad {}_2F_1 \left[\begin{matrix} -\frac{n}{2}, \frac{1-n}{2}; \\ \frac{1}{2}; \end{matrix} -\frac{b^2}{a^2} \right] = \frac{(a^2 + b^2)^{\frac{n}{2}}}{a^n} \cos(n\vartheta)$$

$$\left(n\vartheta \neq \frac{2\kappa + 1}{2} \pi; \kappa \in \mathbb{Z} \right)$$

and

$$(1.12) \quad {}_2F_1 \left[\begin{matrix} -\frac{n}{2}, \frac{1-n}{2}; \\ \frac{3}{2}; \end{matrix} -\frac{b^2}{a^2} \right] = \frac{(a^2 + b^2)^{\frac{n+1}{2}}}{(n+1)a^n b} \sin[(n+1)\vartheta]$$

$$((n+1)\vartheta \neq \kappa\pi; \kappa \in \mathbb{Z}),$$

where \mathbb{Z} denotes the set of integers and ϑ is given by

$$(1.13) \quad \vartheta := \begin{cases} \arctan\left(\frac{b}{a}\right) & (a, b \in \mathbb{R}_+) \\ \pi - \arctan\left(\frac{b}{|a|}\right) & (a \in \mathbb{R}_-; b \in \mathbb{R}_+) \\ \arctan\left(\frac{|b|}{|a|}\right) - \pi & (a, b \in \mathbb{R}_-) \\ -\arctan\left(\frac{|b|}{a}\right) & (a \in \mathbb{R}_+; b \in \mathbb{R}_-), \end{cases}$$

\mathbb{R}_+ and \mathbb{R}_- being the sets of positive and negative real numbers, respectively.

Here, in this paper, we aim at presenting an interesting easily derivable summation formula (2.1) involving the exponential and hypergeometric functions ${}_2F_1$. Moreover, it is easily seen that our summation formula immediately yields Equations (1.11) and (1.12).

2. Main result

The main result to be established here is as in the following theorem.

Theorem 2. The following summation formula holds true.

$$(2.1) \quad \sum_{n=0}^{\infty} (a^2 + b^2)^{\frac{n}{2}} e^{in\vartheta} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{a^n x^n}{n!} {}_2F_1 \left[\begin{matrix} -\frac{n}{2}, \frac{1-n}{2}; \\ \frac{1}{2}; \end{matrix} -\frac{b^2}{a^2} \right]$$

$$+ ib \sum_{n=0}^{\infty} \frac{a^n x^{n+1}}{n!} {}_2F_1 \left[\begin{matrix} -\frac{n}{2}, \frac{1-n}{2}; \\ \frac{3}{2}; \end{matrix} -\frac{b^2}{a^2} \right],$$

provided that the series on the left-hand side is absolutely convergent and a, b, x and ϑ are real numbers, and $i = \sqrt{-1}$.

Proof. We begin by considering

$$(2.2) \quad S := e^{(a+ib)x}.$$

Now let us express $a + ib$ in the polar coordinate and then Euler's formula:

$$(2.3) \quad a + ib = r (\cos \vartheta + i \sin \vartheta) = r e^{i \vartheta},$$

where

$$r = \sqrt{a^2 + b^2} \quad \text{and} \quad \vartheta = \arctan\left(\frac{b}{a}\right).$$

By using the Maclaurin expansion, we find from (2.2) and (2.3) that

$$(2.4) \quad \begin{aligned} S &= e^{(a+ib)x} = e^{(r e^{i \vartheta})x} = \sum_{n=0}^{\infty} \frac{r^n x^n}{n!} e^{i \vartheta n} \\ &= \sum_{n=0}^{\infty} \frac{(a^2 + b^2)^{\frac{n}{2}}}{n!} \cos(n \vartheta) x^n + i \sum_{n=0}^{\infty} \frac{(a^2 + b^2)^{\frac{n}{2}}}{n!} \sin(n \vartheta) x^n. \end{aligned}$$

On the other hand, we have

$$(2.5) \quad \begin{aligned} S &= e^{(a+ib)x} = e^{ax} e^{ibx} = \left(\sum_{m=0}^{\infty} \frac{a^m}{m!} x^m \right) \left(\sum_{n=0}^{\infty} \frac{(ib)^n}{n!} x^n \right) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a^m (ib)^n}{m! n!} x^{m+n} := S_e + S_o, \end{aligned}$$

where, upon separating the second sum into the even and odd parts, for convenience,

$$S_e := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a^m (ib)^{2n}}{m! (2n)!} x^{m+2n} \quad \text{and} \quad S_o := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a^m (ib)^{2n+1}}{m! (2n+1)!} x^{m+2n+1}.$$

Using the following easily-derivable formula (see, for example, [7, p. 6, Equation (28)]):

$$(2.6) \quad (\lambda)_{2n} = 2^{2n} \left(\frac{1}{2}\lambda\right)_n \left(\frac{1}{2}\lambda + \frac{1}{2}\right)_n \quad (n \in \mathbb{N}_0)$$

and the following well known formal manipulation of double series (see, for example, [6, p. 57, Equation (7)]; for other manipulations, see also [2]):

$$(2.7) \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A(n, m) = \sum_{m=0}^{\infty} \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} A(n, m - 2n),$$

we have

$$\begin{aligned}
 (2.8) \quad S_e &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n a^m b^{2n}}{m! (2n)!} x^{m+2n} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n a^m b^{2n}}{m! 2^{2n} n! \left(\frac{1}{2}\right)_n} x^{m+2n} \\
 &= \sum_{m=0}^{\infty} \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^n a^{m-2n} b^{2n}}{(m-2n)! 2^{2n} n! \left(\frac{1}{2}\right)_n} x^m.
 \end{aligned}$$

Using the following formula:

$$(2.9) \quad (n-k)! = \frac{(-1)^k n!}{(-n)_k} \quad (0 \leq k \leq n; n, k \in \mathbb{N}_0)$$

and (2.6), after a little simplification, we get

$$\begin{aligned}
 S_e &= \sum_{m=0}^{\infty} \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^n a^{m-2n} b^{2n} (-m)_{2n}}{m! 2^{2n} n! \left(\frac{1}{2}\right)_n} x^m \\
 &= \sum_{m=0}^{\infty} \frac{a^m x^m}{m!} \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{\left(-\frac{1}{2}m\right)_n \left(-\frac{1}{2}m + \frac{1}{2}\right)_n}{\left(\frac{1}{2}\right)_n n!} (-1)^n \left(\frac{b}{a}\right)^{2n}.
 \end{aligned}$$

Using (1.2) for the inner sum, we finally obtain

$$(2.10) \quad S_e = \sum_{m=0}^{\infty} \frac{a^m x^m}{m!} {}_2F_1 \left[\begin{matrix} -\frac{m}{2}, \frac{1-m}{2} \\ \frac{1}{2} \end{matrix}; -\frac{b^2}{a^2} \right].$$

Similarly as in getting (2.10), we also have

$$(2.11) \quad S_o = i x b \sum_{m=0}^{\infty} \frac{a^m x^m}{m!} {}_2F_1 \left[\begin{matrix} -\frac{m}{2}, \frac{1-m}{2} \\ \frac{3}{2} \end{matrix}; -\frac{b^2}{a^2} \right].$$

Hence, upon replacing S_e and S_o in (2.5) by (2.10) and (2.11), we immediately arrive at the desired formula (2.1). This completes the proof of Theorem 2. \square

Remark 1. Writing the Euler formula $e^{i n \vartheta} = \cos(n\vartheta) + i \sin(n\vartheta)$ in (2.1) and equating the real and imaginary parts, and then comparing the coefficients of x^n in each of two resulting identities, we easily see the formulas in Theorem 1.

3. Applications

Here we consider some interesting applications of the results (1.11) and (1.12). The results to be established are asserted by the following theorem.

Theorem 3. *Each of the following formulas holds true.*

$$(3.1) \quad e^x {}_0F_1 \left[\begin{matrix} -; \\ \frac{1}{2}; \end{matrix} -\frac{b^2}{4a^2}x^2 \right] = \sum_{m=0}^{\infty} \frac{x^m}{m!} \frac{(a^2 + b^2)^{\frac{m}{2}}}{a^m} \cos(m\vartheta);$$

$$(3.2) \quad e^x {}_0F_1 \left[\begin{matrix} -; \\ \frac{3}{2}; \end{matrix} -\frac{b^2}{4a^2}x^2 \right] = \sum_{m=1}^{\infty} \frac{x^{m-1}}{m!} \frac{(a^2 + b^2)^{\frac{m}{2}}}{a^{m-1}b} \sin(m\vartheta);$$

$$(3.3) \quad (1-x)^{-2\alpha} {}_2F_1 \left[\begin{matrix} \alpha, \alpha + \frac{1}{2}; \\ \frac{1}{2}; \end{matrix} -\frac{b^2 x^2}{a^2(1-x)^2} \right] \\ = \sum_{m=0}^{\infty} \frac{(2\alpha)_m x^m}{m!} \frac{(a^2 + b^2)^{\frac{m}{2}}}{a^m} \cos(m\vartheta);$$

$$(3.4) \quad (1-x)^{-2\alpha} {}_2F_1 \left[\begin{matrix} \alpha, \alpha + \frac{1}{2}; \\ \frac{3}{2}; \end{matrix} -\frac{b^2 x^2}{a^2(1-x)^2} \right] \\ = \sum_{m=0}^{\infty} \frac{(2\alpha)_m x^m}{m!} \frac{(a^2 + b^2)^{\frac{m+1}{2}}}{a^m b} [\sin(m+1)\vartheta],$$

where ϑ is the same as given in (1.13).

Proof. Denoting, for simplicity, the left-hand side of (3.1) by L and expressing the two involved functions as series, after a little simplification, we have

$$L = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\left(-\frac{b^2}{a^2}\right)^n x^{m+2n}}{\left(\frac{1}{2}\right)_n m! n! 2^{2n}}.$$

Using (2.7), we obtain

$$L = \sum_{m=0}^{\infty} \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{\left(-\frac{b^2}{a^2}\right)^n x^m}{\left(\frac{1}{2}\right)_n n! 2^{2n} (m-2n)!}.$$

Using the following easily-derivable identities (see Equations (2.6) and (2.9)):

$$(m-2n)! = \frac{m!}{(-m)_{2n}} \quad \text{and} \quad (-m)_{2n} = 2^{2n} \left(-\frac{m}{2}\right)_n \left(-\frac{m}{2} + \frac{1}{2}\right)_n,$$

after a little simplification, we get

$$L = \sum_{m=0}^{\infty} \frac{x^m}{m!} \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{\left(-\frac{m}{2}\right)_n \left(-\frac{m}{2} + \frac{1}{2}\right)_n}{\left(\frac{1}{2}\right)_n n!} \left(-\frac{b^2}{a^2}\right)^n.$$

Expressing the inner series by using the functions ${}_pF_q$ as in (1.2), we have

$$L = \sum_{m=0}^{\infty} \frac{x^m}{m!} {}_2F_1 \left[\begin{matrix} -\frac{m}{2}, \frac{1-m}{2} \\ \frac{1}{2} \end{matrix}; -\frac{b^2}{a^2} \right].$$

Finally, using the known summation formula (1.11), we arrive at the right-hand side of (3.1). This completes the proof of (3.1). The same argument as in proving (3.1) will easily establish the other three formulas (3.2) to (3.4). \square

If we take $b = a$ in Equations (3.1) to (3.4), we get the following known (see [1]) results asserted by the following theorem.

Theorem 4. *Each of the following identities holds true.*

$$(3.5) \quad e^x {}_0F_1 \left[\begin{matrix} - \\ \frac{1}{2} \end{matrix}; -\frac{x^2}{4} \right] = \sum_{m=0}^{\infty} \frac{x^m}{m!} 2^{\frac{m}{2}} \cos \left(\frac{m\pi}{4} \right);$$

$$(3.6) \quad e^x {}_0F_1 \left[\begin{matrix} - \\ \frac{3}{2} \end{matrix}; -\frac{x^2}{4} \right] = \sum_{m=1}^{\infty} \frac{x^{m-1}}{m!} 2^{\frac{m}{2}} \sin \left(\frac{m\pi}{4} \right);$$

$$(3.7) \quad (1-x)^{-2\alpha} {}_2F_1 \left[\begin{matrix} \alpha, \alpha + \frac{1}{2} \\ \frac{1}{2} \end{matrix}; -\frac{x^2}{(1-x)^2} \right] = \sum_{m=0}^{\infty} \frac{(2\alpha)_m x^m}{m!} 2^{\frac{m}{2}} \cos \left(\frac{m\pi}{4} \right);$$

$$(3.8) \quad (1-x)^{-2\alpha} {}_2F_1 \left[\begin{matrix} \alpha, \alpha + \frac{3}{2} \\ \frac{1}{2} \end{matrix}; -\frac{x^2}{(1-x)^2} \right] = \sum_{m=0}^{\infty} \frac{(2\alpha)_m x^m}{(m+1)!} 2^{\frac{m+1}{2}} \sin \left[\frac{(m+1)\pi}{4} \right].$$

Remark 2. Here it is interesting to point out that, in Equations (3.3) and (3.4), if we replace x by $\frac{x}{\alpha}$ and let $\alpha \rightarrow \infty$, after a little simplification, we get the results (3.1) and (3.2), respectively.

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