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LAPLACIAN ON A QUANTUM HEISENBERG MANIFOLD

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Abstract. In this paper we give a definition of the Hodge type Laplacian Δ on a non-commutative manifold which is the smooth dense subalgebra of a C^*-algebra. We prove that the Laplacian on a quantum Heisenberg manifold is an elliptic operator in the sense that $(\Delta+1)^{-1}$ is compact.

1. Introduction

In non-commutative geometry, the Chevalley-Eilenberg complex is used to produce a cyclic cocycle in Connes' cyclic cohomology via a cycle over an algebra A where a Lie-group action on A is given. The most important result in this direction is the integrality of the pairing of a cyclic 2-cocycle and Rieffel's projection in the non-commutative torus A_θ [3, 6, 8].

In this paper, using the same framework, we investigate a metric aspect of this complex. In fact, we define "Laplacian" on a non-commutative manifold which is slightly different with the one given in [10] and establish a Hodge-type theorem of the Laplacian on a quantum Heisenberg manifold. While the non-commutative torus is simpler, quantum Heisenberg manifolds with the non-commutative Heisenberg group action are tractable non-commutative manifolds given by the stirct deformation quantization of the classical Heisenberg manifold [7].

We emphasize that if the group action is commutative and a C^*-algebra A is deformed from the group, this is not so interesting since the metric aspect on A is almost commutative as we observe the non-commutative torus case [3, 5]. Thus it seems natural to consider the Heisenberg group action and a deformation from it. We show that in this case the Laplacian on zero forms or "functions" is diagonalizable and eigenvalues form a discrete set of $\mathbb{R}^+$ which is well-known for a connected, oriented Riemmanian manifold (see, for example, [9]).

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2. Laplacian on a non-commutative manifold

Let G be a finite dimensional Lie-group and $\mathfrak{g}$ its Lie-algebra. Let (A, G, α) be a C^*-dynamical system that consists of a C^*-algebra A together with a homomorphism α of G into the group of automorphisms of A. We always assume that A is equipped with a G-invariant trace τ in the sense that $\tau(\alpha_g(a)) = \tau(a)$ for all $g \in G$ and $a \in A$. It is said that a in A is of C^∞ if and only if $g \to \alpha_g(a)$ from G to the normed space is of C^∞. Then $A^\infty = \{a \in A \mid a \text{ is of } C^\infty\}$ is norm dense in A. In this case we call A^∞ the smooth dense subalgebra of A [4]. Since a C^*-algebra with a smooth dense subalgebra is an analogue of a smooth manifold, $\mathfrak{g}$ plays a role of tangent space or directional derivatives via the map δ which is the representation of $\mathfrak{g}$ in the Lie-algebra of (unbounded) derivations of A^∞ given by

$$\delta_X(a) = \lim_{t \to 0} \frac{1}{t}(\alpha_{g_t}(a) - a)$$

for $X \in \mathfrak{g}$ and $a \in A^\infty$ where g_t is the path in G such that $\dot{g}_0 = X$ [4].

Then to do calculus we consider a A^∞-module of alternating A^∞-valued forms on $\overbrace{\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}^{k}$ denoted by $\Omega^k = (\Lambda^k \mathfrak{g})^* \otimes A^\infty$. The coboundary map from Ω^k to Ω^{k+1} is defined by

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) = &\sum_{j=1}^{k+1} (-1)^{j-1} \delta_{X_j}(\omega(X_1, \ldots, \widehat{X_j}, \ldots, X_{k+1})) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}) \end{aligned}$$

for $k \geq 1$. In particular, the coboundary map from $\Omega^0 = A^\infty$ to $\Omega^1 = \mathrm{Hom}(\mathfrak{g}, A^\infty)$ is defined by $da(X) = \delta_X(a)$ for $X \in \mathfrak{g}$, $a \in A^\infty$.

Proposition 2.1. $d^2 = 0$.

Proof. For $k \geq 1$, it follows from a general Cartan formula. We check the case that

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2.$$

$$\begin{aligned} d(da)(X_1, X_2) &= \delta_{X_1}(da(X_2)) - \delta_{X_2}(da(X_1)) - d(a)([X_1, X_2]) \\ &= \delta_{X_1}(\delta_{X_2}(a)) - \delta_{X_2}(\delta_{X_1}(a)) - \delta_{[X_1, X_2]}(a) \\ &= [\delta_{X_1}, \delta_{X_2}](a) - \delta_{[X_1, X_2]}(a) \\ &= 0 \end{aligned}$$

since $\delta : \mathfrak{g} \to \mathrm{Der}(A^\infty)$ is a Lie-algebra homomorphism, i.e., $\delta_{[X_1, X_2]} = [\delta_{X_1}, \delta_{X_2}]$. Here $\mathrm{Der}(A^\infty)$ means a set of derivations on A^∞. $\square$

(Ω^k, d) is called a "Chevalley-Eilenberg" (chain) complex. For $\omega \in \Omega^p$ and $\eta \in \Omega^q$, we define a product $\wedge$ by

$$\omega \wedge \eta(X_1, \ldots, X_{p+q}) = \sum_{(p,q)\text{-shuffles}} \omega(X_{\sigma(1)}, \ldots, X_{\sigma(p)})\eta(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)})$$

where a (p, q) shuffle means a permutation σ such that $\sigma(1) < \cdots < \sigma(p)$, $\sigma(p+1) < \cdots < \sigma(p+q)$. Since A^∞ is a non-commutative algebra, this product never commutes, i.e., it is not true that $\omega \wedge \eta = (-1)^{pq}\eta \wedge \omega$. However, d is an anti-derivation of degree 1.

Proposition 2.2. *Given $\omega \in \Omega^p$ and $\eta \in \Omega^q$,*

$$d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^p \omega \wedge d(\eta).$$

Proof. It is well-known. See [2, Chapter III]. □

Throughout the article, we set $\{\theta^i\}$ as a dual (orthonormal) basis of an orthonormal basis $\{X_i\}$ of $\mathfrak{g}$. Suppose that $\dim \mathfrak{g} = n$. We introduce a notation for a A^∞-valued k-form in Ω^k. Let I be an index set such that $|I| = k$, i.e., $I = \{i_1 < \cdots < i_k\} \subset \{1, 2, \ldots, n\}$. We define $\theta^I = \theta^{i_1} \wedge \cdots \wedge \theta^{i_k}$. Then given $\omega \in \Omega^k$ we can write as $\omega = \sum_I a_I \theta^I$ where I runs over all size k subsets of $\{1, 2, \ldots, n\}$ and $a_I = \omega(X_{i_1}, \ldots, X_{i_k})$ since $\{\theta^I\}$ is a basis of $(\Lambda^k \mathfrak{g})^*$. As usual, we define the graded (differential) algebra $\Omega = \sum_{k=0} \Omega^k$.

Definition 2.3. The "exterior product" by X is the endomorphism $\varepsilon(X)$ on Ω defined by

$$\varepsilon(X_i)(a_I \theta^I) = a_I \theta^i \wedge \theta^I.$$

The "interior product" by X is the endomorphism $\iota(X)$ on Ω defined by

$$\iota(X_j)(a_I \theta^I) = a_I \iota(X_j)(\theta^I),$$

where the latter is the usual interior product on the exterior algebra of $\mathfrak{g}^*$.

Basically, viewing Ω^k as the tensor product of $(\Lambda^k \mathfrak{g})^*$ and A^∞, the operations $\varepsilon(X)$ and $\iota(X)$ are just $\varepsilon(X) \otimes I$ and $\iota(X) \otimes I$ where the latter $\varepsilon(X)$ and $\iota(X)$ are the usual maps on the exterior algebra of $\mathfrak{g}^*$.

We also extend the maps δ_X for a vector $X \in \mathfrak{g}$ to forms in Ω as follows.

Definition 2.4. The action of $\mathfrak{g}$ on Ω is the map defined by

$$\delta_X(a_I \theta^I) = \delta_X(a_I)\theta^I.$$

Then the derivation δ commutes with ε and ι.

Lemma 2.5. *For $X, Y \in \mathfrak{g}$*

$$\varepsilon(X) \cdot \delta_Y = \delta_Y \cdot \varepsilon(X), \tag{1}$$

$$\iota(X) \cdot \delta_Y = \delta_Y \cdot \iota(X). \tag{2}$$

Now we define an inner product structure on Ω^k as we have the (global) inner product on differential forms induced from a Riemannian metric on the tangent space $T_x(M)$ where M is a Riemannian manifold. We assume that $\mathfrak{g}$ is equipped with a metric $\langle , \rangle_{\mathfrak{g}}$ corresponding to a Riemannian metric. Then the induced metric on $\mathfrak{g}^*$ is also denoted by $\langle , \rangle_{\mathfrak{g}}$. Since we assume $\mathfrak{g}$ has an orthonormal basis, we can give a definition of the inner product without involving $\langle , \rangle_g$.

Definition 2.6. Given two k-forms ω and η, we define an inner product

$$\langle \omega, \eta \rangle = \tau \left(\sum_{i_1 < i_2 < \cdots < i_k} \omega(X_{i_1}, \ldots, X_{i_k})^* \eta(X_{i_1}, \ldots, X_{i_k}) \right),$$

where $\{X_i\}$ is an orthonormal basis of $\mathfrak{g}$.

Since τ plays a role of $\int_M$ dvol, our definition is a non-commutative analogue of the global inner product on forms on a connected, oriented Riemmanian manifold M, which is given by $\int_M g(\omega, \eta)$ dvol where g is a Riemannian metric. We note that for $a, b \in A^\infty$

$$\langle a, b \rangle = \tau(a^* b).$$

Since $C^\infty(M)$ act as multiplications on the Hilbert space $L^2(M, g)$ where M is a Riemannian manifold and its completion under the inner product $\int_M \bar{f} g$ dvol is $L^2(M, g)$, the appropriate completion of A^∞ under τ is the GNS-construction induced from τ. We denote it by $L^2(A^\infty, \tau)$ by the Hilbert space produced by GNS-construction.

Proposition 2.7. *$da = \sum_i \delta_i(a)\theta^i$ where $\delta_i = \delta_{X_i}$.*

Proof. Write $X = \sum_j \lambda_j X_j$ where $\{X_j\}$ is an orthonormal basis of $\mathfrak{g}$. Then $da(X) = \delta_{\sum_j \lambda_j X_j}(a) = \sum \lambda_j \delta_j(a) = \sum \theta^i(\lambda_j X_j)\delta_i(a) = \sum \delta_i(a)\theta^i(X)$. □

Definition 2.8. For $k \geq 0$, we define $d^* : \Omega^{k+1} \to \Omega^k$ as the formal adjoint of $d : \Omega^k \to \Omega^{k+1}$ with respect to the inner product $\langle , \rangle$. Thus it follows that $(d^*)^2 = 0$.

We define a "Dirac" operator on forms as follows, which will be denoted by D.

$$D = d + d^*.$$

And we define the (non-commutative) "Laplacian" on forms using D.

Definition 2.9. We define $\Delta : \Omega^k \to \Omega^k$ to be D^2, i.e., $\Delta = (d + d^*)(d + d^*) = dd^* + d^*d$.

The following is a useful lemma which is a non-commutative version of "integration by parts".

Lemma 2.10. *For any $a, b \in A^\infty$*

$$\tau(\delta_X(a)^* b) = -\tau(a^* \delta_X(b)).$$

Proof. Recall that τ is $\mathfrak{g}$-invariant so that $\tau(\delta_X(a)) = 0$ for all $X \in \mathfrak{g}$ and $a \in A^\infty$. Then, since $\delta_X : A^\infty \to A^\infty$ is a derivation,

$$\tau(\delta_X(a^*b)) = \tau(\delta_X(a)^*b + a^*\delta_X(b)) = 0.$$

So we have $\tau(\delta_X(a)^*b) = -\tau(a^*\delta_X(b))$. □

Proposition 2.11. *$d^*(\omega) = -\sum_i \delta_i(\omega(X_i))$ for $\omega \in \Omega^1$.*

Proof. Suppose ω is a one form, i.e., $\omega \in \Omega^1$. Then we can write $\omega = \sum_i \omega(X_i)\theta^i$.

$$\begin{aligned}
\langle da, \omega\rangle &= \sum_i \tau(\delta_i(a)^*\omega(X_i)) \\
&= \sum_i -\tau(a^*\delta_i(\omega(X_i))) \text{ by Lemma 2.10} \\
&= \tau(a^*(-\sum_i \delta_i(\omega(X_i)))) \\
&= \langle a, d^*(\omega)\rangle.
\end{aligned}$$

Thus $d^*(\omega) = -\sum_i \delta_i(\omega(X_i))$. □

Now we introduce the "Clifford algebra variables/ Dirac symbols"

$$e_i = \varepsilon(X_i) - \iota(X_i),$$

which satisfies

$$\{e_i, e_j\} = e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}.$$

Then on $\Omega^0 \oplus \Omega^1$

$$d + d^* = \sum_i e_i \cdot \delta_i. \tag{3}$$

We remark that the equation (3) does not hold on forms of higher order.

Proposition 2.12. *In particular, the Laplacian on zero forms or "functions" given by*

$$\Delta(a) = d^*(d(a)) = -\sum_i \delta_i(\delta_i(a))$$

for any $a \in A^\infty$.

Note that this definition of Laplacian coincides with the classical definition of the Laplacian on a Riemannian manifold where $A^\infty = C^\infty(M)$, $\mathfrak{g} = \mathbb{R}^n$. But, in general, we do not know how complicated the Laplacian could be. Basically, the coboundary map d in the Chevalley-Eilenberg complex reflects the non-commutativity of the Lie-algebra action. Moreover, the C^*-algebra itself is a non-commutative algebra so that the Laplacian on A^∞-valued k-forms might be quite different with the Laplacian on smooth forms of a manifold. Nonetheless, Proposition 2.12 says that the Laplacian on zero forms is always analogous to the Laplacian on functions on a manifold.

3. A case study: Laplacian on a quantum Heisenberg manifold

In this section, we are going to work out detailed computations on a non-commutative manifold which is a strict deformation quantization of the Heisenberg manifold. We begin with the definition of a quantum Heisenberg manifold which will be denoted by D_h.

Definition 3.1. For any positive integer c let S^c denote the space of C^∞ functions ϕ on $\mathbb{R} \times \mathbb{T} \times \mathbb{Z}$ which satisfy

(a) $\phi(x+k, y, p) = e(ckpy)\phi(x, y, p)$ for all $k \in \mathbb{Z}$ where $e(x)$ is $e^{2\pi i x}$, and
(b) for every polynomial P on $\mathbb{Z}$ and for every partial differential operator $\frac{\partial^{m+n}}{\partial x^m \partial y^n}$ on $\mathbb{R} \times \mathbb{T}$, $P(p)\frac{\partial^{m+n}}{\partial x^m \partial y^n}\phi(x, y, p)$ is bounded on $K \times \mathbb{Z}$ for any compact set K of $\mathbb{R} \times \mathbb{T}$.

For each $\hbar \in \mathbb{R}$ let $D^\infty_\hbar$ denote S^c with product and involution for each $\hbar \in \mathbb{R}$ defined by

(c) $\phi * \psi(x, y, p) = \sum_{q\in\mathbb{Z}} \phi(x - \hbar(q-p)\mu, y - \hbar(q-p)\nu, q)\psi(x - \hbar q\mu, y - \hbar q\nu, p-q)$,
(d) $\phi^*(x, y, p) = \overline{\phi}(x, y, -p)$,

and with the norm coming from the representation on $L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$ defined by

$$\phi(f)(x, y, p) = \sum_q \phi(x - \hbar(q-2p)\mu, y - \hbar(q-2p)\nu, q)f(x, y, p-q),$$

where μ, ν are non-zero real numbers. Then the norm closure of the involutive algebra $D^\infty_\hbar$, to be denoted by $D_\hbar$, is called a quantum Heisenberg manifold ([7, Theorem 5.5]). Since we are going to work with the fixed parameters μ, ν, we dropped them in the definition.

Let G be the Heisenberg group, parametrized by

$$(x, y, z) = \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix},$$

so that when we identify G with $\mathbb{R}^3$ the product is given by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + yx').$$

Then there is a canonical action of G on $D_\hbar$, or $D^\infty_\hbar$ given by

$$\alpha_{(r,s,t)}(\phi)(x, y, p) = e(p(t + cs(x-r)))\phi(x - r, y - s, p).$$

In addition, we have a faithful normal trace τ defined by

$$\tau(\phi) = \int_0^1 \int_{\mathbb{T}} \phi(x, y, 0)dydx \quad \text{for } \phi \in D^\infty_\hbar.$$

It is easily checked that τ is invariant under the action of G. Let's consider the GNS-representation of $(D^\infty_\hbar, \tau)$, denoted by $L^2(D^\infty_\hbar, \tau)$.

Lemma 3.2 ([1, Proposition 6]). *$L^2(D_\hbar^\infty, \tau)$ is unitarily equivalent with $L^2([0,1] \times [0,1] \times \mathbb{Z})$.*

Proof. We define a map $\Gamma : L^2(D_\hbar^\infty, \tau) \to L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$ by

$$\Gamma\phi(x,y,p) = \begin{cases} e(-cxyp)\phi(x,y,p) & \text{for } y < 1, \\ \phi(x,y,p) & \text{for } y = 1, \end{cases}$$

for $\phi \in L^2(D_\hbar^\infty, \tau)$. Note that $\Gamma\phi(x+k,y,p) = \Gamma\phi(x,y,p)$. It follows that $L^2(D_\hbar^\infty, \tau) \simeq L^2(\mathbb{T} \times \mathbb{T} \times \mathbb{Z}) \simeq L^2([0,1] \times [0,1] \times \mathbb{Z})$. $\square$

Since τ is invariant under the Heisenberg group action, the action is lifted to $L^2(D_\hbar^\infty, \tau)$. We shall denote the action by the same symbol. To compute the Laplacian on $D_\hbar^\infty$, we need to transform $L^2([0,1] \times [0,1] \times \mathbb{Z})$ under a map $\eta : L^2([0,1] \times [0,1] \times \mathbb{Z}) \to L^2([0,1] \times [0,1] \times \mathbb{Z})$ given by

$$\eta(f)(x,y,p) = \begin{cases} e(-cxyp)f(x,y,p) & \text{for } y < 1, \\ f(x,y,p) & \text{for } y = 1. \end{cases}$$

For a technical reason, consider a particular basis of $\mathfrak{g}$ given by

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; X_3 = \begin{pmatrix} 0 & 0 & c\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some $\alpha > 1$. Then δ_i's on $L^2([0,1] \times [0,1] \times \mathbb{Z})$ are given by

$$\delta_1(f)(x,y,p) = -2\pi icypf(x,y,p) - \frac{\partial f}{\partial x}(x,y,p), \tag{4}$$

$$\delta_2(f)(x,y,p) = -\frac{\partial f}{\partial y}(x,y,p), \tag{5}$$

$$\delta_3(f)(x,y,p) = 2\pi ipc\alpha f(x,y,p) \tag{6}$$

under Γ and η [1, Proposition 9].

Let T and S be (unbounded) operators on a Hilbert space H. We say that S is T-bounded, with relative bound $s \geq 0$, if

(1) $\mathrm{dom}(T) \subset \mathrm{dom}(S)$,
(2) for some $b \geq 0$ and for all $\xi \in H$,

$$\|S\xi\| \leq s\|T\xi\| + b\|\xi\|.$$

For example, a bounded operator S is T-bounded for any T, with relative bound $s = 0$.

Lemma 3.3 (Kato-Rellich). *Let T be a self-adjoint operator and S be a symmetric operator. If S is T-bounded with relative bound < 1, then $T + S$ is a self-adjoint operator on* $\mathrm{dom}(T)$.

Now we are ready to prove the main theorem which is a version of noncommutative Hodge theory for a quantum Heisenberg manifold. We borrow the idea of our proof from [1, Proposition 9], though the setting and aim are quite different.

Theorem 3.4. *Suppose that δ_i's are given as above (see equations (3), (4), (5)). Let the "Laplacian" Δ be defined by $-\sum_i \delta_i^2$ on $D_\hbar^\infty$. Then there exists an orthonormal basis of $L^2(D_\hbar^\infty, \tau)$ consisting of eigenvectors of the Laplacian. All the eigenvalues are positive, except that zero is an eigenvalue with multiplicity one. Each eigenvalue has finite multiplicity, and the eigenvalues accumulate only at infinity.*

Proof. First, note that the domain of D on $L^2([0,1]\times[0,1]\times\mathbb{Z})$ is given by all those square integrable functions f that satisfy the boundary conditions

(i) $f(0,y,p) = f(1,y,p)$,
(ii) $f(x,0,p) = f(x,1,p)$,
(iii) $\frac{\partial}{\partial x}f, \frac{\partial}{\partial y}f, pf$ are square integrable.

Let $T = -e_1 \cdot \frac{\partial}{\partial x} - e_2 \cdot \frac{\partial}{\partial y} + e_3 \cdot 2\pi i c\alpha M_p$ and $S = -e_1 \cdot 2\pi i c M_{yp}$ where $M_p(f)(x,y,p) = pf(x,y,p)$, and $M_{yp}(f)(x,y,p) = ypf(x,y,p)$. Then $D = d + d^* = T + S$ and $\mathrm{dom}(T) = \mathrm{dom}(D) \subset \mathrm{dom}(S)$.

Viewing $L^2([0,1]\times[0,1]\times\mathbb{Z})$ as $L^2([0,1]\times[0,1]) \otimes l^2(\mathbb{Z})$, we can see $T^2 = -(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \otimes \mathrm{id} - \mathrm{id} \otimes (4\pi^2 c^2\alpha^2 M_{p^2})$. Observe that after taking Fourier transform on the first two variables, T^2 is nothing but $N_1^2 + N_2^2 + 4\pi^2c^2\alpha^2 N_3^2$ where each N_i is the number (multiplication) operator on the appropriate copy of $\mathbb{Z}$. It follows that $(T^2+1)^{-1}$ is compact operator or T has compact resolvents. In other words, $(T\pm i)^{-1}$ are compacts. Since S is T-bounded with relative bound less that $\frac{1}{\alpha} < 1$, D is self-adjoint on $\mathrm{dom}(T)$ by the Kato-Rellich lemma. From

$$-((D\pm i)^{-1} - (T\pm i)^{-1}) = (D\pm i)^{-1}(D-T)(T\pm i)^{-1},$$

$(D\pm i)^{-1}$ are compacts too. Consequently, $(\Delta+1)^{-1} = (D+i)^{-1}(D-i)^{-1}$ is compact. □

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