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INVOLUTIONS ON SURFACES OF GENERAL TYPE WITH $p_g = 0$ I. THE COMPOSED CASE

YongJoo Shin

Abstract. Let S be a minimal surface of general type with $p_g(S) = q(S) = 0$ having an involution σ over the field of complex numbers. It is well known that if the bicanonical map φ of S is composed with σ, then the minimal resolution W of the quotient S/σ is rational or birational to an Enriques surface. In this paper we prove that the surface W of S with $K_S^2 = 5, 6, 7, 8$ having an involution σ with which the bicanonical map φ of S is composed is rational. This result applies in part to surfaces S with $K_S^2 = 5$ for which φ has degree 4 and is composed with an involution σ. Also we list the examples available in the literature for the given K_S^2 and the degree of φ.

1. Introduction

An algebraic surface is one of crucial objects in algebraic geometry. Classification and construction are main parts in the studies of algebraic surfaces. Construction of algebraic surfaces for given invariants is especially one of important problems. We refer a recent survey [1] for algebraic surfaces of general type with vanishing geometric genus.

One of classical methods to construct an algebraic surface is to use a double covering. Campedelli [6] uses a double covering to construct a minimal surface of general type with $p_g = q = 0$, $K^2 = 2$ over a rational surface. Keum [9] and Naie [21] also use a double covering to construct a minimal surface of general type with $p_g = q = 0$, $K^2 = 3$ over a surface birational to an Enriques surface.

In this view we investigate involutions on a given surface, especially a minimal surface of general type with $p_g = q = 0$. Keum and Lee [11] study on numerical Godeaux surfaces with involutions when the bicanonical system has no base components. Without the assumption in [11] Calabri, Ciliberto and Mendes Lopes [4] extend the result of Keum and Lee [11]. Calabri, Mendes

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Lopes and Pardini [5] consider numerical Campedelli surfaces with involutions. And Rito [26] treats involutions on a minimal surface of general type with $p_g = q = 0$ and $K^2 = 3$ with which the bicanonical map is not composed. Rito describes a classification of branch divisors, birational types of quotients and examples. Lee and the author [13] deal with involutions on a minimal surface of general type with $p_g = q = 0$ and $K^2 = 7$. We list possible branch divisors and birational types of quotients. Dolgachev, Mendes Lopes and Pardini [7] give a classification of birational types of quotients for a minimal surface of general type with $p_g = q = 0$ and $K^2 = 8$ by involutions.

The bicanonical map can be composed or not with involutions (cf. [4], [27] and [25]). In this paper we consider involutions on a minimal surface S of general type with $p_g(S) = q(S) = 0$ with which the bicanonical map φ of S is composed. Let σ be one of such involutions on S, and we denote that W is the minimal resolution of the quotient S/σ and d is the degree of the bicanonical map φ.

Xiao [32] shows that W is rational for $K_S^2 = 5, 6$ with $d = 2$ and $K_S^2 = 7, 8$. Mendes Lopes and Pardini [15] prove that a minimal surface S of general type with $p_g(S) = 0$, $K_S^2 = 6$ and $d = 4$ is a Burniat surface. It implies that W is rational because $H^0(W, \mathcal{O}_W(2K_W)) = 0$. However it was not previously known whether W is rational or not if S is a minimal surface of general type with $p_g(S) = 0$, $K_S^2 = 5$ and $d = 4$.

The following theorem includes the assertion that W is rational for a minimal surface S of general type with $p_g(S) = 0$, $K_S^2 = 5$ and $d = 4$.

Theorem 1.1. *Let S be a minimal surface of general type with $p_g = q = 0$ having an involution σ. Assume that the bicanonical map φ is composed with σ. Then the minimal resolution W of the quotient S/σ is rational for $K_S^2 = 5, 6, 7, 8$.*

The paper is organized as follows: In Section 3 we give a proof of Theorem 1.1 and in Section 4 we consider the examples available in the literature.

2. Notation and conventions

In this section we fix notations that will be used in this paper. In this paper, we work over the field of complex numbers.

Let X be a smooth projective surface. We set:

K_X: the canonical divisor of X;

$\kappa(X)$: the Kodaira dimension of X;

$p_g(X)$: the geometric genus of X, that is, $h^0(X, \mathcal{O}_X(K_X))$;

$q(X)$: the irregularity of X, that is, $h^1(X, \mathcal{O}_X)$;

$\equiv$: the linear equivalence of divisors on a surface;

$\sim$: the birational equivalence between surfaces.

We usually omit the sign $\cdot$ of the intersection product of two divisors on a surface.

Let S be a minimal surface of general type with $p_g(S) = q(S) = 0$ having an involution σ. Then there is a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\epsilon} & S \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ W & \xrightarrow{\eta} & \Sigma \end{array}$$

In the above diagram π is the quotient map induced by the involution σ. And ϵ is the blow-up of S at k isolated fixed points arising from the involution σ. Also, $\tilde{\pi}$ is induced by the quotient map π and η is the minimal resolution of the k ordinary double points made by the quotient map π. We denote the k disjoint (-1)-curves on V (resp. the k disjoint (-2)-curves on W) related to the k isolated fixed points on S (resp. the k ordinary double points on Σ) as E_i (resp. N_i), $i = 1, \dots, k$. And, there is a fixed divisor R of σ on S which is the union of smooth curves. We set $R_0 := \epsilon^*(R)$ and $B_0 := \tilde{\pi}(R_0)$.

The map $\tilde{\pi}$ is a double cover branched on $B := B_0 + \sum_{i=1}^{k} N_i$. Thus there exists a divisor L on W such that $2L \equiv B$ and

$$\tilde{\pi}_*\mathcal{O}_V = \mathcal{O}_W \oplus \mathcal{O}_W(-L).$$

Moreover, $K_V \equiv \tilde{\pi}^*(K_W + L)$ and $K_S \equiv \pi^* K_\Sigma + R$.

Also we denote:

$\rho\colon W \longrightarrow W'$ is a morphism, where W' is a minimal model of W;

$\rho_j\colon W_{j-1} \longrightarrow W_j$, $j = 1, 2, \dots, l$ are contractions of (-1)-curves to decompose ρ (i.e., $\rho = \rho_l \circ \dots \circ \rho_2 \circ \rho_1$), where $W_0 = W$, $W_l = W'$;

$B'{:=}\rho(B)$; $B_0'{:=}\rho(B_0)$; $L'{:=}\rho_*(L)$;

And the bicanonical map φ is composed with an involution σ if $\varphi \circ \sigma = \varphi$;

$d :=$ the degree of φ; $Z :=$ the image of S by φ.

3. Rationality of the quotient

In this section we prove that W is rational for $K_S^2 = 5, 6, 7, 8$.

Remark 3.1. (1) Assume that B' has singularities with multiplicity r_i, $i = 1, \dots, s$. Then $B = \rho^*(B') - \sum_{i=1}^{s} r_i H_i$, where H_i is an exceptional curve with self intersection number -1. Moreover r_i is an even integer since B and B' are even divisors.

(2) (i) $\chi(2K_{W'} + L') = \chi(K_{W'} + L') + K_{W'}L' + K_{W'}^2$ by Riemman-Roch theorem. If we apply the above (1), then

(ii) $\chi(K_W + L) = \chi(K_{W'} + L') - \frac{1}{8}\sum_{i=1}^{s} r_i(r_i - 2)$,

(iii) $\chi(2K_W + L) = \chi(2K_{W'} + L') - \frac{1}{8}\sum_{i=1}^{s}(r_i - 2)(r_i - 4)$.

Lemma 3.2. *If the following conditions hold*:

(i) *the bicanonical morphism φ is composed with σ,*

(ii) *W is birational to an Enriques surface,*

(iii) $\chi(K_W + L) \geq 0$,

then B_0' may have only double points.

Proof. Since φ is the bicanonical morphism, φ is composed with σ if and only if one of summand of $H^0(W,\mathcal{O}_W(2K_W+L))\bigoplus H^0(W,\mathcal{O}_W(2K_W+B))$ is zero. Since $\tilde{\pi}^*(2K_W+B_0)=\epsilon^*(2K_S)$, $2K_W+B_0$ is nef and big, and so

$$0\neq H^0(W,\mathcal{O}_W(2K_W+B_0))\subset H^0(W,\mathcal{O}_W(2K_W+B)).$$

Thus $H^0(W,\mathcal{O}_W(2K_W+L))=0$, and it induces

$$\chi(2K_W+L)=0 \tag{1}$$

by Kawamata-Viehweg Vanishing Theorem because $K_W+L=\frac{1}{2}(2K_W+B_0)+\frac{1}{2}\sum_{i=1}^{k}N_i$.

Moreover $\chi(2K_{W'}+L')=\chi(K_{W'}+L')$ by the given condition (ii) and Remark 3.1(2)(i). Thus we get

$$\begin{aligned}\chi(K_W+L)&=\frac{1}{8}\sum_{i=1}^{s}(r_i-2)(r_i-4)-\frac{1}{8}\sum_{i=1}^{s}r_i(r_i-2)\\&=\frac{1}{2}\sum_{i=1}^{s}(-r_i+2)\geq 0\end{aligned}$$

by the given condition (iii), and Remark 3.1(2)(ii) and (iii). Therefore $r_i=2$ for all i. This means that B_0' may have only double points. □

Lemma 3.3. *$p_g(S)=q(S)=0$ implies $\chi(K_W+L)=0$.*

Proof. The standard double cover formula, $\chi(\mathcal{O}_V)=2\chi(\mathcal{O}_W)+\frac{1}{2}(L^2+K_WL)$ induces

$$L^2+K_WL=-2 \tag{2}$$

because of the condition $p_g(S)=q(S)=0$. Moreover $\chi(2K_W+L)=1+K_W(K_W+L)+\frac{1}{2}L(K_W+L)$ by Riemann-Roch theorem and $p_g(S)=q(S)=0$, and so

$$\chi(2K_W+L)=K_W(K_W+L). \tag{3}$$

Thus $\chi(K_W+L)=0$ because $\chi(2K_W+L)=\chi(K_W+L)+K_WL+K_W^2$ as Remark 3.1(2)(i). □

Note that $k=K_S^2+4$ because $k=K_S^2-K_V^2=K_S^2-2(K_W+L)^2=K_S^2-2(K_W(K_W+L)+L(K_W+L))=K_S^2-2(0-2)=K_S^2+4$ by (1) in the proof of Lemma 3.2 and (2), (3) in the proof of Lemma 3.3.

Lemma 3.4. *If W is birational to an Enriques surface and not a minimal surface, then there exists a (-1)-curve E on W such that $EN_i=1$ for some $i\in\{1,2,\ldots,k\}$.*

Proof. Suppose that there does not exist a (-1)-curve such that $EN_i\geq 1$ for any i. The Enriques surface W' has disjoint (-2)-curves whose a number is bigger than $k=K_S^2+4\geq 9$ for $K_S^2=5,6,7,8$. However an Enriques surface has at most eight (-2)-curves. It induces a contradiction.

We prove that $EN_i = 1$. Let $\rho_1 \colon W \longrightarrow W_1$ be a contraction of E. Denote that $\alpha := EN_i \geq 1$, and then $(\rho_1(N_i))^2 = -2 + \alpha^2$ and $\rho_1(N_i)K_{W_1} = -\alpha$. Suppose $\alpha \geq 2$, and so $(\rho_1(N_i))^2 > 0$. It gives that $\rho_2 \circ \rho_1(N_i)$ is a curve on W_2. Moreover $\rho_l \circ \cdots \circ \rho_2 \circ \rho_1(N_i)$ is a curve on $W_l = W'$. Thus we consider that $\rho_l \circ \cdots \circ \rho_2 \circ \rho_1(N_i)K_{W'} \leq \rho_1(N_i)K_{W_1}$. In this inequality the left side is zero since W' is an Enriques surface and the right side is $-\alpha$. It means that $0 \leq -\alpha$ which is a contradiction because $\alpha \geq 2$. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Corollary 3.7(v) in [4] W is rational or birational to an Enriques surface. Suppose that W is birational to an Enriques surface. In this case B_0' may have only double points by Lemma 3.2 and Lemma 3.3.

Now we claim that W is minimal. Suppose that W is not minimal. Then there exists a (-1)-curve E on W such that $EN_i = 1$ for some $i \in \{1, 2, \ldots, k\}$ by Lemma 3.4.

If there exist distinct i_1, $i_2 \in \{1, 2, \ldots, k\}$ such that $EN_{i_1} = EN_{i_2} = 1$, then W_1 has two (-1)-curves $\rho_1(N_{i_1})$, $\rho_1(N_{i_2})$ meeting only one point. Let $\rho_2 \colon W_1 \longrightarrow W_2$ be a contraction of $\rho_1(N_{i_1})$. Then $\rho_2 \circ \rho_1(N_{i_2})$ is a smooth rational curve with $(\rho_2 \circ \rho_1(N_{i_2}))^2 = 0$ on W_2. But it induces a contradiction because $-2m = (mK_{W_2})(\rho_2 \circ \rho_1(N_{i_2})) \geq 0$ for a sufficiently large integer m since $\kappa(W_2) = 0$. Thus we denote the special (-2)-curve which meets with E at only one point on W as N_1 after relabelling $N_i \in \{1, 2, \ldots, k\}$.

We show $B_0E = 1$. Since B_0' may have only double points, $B_0E \leq 2$. Suppose $B_0E = 0$, and then $2g(\tilde{\pi}^*(E)) - 2 = 2(2g(E) - 2) + 1 = -3$ by Hurwitz's formula which is a contradiction. Suppose $B_0E = 2$, and then B_0' has a singularity with multiplicity ≥ 3 because $EN_1 = 1$ and N_1 is a (-2)-curve. It is a contradiction.

The remaining case is that $B_0E = EN_1 = 1$ on W. In this case $\tilde{\pi}^*(E)$ is (-2)-curve, and so since $\tilde{\pi}^*(E)E_1 = 1$ and E_1 is a (-1)-curve (Note $\tilde{\pi}(E_1) = N_1$) S has a (-1)-curve $\epsilon(\tilde{\pi}^*(E))$ which is a contradiction because S is minimal. So the claim is proved.

Thus W is an Enriques surface having disjoint (-2)-curves whose a number is bigger than $k = K_S^2 + 4 \geq 9$ for $K_S^2 = 5, 6, 7, 8$. However an Enriques surface has at most eight (-2)-curves. It induces a contradiction. □

4. Examples

Let S be a minimal surface of general type with $p_g = q = 0$ having an involution σ. Assume that the bicanonical map φ of S is composed with the involution σ, and the bicanonical system $|2K_S|$ is base point free.

When we consider the bicanonical map φ of a minimal surface of general type it is not a morphism in general. We know that φ is a morphism for $K^2 \geq 5$ in [2], [24]. On the other hand, Mendes Lopes and Pardini [20] give examples of numerical Campedelli surfaces with fundamental group of order 9 of which the bicanonical systems have two base points. These examples are only known surfaces of general type with $p_g = q = 0$ and $K^2 > 1$ having base points in the bicanonical system.

Since S is a minimal surface of general type with $p_g = q = 0$, we have $1 \leq K_S^2 \leq 9$ by Bogomolov-Miyaoka-Yau inequality. For $K_S^2 = 1$ the dimension of Z is 1 which is not a surface by Riemann-Roch formula, and for $K_S^2 = 9$ S cannot have an involution by Theorem 4.2 in [7] or Corollary 2.5 in [10]. We exclude the cases of $K_S^2 = 1$ and 9.

Since φ is composed with σ, d is even, and so $d = 2$ if $K_S^2 = 7, 8$ and $d = 2, 4$ if $3 \leq K_S^2 \leq 6$ by Theorem 1.2 in [19]. Moreover $d = 8$ if $K_S^2 = 2$. Indeed, if $K_S^2 = 2$, then Z is a surface contained in $\mathbb{P}^2$ by Theorem 1 in [31] and Riemann-Roch Theorem, and so $Z = \mathbb{P}^2$. Thus $d = d(\text{the degree of } Z) = (2K_S)^2 = 8$ if $K_S^2 = 2$ because of the morphism φ since $|2K_S|$ is base point free.

W should be rational or birational to an Enriques surface by Corollary 3.7 (v) in [4]. Precisely W is rational for the cases $K_S^2 = 5, 6, 7, 8$ by Theorem 1.1, and W is rational for $K_S^2 = 4$ with $d = 2$ by Theorem 5.1 in [16]. For the other cases W is rational or birational to an Enriques surface.

Z is rational for $K_S^2 = 4, 5, 6, 7, 8$ and is rational or birational to an Enriques surface for $K_S^2 = 3$ with $d = 2$ and is rational for $K_S^2 = 3$ with $d = 4$ by Theorem 5.1 in [16]. Moreover Z is rational for $K_S^2 = 2$ because Z is equal to $\mathbb{P}^2$.

We provide the following table containing the examples available in the literature for each case.

K_S^2	d	W	Z	Examples
2	8	rational	rational	Campedelli (1932) [6], Burniat (1966) [3], Peters (1977) [23], Kulikov (2004) [12]
	8	$\sim$ Enriques surface	rational	Kulikov (2004) [12]
3	2	rational	rational	Rito (2010) [26]
	2	$\sim$ Enriques surface	$\sim$ Enriques surface	Mendes Lopes, Pardini (2004) [17]
	4	rational	rational	Burniat (1966) [3], Peters (1977) [23]
	4	$\sim$ Enriques surface	rational	Keum (1988) [9], Naie (1994) [21]
4	2	rational	rational	Werner (2010) [30], Rito (2011) [28]
	4	rational	rational	Burniat (1966) [3], Peters (1977) [23]
	4	$\sim$ Enriques surface	rational	Keum (1988) [9], Naie (1994) [21]
5	2	rational	rational	Werner (2010) [30], Rito (2011) [28]
	4	rational	rational	Burniat (1966) [3], Peters (1977) [23]
6	2	rational	rational	Inoue (1994) [8], Mendes Lopes, Pardini (2004) [18], Werner (2010) [30], Rito (2011) [28]
	4	rational	rational	Burniat (1966) [3], Peters (1977) [23]
7	2	rational	rational	Inoue (1994) [8], Mendes Lopes, Pardini (2001) [14], Werner (2010) [30], Rito (2011) [28]
8	2	rational	rational	Mendes Lopes, Pardini (2001) [14], Pardini (2003) [22]

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Department of Mathematics
Sogang University
Seoul 121-742, Korea
E-mail address: shinyongjoo@gmail.com