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HARDILY RANKED BIGROUPOIDS

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Abstract. The notion of hardily ranked bigroupoids is introduced and related properties are investigated. By considering congruence relations on a hardily ranked bigroupoid, the quotient structure of hardily ranked bigroupoids is discussed.

1. Introduction

Alshehri et al. [1] introduced the notion of ranked bigroupoids and discussed $(X, *, \&)$-self-(co)derivations. Jun et al. [2] investigated further properties on $(X, *, \&)$-self-(co)derivations, and provided conditions for an $(X, *, \&)$-self-(co)derivation to be regular. They introduced the notion of ranked $*$-subsystems, and investigated related properties. Jun et al. [3] discussed the generalization of coderivations of ranked bigroupoids, and introduced the notion of generalized coderivations in ranked bigroupoids. Combining a generalized self-coderivation with a rankomorphism, they obtained new generalized coderivations of ranked bigroupoids. From the notion of $(X, *, \&)$-derivation, they induced the existence of a rankomorphism of ranked bigroupoids.

In this paper, we introduce the notion of hardily ranked bigroupoids, and investigate related properties. By considering congruence relations on a hardily ranked bigroupoid, we discuss the quotient structure of hardily ranked bigroupoids.

2. Preliminaries

Let X be a set with a distinguished element 0. For any binary operation $\natural$ on X, we consider the following axioms:

(2.1) $$x\natural y = 0 \text{ and } y\natural x = 0 \text{ imply } x = y,$$

(2.2) $$x\natural(y\natural x) = 0,$$

(2.3) $$(x\natural(y\natural z))\natural((x\natural y)\natural(x\natural z)) = 0,$$

(2.4) $$x\natural x = 0 = x\natural 0,\ 0\natural x = x,$$

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$$x\natural(y\natural z) = y\natural(x\natural z), \tag{2.5}$$

$$x\natural(y\natural z) = (x\natural y)\natural(x\natural z), \tag{2.6}$$

$$x\natural y = 0 \ \Rightarrow (z\natural x)\natural(z\natural y) = 0,\ (y\natural z)\natural(x\natural z) = 0, \tag{2.7}$$

A *ranked bigroupoid* (see [1]) is an algebraic system $(X, *, \bullet)$ where X is a non-empty set and "$*$" and "$\bullet$" are binary operations defined on X. We may consider the first binary operation $*$ as the major operation, and the second binary operation $\bullet$ as the minor operation.

3. Hardily ranked bigroupoids

Definition 3.1. Let $(X, *, \&)$ be a ranked bigroupoid with a distinguished element 0. Then $(X, *, \&)$ is called a *hardily ranked bigroupoid* if it satisfies:

(1) X is a semigroup under the minor operation $(\&)$ in which the minor operation $(\&)$ is distributive (on both sides) over the major operation $(*)$, that is,

$$x\&(y * z) = (x\&y) * (x\&z),\ (x * y)\&z = (x\&z) * (y\&z) \tag{3.1}$$

for all $x, y, z \in X$,

(2) The major operation $(*)$ satisfies axioms (2.1), (2.2) and (2.3).

Example 3.2. Consider a set $X = \{0, a, b, c\}$ with a major operation $(*)$ and a minor operation $(\&)$ which are given as follows:

$$x * y = \begin{cases} a & \text{if } (x, y) \in \{(0, a), (b, a), (c, a)\}, \\ b & \text{if } (x, y) \in \{(0, b), (a, b), (c, b)\}, \\ c & \text{if } (x, y) \in \{(0, c), (a, c), (b, c)\}, \\ 0 & \text{otherwise} \end{cases}$$

and

&	0	a	b	b
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

It is easy to verify that $(X, *, \&)$ is a hardily ranked bigroupoid.

Proposition 3.3. *Every hardily ranked bigroupoid $(X, *, \&)$ satisfies the axioms* (2.4), (2.5), (2.6) *and* (2.7).

Proof. It is easy, and so we omit the proof. ☐

Proposition 3.4. *Let $(X, *, \&)$ be a hardily ranked bigroupoid. Then*

(1) $(\forall x \in X)\,(0\&x = x\&0 = 0)$.

(2) $(\forall x, y \in X)\,(x * y = 0 \ \Rightarrow \ (x\&z) * (y\&z) = 0,\ (z\&x) * (z\&y) = 0)$.

Proof. (1) Using (2.4) and (3.1), we have

$$x\&0 = x\&(0 * 0) = (x\&0) * (x\&0) = 0$$

and

$$0\&x = (0 * 0)\&x = (0\&x) * (0\&x) = 0$$

for all $x \in X$.

(2) Let $x, y \in X$ be such that $x * y = 0$. Then

$$(z\&x) * (z\&y) = z\&(x * y) = z\&0 = 0$$

and

$$(x\&z) * (y\&z) = (x * y)\&z = 0\&z = 0$$

for all $z \in X$. □

Let $\triangle$ be a new operation on a hardily ranked bigroupoid $(X, *, \&)$ which is defined by

$$(\forall x, y \in X)\,(x\triangle y = (y * x) * x)\,.$$

Proposition 3.5. *Every hardily ranked bigroupoid* $(X, *, \&)$ *satisfies the following conditions*:

(1) $(\forall x, y, z \in X)\,(x\&(y\triangle z) = (x\&z)\triangle(y\&z))\,,$
(2) $(\forall x, y \in X)\,((x * y)\triangle x = 0,\ (x * y)\triangle y = x * y)\,,$
(3) $(\forall x, y \in X)\,((x * y)\triangle(y * x) = 0)\,.$

Proof. (1) Using (3.1), we get

$$\begin{aligned} x\&(y\triangle z) &= x\&((z * y) * y) = (x\&(z * y)) * (x\&y) \\ &= ((x\&z) * (x\&y)) * (x\&y) = (x\&y)\triangle(x\&z) \end{aligned}$$

for all $x, y, z \in X$.

(2) For any $x, y \in X$, we have

$$\begin{aligned} (x * y)\triangle x &= (x * (x * y)) * (x * y) = ((x * x) * (x * y)) * (x * y) \\ &= (0 * (x * y)) * (x * y) = (x * y) * (x * y) = 0 \end{aligned}$$

by (2.6) and (2.4). Using (2.2) and (2.4), we obtain

$$(x * y)\triangle y = (y * (x * y)) * (x * y) = 0 * (x * y) = x * y$$

for all $x, y \in X$.

(3) Using (2.5), (2.6) and (2.2), we get

$$\begin{aligned} (x * y)\triangle(y * x) &= ((y * x) * (x * y)) * (x * y) \\ &= (x * ((y * x) * y)) * (x * y) \\ &= ((x * (y * x)) * (x * y)) * (x * y) \\ &= (0 * (x * y)) * (x * y) = (x * y) * (x * y) = 0 \end{aligned}$$

for all $x, y \in X$. □

Definition 3.6. Let ∂ be a relation on a hardily ranked bigroupoid $(X, *, \&)$. Then ∂ is said to be

(i) *right compatible* if it satisfies:

$$(\forall x, y, z \in X)\left((x, y) \in \partial \ \Rightarrow \left(\begin{array}{c} (x * z, y * z) \in \partial, \\ (x\&z, y\&z) \in \partial \end{array}\right)\right). \tag{3.2}$$

(ii) *left compatible* if it satisfies:

$$(\forall x, y, z \in X)\left((x, y) \in \partial \ \Rightarrow \left(\begin{array}{c} (z * x, z * y) \in \partial, \\ (z\&x, z\&y) \in \partial \end{array}\right)\right). \tag{3.3}$$

(iii) *compatible* if it satisfies:

$$(\forall x, y, a, b \in X)\left((x, y), (a, b) \in \partial \ \Rightarrow \left(\begin{array}{c} (x * a, y * b) \in \partial, \\ (x\&a, y\&b) \in \partial \end{array}\right)\right). \tag{3.4}$$

A compatible equivalence relation is called a *congruence relation*.

Proposition 3.7. *Let ∂ be an equivalence relation on a hardily ranked bigroupoid $(X, *, \&)$. Then ∂ is a congruence relation on X if and only if it is both a left and right compatible relation.*

Proof. Suppose that ∂ is a congruence relation on X. Let $x, y, z \in X$ be such that $(x, y) \in \partial$. Since $(z, z) \in \partial$, it follows from (3.4) that $(x * z, y * z) \in \partial$ and $(x\&z, y\&z) \in \partial$. Hence ∂ is a right compatible relation on X. Similarly, we know that ∂ is a left compatible relation on X.

Conversely, assume that ∂ is both a left and right compatible relation on X. Let $x, y, a, b \in X$ be such that $(x, y) \in \partial$ and $(a, b) \in \partial$. The right compatibility of ∂ implies that $(x*a, y*a) \in \partial$ and $(x\&a, y\&a) \in \partial$, and the left compatibility of ∂ induces that $(y * a, y * b) \in \partial$ and $(y\&a, y\&b) \in \partial$. Using the transitivity of ∂, we have $(x * a, y * b) \in \partial$ and $(x\&a, y\&b) \in \partial$. Therefore ∂ is a congruence relation on X. □

For any equivalence relation ∂ on a hardily ranked bigroupoid $(X, *, \&)$ and an element x of X, we consider the following sets:

$$x_\partial := \{y \in X \mid (x, y) \in \partial\} \quad \text{and} \quad X/\partial := \{x_\partial \mid x \in X\}.$$

Theorem 3.8. *Let ∂ be a congruence relation on a hardily ranked bigroupoid $(X, *, \&)$. Define both a major operation "$*_\partial$" and a minor operation "$\&_\partial$" as follows*:

$$x_\partial *_\partial y_\partial = (x * y)_\partial \quad \textit{and} \quad x_\partial \&_\partial y_\partial = (x\&y)_\partial$$

for all $x_\partial, y_\partial \in X/\partial$. Then $(X/\partial, *_\partial, \&_\partial)$ is a hardily ranked bigroupoid.

Proof. The operations are well-defined because ∂ is a congruence relation on $(X, *, \&)$. It is easy to see that X/∂ is a semigroup under the minor operation

$\&_\partial$ and the major operation "$*_\partial$" satisfies axioms (2.1), (2.2) and (2.3). Let $x_\partial, y_\partial, z_\partial \in X/\partial$. Then

$$\begin{aligned} x_\partial \&_\partial (y_\partial *_\partial z_\partial) &= x_\partial \&_\partial (y * z)_\partial = (x\&(y * z))_\partial \\ &= ((x\&y) * (x\&z))_\partial = (x\&y)_\partial *_\partial (x\&z)_\partial \\ &= (x_\partial \&_\partial y_\partial) *_\partial (x_\partial \&_\partial z_\partial) \end{aligned}$$

and

$$\begin{aligned} (x_\partial *_\partial y_\partial) \&_\partial z_\partial &= (x * y)_\partial \&_\partial z_\partial = ((x * y)\&z)_\partial \\ &= ((x\&z) * (y\&z))_\partial = (x\&z)_\partial *_\partial (y\&z)_\partial \\ &= (x_\partial \&_\partial z_\partial) *_\partial (y_\partial \&_\partial z_\partial) . \end{aligned}$$

Therefore $(X/\partial, *_\partial, \&_\partial)$ is a hardily ranked bigroupoid. □

Given ranked bigroupoids $(X, *, \&)$ and $(Y, \bullet, \omega)$, a map $f : (X, *, \&) \to (Y, \bullet, \omega)$ is called a

(1) *major rankomorphism* if it satisfies

(3.5) $$(\forall x, y \in X) \, (f(x * y) = f(x) \bullet f(y)) \, ,$$

(2) *minor rankomorphism* if it satisfies

(3.6) $$(\forall x, y \in X) \, (f(x\&y) = f(x) \omega f(y)) \, .$$

If f is both a major rankomorphism and a minor rankomorphism, we say that f is a *rankomorphism* (see [1]).

Proposition 3.9. *Let $f : (X, *, \&) \to (Y, \bullet, \omega)$ be a rankomorphism of hardily ranked bigroupoids. Then*

(1) $f(0) = 0$.
(2) $(\forall x, y \in X) \, (x * y = 0 \ \Rightarrow \ f(x) \bullet f(y) = 0)$.
(3) $(\forall x, y \in X) \, (f(x \triangle y) = f(x) \triangle f(y))$.
(4) $f^{-1}(0) = \{0\} \ \Rightarrow \ x * y = 0$ *for all* $x, y \in X$ *with* $f(x) \bullet f(y) = 0$.

Proof. (1) $\sim$ (3) are straightforward.

(4) Assume that $f^{-1}(0) = \{0\}$ and let $x, y \in X$ be such that $f(x) \bullet f(y) = 0$. Then $f(x * y) = f(x) \bullet f(y) = 0$, and so $x * y = 0$. □

Theorem 3.10. *Let ∂ be a congruence relation on a hardily ranked bigroupoid $(X, *, \&)$. The mapping*

$$f^\sharp : X \to X/\partial, \ x \mapsto x_\partial$$

is an onto rankomorphism.

Proof. Let $x, y \in X$. Then

$$f^\sharp(x * y) = (x * y)_\partial = x_\partial *_\partial y_\partial = f^\sharp(x) *_\partial f^\sharp(y)$$

and

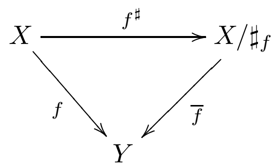
$$f^\sharp(x\&y) = (x\&y)_\partial = x_\partial \&_\partial y_\partial = f^\sharp(x) \&_\partial f^\sharp(y).$$

Hence $f^\sharp$ is a rankomorphism. Obviously, $f^\sharp$ is onto. □

Theorem 3.11. *Let $f:(X,*,\&)\to(Y,\bullet,\omega)$ be a rankomorphism of hardily ranked bigroupoids. Consider the following set*:

$$\sharp_f := \{(x,y)\in X\times X \mid f(x)=f(y)\}.$$

(1) *$\sharp_f$ is a congruence relation on $(X,*,\&)$.*
(2) *There exists a unique one-one rankomorphism $\overline{f}: X/\sharp_f \to Y$ such that the following diagram commutes*:



Proof. (1) It is clear that $\sharp_f$ is an equivalence relation on $(X,*,\&)$. Let $a,b,x,y \in X$ be such that $(a,b),(x,y)\in\sharp_f$. Then $f(a)=f(b)$ and $f(x)=f(y)$, which imply that

$$f(x*a)=f(x)\bullet f(a)=f(y)\bullet f(b)=f(y*b),$$
$$f(x\&a)=f(x)\omega f(a)=f(y)\omega f(b)=f(y\&b).$$

Hence $(x*a, y*b)\in\sharp_f$ and $(x\&a, y\&b)\in\sharp_f$. Therefore $\sharp_f$ is a congruence relation on $(X,*,\&)$.

(2) Let $\overline{f}: X/\sharp_f\to Y$ be a map defined by $\overline{f}\left(x_{\sharp_f}\right)=f(x)$ for all $x\in X$. Then $\overline{f}$ is a well-defined map. For any $x_{\sharp_f}, y_{\sharp_f}\in X/\sharp_f$, we have

$$\begin{aligned}\overline{f}\left(x_{\sharp_f} *_{\sharp_f} y_{\sharp_f}\right) &= \overline{f}\left((x*y)_{\sharp_f}\right)=f(x*y)\\ &= f(x)\bullet f(y)=\overline{f}\left(x_{\sharp_f}\right)\bullet\overline{f}\left(y_{\sharp_f}\right)\end{aligned}$$

and

$$\begin{aligned}\overline{f}\left(x_{\sharp_f}\&_{\sharp_f} y_{\sharp_f}\right) &= \overline{f}\left((x\&y)_{\sharp_f}\right)=f(x\&y)\\ &= f(x)\omega f(y)=\overline{f}\left(x_{\sharp_f}\right)\omega\overline{f}\left(y_{\sharp_f}\right).\end{aligned}$$

Hence $\overline{f}$ is a rankomorphism. Clearly, $\overline{f}$ is one-one. Let $g: X/\sharp_f\to Y$ be a rankomorphism such that $g\circ f^\sharp = f$. Then

$$g\left(x_{\sharp_f}\right)=g\left(f^\sharp(x)\right)=f(x)=\overline{f}\left(x_{\sharp_f}\right)$$

for all $x_{\sharp_f}\in X/\sharp_f$. Thus $g=\overline{f}$, which shows that $\overline{f}$ is unique. □

Corollary 3.12. *For two congruence relations ∂ and ρ on a hardily ranked bigroupoid $(X,*,\&)$ with $\partial\subseteq\rho$, the set*

$$\rho/\partial := \{(x_\partial, y_\partial)\in X/\partial\times X/\partial \mid (x,y)\in\rho\}$$

is a congruence relation on X/∂, and there exists a one-one and onto rankomorphism from $\frac{X/\partial}{\rho/\partial}$ to X/ρ.

Proof. Let $f : X/\partial \to X/\rho$ be a map defined by $f(x_\partial) = x_\rho$ for all $x_\partial \in X/\partial$. Then f is well-defined onto rankomorphism because of $\partial \subseteq \rho$. According to Theorem 3.11, it is sufficient to show that $\rho/\partial = \sharp_f$. If $(x_\partial, y_\partial) \in \rho/\partial$, then $(x, y) \in \rho$ and thus $x_\rho = y_\rho$. Thus $f(x_\partial) = x_\rho = y_\rho = f(y_\partial)$, which shows that $(x_\partial, y_\partial) \in \sharp_f$. Now, if $(x_\partial, y_\partial) \in \sharp_f$, then $x_{rho} = f(x_\partial) = f(y_\partial) = y_\rho$, that is, $(x_\partial, y_\partial) \in \rho/\partial$. Therefore $\rho/\partial = \sharp_f$. This completes the proof. □

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