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# ANALYTIC CONTINUATION OF WEIGHTED $q$-GENOCCHI NUMBERS AND POLYNOMIALS

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Abstract. In the present paper, we analyse analytic continuation of weighted $q$-Genocchi numbers and polynomials. A novel formula for weighted $q$-Genocchi-zeta function $\widetilde{\zeta}_{G,q}(s \mid \alpha)$ in terms of nested series of $\widetilde{\zeta}_{G,q}(n \mid \alpha)$ is derived. Moreover, we introduce a novel concept of dynamics of the zeros of analytically continued weighted $q$-Genocchi polynomials.

## 1. Introduction

In this paper, we use notations like $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$, where $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the field of real numbers and $\mathbb{C}$ also denotes the set of complex numbers. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number or a $p$-adic number.

Throughout this work, we will assume that $q \in \mathbb{C}$ with $|q| < 1$. The $q$-integer symbol $[x : q]$ denotes as

$$[x : q] = \frac{q^x - 1}{q - 1} \text{ (see [1-10]).}$$

Firstly, analytic continuation of $q$-Euler numbers and polynomials was investigated by Kim in [8]. He gave a new concept of dynamics of the zeros of analytically continued $q$-Euler polynomials. Actually, we are motivated from his excellent paper which is "Analytic continuation of $q$-Euler numbers and polynomials, Applied Mathematics Letters 21 (2008), 1320–1323". By the same motivation, we also procure the analytic continuation of weighted $q$-Genocchi numbers and polynomials as parallel to his article. However, we give some interesting identities by using generating function of weighted $q$-Genocchi polynomials.

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## 2. Some properties of the weighted $q$-Genocchi numbers and polynomials

For $\alpha \in \mathbb{N} \cup \{0\}$, the weighted $q$-Genocchi polynomials are defined by means of the following generating function:

For $x \in \mathbb{C}$,

$$\sum_{n=0}^{\infty} \widetilde{G}_{n,q}(x \mid \alpha) \frac{t^n}{n!} = [2:q]\, t \sum_{n=0}^{\infty} (-1)^n q^n e^{t[n+x:q^\alpha]}. \tag{2.1}$$

In the special case, $x = 0$ in (2.1), $\widetilde{G}_{n,q}(0 \mid \alpha) := \widetilde{G}_{n,q}(\alpha)$ are called the weighted $q$-Genocchi numbers. By (2.1), we readily derive the following:

$$\frac{\widetilde{G}_{n+1,q}(x \mid \alpha)}{n+1} = \frac{[2:q]}{[\alpha:q]^n (1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{1+q^{\alpha l+1}}, \tag{2.2}$$

where $\binom{n}{l}$ is the binomial coefficient. By expression (2.1), we see that

$$\widetilde{G}_{n,q}(x \mid \alpha) = q^{-\alpha x} \left( q^{\alpha x} \widetilde{G}_q(\alpha) + [x:q^\alpha] \right)^n, \tag{2.3}$$

with the usual convention of replacing $\left(\widetilde{G}_q(\alpha)\right)^n$ by $\widetilde{G}_{n,q}(\alpha)$ is used (for details, see [1], [2]).

Let $\widetilde{T}_q^{(\alpha)}(x,t)$ be the generating function of weighted $q$-Genocchi polynomials as follows:

$$\widetilde{T}_q^{(\alpha)}(x,t) = \sum_{n=0}^{\infty} \widetilde{G}_{n,q}(x \mid \alpha) \frac{t^n}{n!}. \tag{2.4}$$

Then, we easily notice that

$$\widetilde{T}_q^{(\alpha)}(x,t) = [2:q]\, t \sum_{n=0}^{\infty} (-1)^n q^n e^{t[n+x:q^\alpha]}. \tag{2.5}$$

From expressions (2.4) and (2.5), we procure the followings:

For $k$ (=even) and $n, \alpha \in \mathbb{N} \cup \{0\}$, we have

$$q^k \frac{\widetilde{G}_{n+1,q}(k \mid \alpha)}{n+1} - \frac{\widetilde{G}_{n+1,q}(\alpha)}{n+1} = [2:q] \sum_{l=0}^{k-1} (-1)^{l+1} q^l [l:q^\alpha]^n. \tag{2.6}$$

For $k$ (=odd) and $n, \alpha \in \mathbb{N} \cup \{0\}$, we have

$$q^k \frac{\widetilde{G}_{n+1,q}(k \mid \alpha)}{n+1} + \frac{\widetilde{G}_{n+1,q}(\alpha)}{n+1} = [2:q] \sum_{l=0}^{k-1} (-1)^{l} q^l [l:q^\alpha]^n. \tag{2.7}$$

Via Eq.(2.5), we easily obtain the following:

$$\widetilde{G}_{n,q}(x \mid \alpha) = q^{-\alpha x} \sum_{j=0}^{n} \binom{n}{j} q^{\alpha j x} \widetilde{G}_{j,q}(\alpha) [x:q^\alpha]^{n-j}. \tag{2.8}$$

From (2.6)-(2.8), we get the following:

(2.9)

$$[2:q]\sum_{l=0}^{k-1}(-1)^{l+1}q^{l}[l:q^{\alpha}]^{n}$$
$$=\left(q^{k(1+\alpha n)}-1\right)\frac{\widetilde{G}_{n+1,q}(\alpha)}{n+1}+\frac{q^{(1-\alpha)k}}{n+1}\sum_{j=0}^{n}\binom{n+1}{j}q^{\alpha jk}\widetilde{G}_{j,q}(\alpha)[k:q^{\alpha}]^{n+1-j},$$

where $k$ is an even positive integer. If $k$ is an odd positive integer. Then, we can derive the following equality:

(2.10)

$$[2:q]\sum_{l=0}^{k-1}(-1)^{l}q^{l}[l:q^{\alpha}]^{n}$$
$$=\left(q^{k(1+\alpha n)}+1\right)\frac{\widetilde{G}_{n+1,q}(\alpha)}{n+1}+\frac{q^{(1-\alpha)k}}{n+1}\sum_{j=0}^{n}\binom{n+1}{j}q^{\alpha jk}\widetilde{G}_{j,q}(\alpha)[k:q^{\alpha}]^{n+1-j}.$$

## 3. On the weighted $q$-Genocchi-zeta function

The famous Genocchi polynomials are defined by

$$\frac{2t}{e^{t}+1}e^{xt}=\sum_{n=0}^{\infty}G_{n}(x)\frac{t^{n}}{n!},\ \ |t|<\pi(\text{cf. [6]}).\tag{3.1}$$

For $s\in\mathbb{C}$, $x\in\mathbb{R}$ with $0\leq x<1$, Genocchi-Zeta function is given by

$$\zeta_{G}(s,x)=2\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(n+x)^{s}},\tag{3.2}$$

and

$$\zeta_{G}(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^{s}}.\tag{3.3}$$

By (3.1), (3.2) and (3.3), Genocchi-zeta functions are related to the Genocchi numbers as follows:

$$\zeta_{G}(-n)=\frac{G_{n+1}}{n+1}.$$

Moreover, it is simple to see

$$\zeta_{G}(-n,x)=\frac{G_{n+1}(x)}{n+1}.$$

The weighted $q$-Genocchi Hurwitz-zeta type function is defined by

$$\widetilde{\zeta}_{G,q}(s,x\mid\alpha)=[2:q]\sum_{m=0}^{\infty}\frac{(-1)^{m}q^{m}}{[m+x:q^{\alpha}]^{s}}.$$

Similarly, weighted $q$-Genocchi-zeta function is given by

$$\widetilde{\zeta}_{G,q}\left(s \mid \alpha\right) = \left[2:q\right] \sum_{m=1}^{\infty} \frac{\left(-1\right)^{m} q^{m}}{\left[m:q^{\alpha}\right]^{s}}.$$

For $n, \alpha \in \mathbb{N} \cup \{0\}$, we have

$$\widetilde{\zeta}_{G,q}\left(-n \mid \alpha\right) = \frac{\widetilde{G}_{n+1,q}\left(\alpha\right)}{n+1}.$$

We now consider the function $\widetilde{G}_{q}\left(n:\alpha\right)$ as the analytic continuation of weighted $q$-Genocchi numbers. All the weighted $q$-Genocchi numbers agree with $\widetilde{G}_{q}\left(n:\alpha\right)$, the analytic continuation of weighted $q$-Genocchi numbers evaluated at $n$. For $n \geq 0$, $\widetilde{G}_{q}\left(n:\alpha\right) = \widetilde{G}_{n,q}\left(\alpha\right)$.

We can now state $\widetilde{G}'_{q}\left(s:\alpha\right)$ in terms of $\widetilde{\zeta}_{G,q}\left(s \mid \alpha\right)$, the derivative of $\widetilde{\zeta}_{G,q}\left(s:\alpha\right)$

$$\frac{\widetilde{G}_{q}\left(s+1:\alpha\right)}{s+1} = \widetilde{\zeta}_{G,q}\left(-s \mid \alpha\right), \; \frac{\widetilde{G}'_{q}\left(s+1:\alpha\right)}{s+1} = \widetilde{\zeta}_{G,q}\left(-s \mid \alpha\right).$$

For $n, \alpha \in \mathbb{N} \cup \{0\}$

$$\frac{\widetilde{G}'_{q}\left(2n+1:\alpha\right)}{2n+1} = \widetilde{\zeta}_{G,q}\left(-2n \mid \alpha\right).$$

This is suitable for the differential of the functional equation and so supports the coherence of $\widetilde{G}_{q}\left(s:\alpha\right)$ and $\widetilde{G}'_{q}\left(s:\alpha\right)$ with $\widetilde{G}_{n,q}\left(\alpha\right)$ and $\widetilde{\zeta}_{G,q}\left(s \mid \alpha\right)$. From the analytic continuation of weighted $q$-Genocchi numbers, we derive as follows:

$$\frac{\widetilde{G}_{q}\left(s+1:\alpha\right)}{s+1} = \widetilde{\zeta}_{G,q}\left(-s \mid \alpha\right) \text{ and } \frac{\widetilde{G}_{q}\left(-s+1:\alpha\right)}{-s+1} = \widetilde{\zeta}_{G,q}\left(s \mid \alpha\right).$$

Moreover, we derive the following: For $n \in \mathbb{N} - \{1\}$

$$\frac{\widetilde{G}_{-n+1,q}\left(\alpha\right)}{-n+1} = \frac{\widetilde{G}_{q}\left(-n+1:\alpha\right)}{-n+1} = \widetilde{\zeta}_{G,q}\left(n \mid \alpha\right).$$

The curve $\widetilde{G}_{q}\left(s:a\right)$ review quickly the points $\widetilde{G}_{-s,q}\left(\alpha\right)$ and grows $\sim n$ asymptotically $\left(-n\right) \to -\infty$. The curve $\widetilde{G}_{q}\left(s:a\right)$ review quickly the point $\widetilde{G}_{q}\left(-s:a\right)$. Then, we procure the following:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\widetilde{G}_{q}\left(-n+1:\alpha\right)}{-n+1} &= \lim_{n\to\infty} \widetilde{\zeta}_{G,q}\left(n \mid \alpha\right) \\
&= \lim_{n\to\infty} \left(\left[2:q\right] \sum_{m=1}^{\infty} \frac{\left(-1\right)^{m} q^{m}}{\left[m:q^{\alpha}\right]^{n}}\right) \\
&= \lim_{n\to\infty} \left(-q\left[2:q\right] + \left[2:q\right] \sum_{m=2}^{\infty} \frac{\left(-1\right)^{m} q^{m}}{\left[m:q^{\alpha}\right]^{n}}\right) \\
&= -q^{2}\left[2:q^{-1}\right].
\end{aligned}$$

From this, we easily note that

$$\frac{\widetilde{G}_q(-n+1:\alpha)}{-n+1}=\widetilde{\zeta}_{G,q}(n\mid\alpha)\mapsto\frac{\widetilde{G}_q(-s+1:\alpha)}{-s+1}=\widetilde{\zeta}_{G,q}(s\mid\alpha).$$

## 4. Analytic continuation of the weighted $q$-Genocchi polynomials

For coherence with the redefinition of $\widetilde{G}_{n,q}(\alpha)=\widetilde{G}_q(n:\alpha)$, we have

$$\widetilde{G}_{n,q}(x\mid\alpha)=q^{-\alpha x}\sum_{k=0}^{n}\binom{n}{k}q^{\alpha kx}\widetilde{G}_{k,q}(\alpha)\left[x:q^{\alpha}\right]^{n-k}.$$

Let $\Gamma(s)$ be Euler-gamma function. Then the analytic continuation can be get as

$$\begin{aligned}
&n\mapsto s\in\mathbb{R},\ x\mapsto w\in\mathbb{C},\\
&\widetilde{G}_{n,q}(\alpha)\mapsto\widetilde{G}_q(k+s-[s]:\alpha)=\widetilde{\zeta}_{G,q}(-(k+s-[s])\mid\alpha),\\
&\tbinom{n}{k}=\tfrac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)}\mapsto\frac{\Gamma(s+1)}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)}\\
&\widetilde{G}_{s,q}(w\mid\alpha)\mapsto\widetilde{G}_q(s,w:\alpha)\\
&\quad=q^{-\alpha w}\sum_{k=-1}^{[s]}\frac{\Gamma(s+1)\widetilde{G}_q(k+(s-[s]):\alpha)\,q^{\alpha w(k+(s-[s]))}}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)}\left[w:q^{\alpha}\right]^{[s]-k}\\
&\quad=q^{-\alpha w}\sum_{k=0}^{[s]+1}\frac{\Gamma(s+1)\widetilde{G}_q(-1+k+(s-[s]):\alpha)\,q^{\alpha w(k-1+(s-[s]))}}{\Gamma(k+(s-[s]))\Gamma(2+[s]-k)}\left[w:q^{\alpha}\right]^{[s]+1-k}.
\end{aligned}$$

Here $[s]$ gives the integer part of s, and so $s-[s]$ gives the fractional part.

Deformation of the curve $\widetilde{G}_q(1,w:\alpha)$ into the curve of $\widetilde{G}_q(2,w:\alpha)$ is by means of the real analytic cotinuation $\widetilde{G}_q(s,w:\alpha)$, $1\leq s\leq 2$, $-0.5\leq w\leq 0.5$.

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