

GLOBAL EXISTENCE FOR VOLTERRA-FREDHOLM TYPE FUNCTIONAL IMPULSIVE INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we study the global existence of solutions for the initial value problems for Volterra-Fredholm type functional impulsive integrodifferential equations. Using the Leray-Schauder Alternative, we derive conditions under which a solution exists globally.

1. INTRODUCTION

Various evolutionary processes from fields as diverse as physics, population dynamics, aeronautics, economics and engineering are characterized by the fact that they undergo abrupt changes of state at certain moments of time between intervals of continuous evolution. Because the duration of these changes are often negligible compared to the total duration of the process, such changes can be reasonably well-approximated as being instantaneous changes of state, or in the form of impulses. These process tend to more suitably modeled by impulsive differential equations, which allow for discontinuities in the evolution of the state. For more details on this theory and on its applications we refer to the monographs of Lakshmikantham et al. [15], Samoilenko and Perestyuk [24], Rogovchenko [22, 23] and Hernández [11–13] for the case of ordinary and partial differential functional differential equations with impulses. Similarly, for more on ordinary and partial impulsive functional differential equations we refer to [1, 2, 5–7, 10, 14].

Recently, in [20], the authors studied the global existence for first order Mixed Volterra neutral functional integrodifferential equations in Banach spaces by using Leray-Schauder nonlinear alternative or Krasnoselskii schaefer fixed point theorem. In [4], the authors studied initial and boundary value problems for nonconvex valued multivalued functional differential inclusions by using a fixed point theorem for contraction multivalued maps due to Covitz and Nadler

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and Schaefer's theorem combined with lower semicontinuous multivalued operators with decomposable values. Also in [3], the authors studied second order impulsive functional differential inclusions by using Schaefer's theorem combined with a selection of theorem of Bressan and Colombo for lower semicontinuous multivalued operators with decomposable values. For recent results on local and global existence for ordinary, functional or neutral integrodifferential equations see [8, 16, 18, 19, 21, 25–34].

In this paper, we study the global existence of solutions for the initial value problems for Volterra-Fredholm type functional impulsive integrodifferential equations of the form

$$(\rho(t)x'(t))' = f\left(t, x_t, x'(t), \int_0^t a(t, s)g(s, x_s, x'(s))ds, \int_0^T b(t, s)h(s, x_s, x'(s))ds\right),$$

$$t \in I = [0, T] \setminus \{t_1, \dots, t_m\}, \quad (1.1)$$

$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, \dots, m, \quad (1.2)$$

$$\Delta x'|_{t=t_k} = J_k(x(t_k^-)), \quad k = 1, \dots, m, \quad \text{and} \quad (1.3)$$

$$x_0 = \phi, \quad t \in [-r, 0], \quad \text{and} \quad x'(0) = \eta, \quad (1.4)$$

where $f : I \times D \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $D = \{\psi : [-r, 0] \rightarrow \mathbb{R}^n; \psi \text{ is continuous everywhere except for a finite number of points } \tilde{t} \text{ at which } \psi(\tilde{t}^-) \text{ and } \psi(\tilde{t}^+) \text{ are exist with } \psi(\tilde{t}^-) = \psi(\tilde{t}^+)\}$, $g : I \times D \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $h : I \times D \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $a : I \times I \rightarrow \mathbb{R}$, $b : I \times I \rightarrow \mathbb{R}$ are continuous functions, ρ is a continuous positive function, $\phi \in D$ and $\eta \in \mathbb{R}^n$, $0 < r < \infty$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $I_k, J_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, ($k = 1, \dots, m$). $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at $t = t_k$, respectively, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, and $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$.

For any continuous function x defined on $[-r, T] \setminus \{t_1, \dots, t_m\}$ and any $t \in [0, T]$ we denote by x_t the element of D defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$, where $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t .

This paper will be organized as follows. In Section 2 we will recall briefly some basic definitions and preliminary facts which are used throughout this paper. In Section 3 we shall present and prove our main results for the problem (1.1)-(1.4).

2. PRELIMINARIES

In this section, we introduce some basic definitions, notations and preliminary facts which are used throughout this paper.

Let $C([-r, 0], \mathbb{R}^n)$ be the Banach space of continuous functions from $[-r, 0]$ into \mathbb{R}^n endowed with the norm

$$\|\phi\| = \sup \{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

and $C([0, T], \mathbb{R}^n)$ denote the Banach space of all continuous functions from $[0, T]$ into \mathbb{R}^n normed by

$$\|x\|_\infty = \sup \{|x(t)| : t \in [0, T]\}.$$

For convenience we put:

$$\begin{aligned} \|x\|_r &= \sup\{|x(t)| : -r \leq t \leq T\}, \\ \|x\|_0 &= \sup\{|x(t)| : t \in I\}, \\ \|x\|_1 &= \sup\{|x'(t)| : t \in I\}, \\ \|x\|^* &= \max\{\|x\|_r, \|x\|_1\}, \\ \|x\|_T &= \max\{\|x\|_0, \|x\|_1\}. \end{aligned}$$

In order to define the solution of (1.1)-(1.4), we introduce the following space:

$$PC = \left\{ x : [0, T] \rightarrow \mathbb{R}^n : x_k \in C([t_k, t_{k+1}], \mathbb{R}^n), k = 0, \dots, m \text{ and there exist } x(t_k^-) \text{ and } x(t_k^+) \text{ with } x(t_k^-) = x(t_k^+), k = 1, \dots, m \right\}$$

which is a Banach space with the norm

$$\|x\|_{PC} = \max\{\|x_k\|_{(t_k, t_{k+1}]}, k = 0, \dots, m\},$$

where x_k is the restriction of x to $(t_k, t_{k+1}]$, $k = 0, \dots, m$.

Set $\Omega = D \cup PC$. Then Ω is a Banach space normed by

$$\|x\|_{\Omega} = \max\{\|x\|_D, \|x\|_{PC}\}, \text{ for each } x \in \Omega.$$

and

$$PC^1 = \left\{ x : [0, T] \rightarrow \mathbb{R}^n : x_k \in C^1([t_k, t_{k+1}], \mathbb{R}^n), k = 0, \dots, m \text{ and there exist } x'(t_k^-) \text{ and } x'(t_k^+) \text{ with } x'(t_k^-) = x'(t_k^+), k = 1, \dots, m \right\}$$

which is a Banach space with the norm

$$\|x\|_{PC^1} = \max\{\|x_k\|_{(t_k, t_{k+1}]}, k = 0, \dots, m\},$$

where x_k is the restriction of x to $(t_k, t_{k+1}]$, $k = 0, \dots, m$.

Set $\Omega^1 = D \cup PC^1$. Then Ω^1 is a Banach space normed by

$$\|x\|_{\Omega^1} = \max\{\|x\|_D, \|x\|_{PC^1}\}, \text{ for each } x \in \Omega^1.$$

Definition 2.1. A function $x \in \Omega \cap \Omega^1$ is called solution of the initial value problem (1.1)-(1.4) if x satisfies the following integral equation

$$\begin{aligned} x(t) &= \phi(0) + \rho(0) \int_0^t \frac{ds}{\rho(s)} \eta + \int_0^t \frac{1}{\rho(s)} \int_0^s f(\tau, x_\tau, x'(\tau), \\ &\int_0^\tau a(\tau, \sigma) g(\sigma, x_\sigma, x'(\sigma)) d\sigma, \int_0^T b(\tau, \sigma) h(\sigma, x_\sigma, x'(\sigma)) d\sigma) d\tau ds \\ &+ \sum_{0 < t_k < t} \left[I_k(x(t_k^-)) + J_k(x(t_k^-)) \int_{t_k}^t \frac{\rho(t_k)}{\rho(s)} ds \right], t \in I. \end{aligned}$$

The considerations of this paper are based on the following fixed point result [9].

Lemma 2.1 (Leray-Schauder's Alternative Theorem). *Let S be a closed convex subset of a normed linear space E and assume that $0 \in S$. If $F : S \rightarrow S$ be a completely continuous operator, i.e. it is continuous and the image of any bounded set is included in a compact set and let*

$$\Phi(F) = \{x \in S : x = \lambda Fx, \text{ for some } 0 < \lambda < 1\}.$$

Then either $\Phi(F)$ is unbounded or F has a fixed point.

3. GLOBAL EXISTENCE

In this section, we present the global existence results for the initial value problem (1.1)-(1.4).

Theorem 3.1. *Let $f : I \times D \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g : I \times D \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $h : I \times D \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $a : I \times I \rightarrow \mathbb{R}$, $b : I \times I \rightarrow \mathbb{R}$ are continuous functions.*

Assume that

H_g *There exists a continuous function $m_1 : I \rightarrow [0, \infty)$ such that*

$$|g(t, \phi, \psi)| \leq m_1(t)\Omega_1(\|\phi\| + |\psi|), \quad t \in I, \quad \phi \in D, \quad \psi \in \mathbb{R}^n,$$

where $\Omega_1 : [0, \infty) \rightarrow (0, \infty)$ is a continuous nondecreasing function.

H_h *There exists a continuous function $m_2 : I \rightarrow [0, \infty)$ such that*

$$|h(t, \phi, \psi)| \leq m_2(t)\Omega_2(\|\phi\| + |\psi|), \quad t \in I, \quad \phi \in D, \quad \psi \in \mathbb{R}^n,$$

where $\Omega_2 : [0, \infty) \rightarrow (0, \infty)$ is a continuous nondecreasing function.

H_f *There exists an integrable function $p : I \rightarrow [0, \infty)$ such that*

$$|f(t, u, v, w, y)| \leq p(t)\Omega_3(\|u\| + |v| + |w| + |y|), \quad t \in I, \quad u \in D, \quad v, w, y \in \mathbb{R}^n,$$

where $\Omega_3 : [0, \infty) \rightarrow (0, \infty)$ is a continuous nondecreasing function.

H_a *There exists a constant L_1 such that*

$$|a(t, s)| \leq L_1 \quad \text{for } t \geq s \geq 0.$$

H_b *There exists a constant L_2 such that*

$$|b(t, s)| \leq L_2 \quad \text{for } t \geq s \geq 0.$$

H_I *There exist constants c_k such that $|I_k(x)| \leq c_k|x|$, $k = 1, \dots, m$ for each $x \in \mathbb{R}^n$.*

H_J *There exist constants d_k such that $|J_k(x)| \leq d_k|x|$, $k = 1, \dots, m$ for each $x \in \mathbb{R}^n$.*

Then if

$$\int_0^T \widehat{m}(s)ds < \int_c^{+\infty} \frac{ds}{2\Omega_3(s) + \Omega_1(s) + \Omega_2(s)}$$

where

$$\widehat{m}(t) = \max \left\{ \frac{1}{R} \int_0^t p(\tau) d\tau, \frac{1}{R} p(t), L_1 m_1(t), L_2 m_2(t) \right\}, \quad R = \min \{ \rho(t) : t \in I \} \text{ and}$$

$$c = \|\phi\| + |\eta| \rho(0) \int_0^T \frac{ds}{\rho(s)} + \frac{|\eta| \rho(0)}{R} + \sum_{i=1}^m c_k |x(t_k^-)| + \sum_{i=1}^m \frac{T - t_k}{R} \rho(t_k) d_k |x(t_k^-)|$$

the initial value problem (1.1)-(1.4) has at least one solution on $[-r, T]$.

Proof. To prove the existence of a solution of the initial value problem (1.1)-(1.4) we apply Lemma 4.1. First we obtain the *a priori bounds* for the solutions of the initial value problem (1.1) $_{\lambda}$ – (1.4), $\lambda \in (0, 1)$, where (1.1) $_{\lambda}$ stands for the equation

$$(\rho(t)x'(t))' = \lambda f \left(t, x_t, x'(t), \int_0^t a(t, s)g(s, x_s, x'(s))ds, \int_0^T b(t, s)h(s, x_s, x'(s))ds \right),$$

$$t \in I = [0, T].$$

Let x be a solution of the initial value problem (1.1) $_{\lambda}$ – (1.4). From

$$x(t) = \lambda \phi(0) + \lambda \rho(0) \int_0^t \frac{ds}{\rho(s)} \eta + \lambda \int_0^t \frac{1}{\rho(s)} \int_0^s f \left(\tau, x_{\tau}, x'(\tau), \int_0^{\tau} a(\tau, \sigma)g(\sigma, x_{\sigma}, x'(\sigma))d\sigma, \int_0^T b(\tau, \sigma)h(\sigma, x_{\sigma}, x'(\sigma))d\sigma \right) d\tau ds$$

$$+ \lambda \sum_{0 < t_k < t} \left[I_k(x(t_k^-)) + J_k(x(t_k^-)) \int_{t_k}^t \frac{\rho(t_k)}{\rho(s)} ds \right], \quad t \in I.$$

we have, for every $t \in I$,

$$|x(t)| \leq \|\phi\| + |\eta| \rho(0) \int_0^t \frac{ds}{\rho(s)} + \lambda \int_0^t \frac{1}{\rho(s)} \int_0^s p(\tau) \Omega_3 \left(\|x_{\tau}\| + |x'(\tau)| + L_1 \int_0^{\tau} m_1(\sigma) (\times) \Omega_1 (\|x_{\sigma}\| + |x'(\sigma)|) d\sigma + L_2 \int_0^{\tau} m_2(\sigma) \Omega_2 (\|x_{\sigma}\| + |x'(\sigma)|) d\sigma \right) d\tau ds$$

$$+ \sum_{i=1}^m \left\{ c_k |x(t_k^-)| + \frac{T - t_k}{R} \rho(t_k) d_k |x(t_k^-)| \right\}.$$

We consider the function μ given by

$$\mu(t) = \sup \{ |x(s)| : -r \leq s \leq t \}, \quad t \in I.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in [0, t]$, by the previous inequality we have, for every $t \in I$,

$$\begin{aligned} |\mu(t)| &\leq \|\phi\| + |\eta|\rho(0) \int_0^T \frac{ds}{\rho(s)} + \int_0^t \frac{1}{\rho(s)} \int_0^s p(\tau)\Omega_3\left(\mu(\tau) + |x'(\tau)| + L_1 \int_0^\tau m_1(\sigma) \right. \\ &\quad \left. (\times)\Omega_1(\mu(\sigma) + |x'(\sigma)|)d\sigma + L_2 \int_0^T m_2(\sigma)\Omega_2(\mu(\sigma) + |x'(\sigma)|)d\sigma\right) d\tau ds \\ &\quad + \sum_{i=1}^m c_k |x(t_k^-)| + \sum_{i=1}^m \frac{T-t_k}{R} \rho(t_k) d_k |x(t_k^-)|. \end{aligned}$$

If $t^* \in [-r, 0]$ then $\mu(t) = \|\phi\|$ and the previous inequality obvious holds.

Denoting by $u(t)$ the right hand side of the above inequality we have,

$$u(0) = \|\phi\| + |\eta|\rho(0) \int_0^T \frac{ds}{\rho(s)} + \sum_{i=1}^n c_k |x(t_k^-)| + \sum_{i=1}^m \frac{T-t_k}{R} \rho(t_k) d_k |x(t_k^-)|.$$

and

$$\begin{aligned} u'(t) &= \frac{1}{\rho(t)} \int_0^t p(\tau)\Omega_3\left(\mu(\tau) + |x'(\tau)| + L_1 \int_0^\tau m_1(\sigma)\Omega_1(\mu(\sigma) + |x'(\sigma)|)d\sigma \right. \\ &\quad \left. + L_2 \int_0^T m_2(\sigma)\Omega_2(\mu(\sigma) + |x'(\sigma)|)d\sigma\right) d\tau \\ &\leq \frac{1}{R} \int_0^t p(\tau)\Omega_3\left(u(\tau) + |x'(\tau)| + L_1 \int_0^\tau m_1(\sigma)\Omega_1(u(\sigma) + |x'(\sigma)|)d\sigma \right. \\ &\quad \left. + L_2 \int_0^T m_2(\sigma)\Omega_2(u(\sigma) + |x'(\sigma)|)d\sigma\right) d\tau, \quad t \in I. \end{aligned}$$

Therefore if

$$v(t) = \sup\{|x'(s)| : s \in I\}, \quad t \in I,$$

we obtain

$$\begin{aligned} u'(t) &\leq \frac{1}{R} \int_0^t p(\tau)\Omega_3\left(u(\tau) + v(\tau) + L_1 \int_0^\tau m_1(\sigma)\Omega_1(u(\sigma) + v(\sigma))d\sigma \right. \\ &\quad \left. + L_2 \int_0^T m_2(\sigma)\Omega_2(u(\sigma) + v(\sigma))d\sigma\right) d\tau, \quad t \in I. \end{aligned}$$

On the other hand, by

$$\begin{aligned} x'(t) &= \lambda \frac{\rho(0)\eta}{\rho(t)} + \lambda \frac{1}{\rho(t)} \int_0^t f\left(\tau, x_\tau, x'(\sigma), \int_0^\tau a(\tau, \sigma)g(\sigma, x_\sigma, x'(\sigma))d\sigma, \right. \\ &\quad \left. \int_0^T b(\tau, \sigma)h(\sigma, x_\sigma, x'(\sigma))d\sigma\right) d\tau, \end{aligned}$$

for any $t \in I$ and every $s \in [0, t]$, we obtain

$$|x'(t)| \leq \frac{\rho(0)|\eta|}{R} + \frac{1}{R} \int_0^t p(\tau)\Omega_3(u(\tau) + |x'(\tau)| + L_1 \int_0^\tau m_1(\sigma)\Omega_1(u(\sigma) + |x'(\sigma)|)d\sigma + L_2 \int_0^T m_2(\sigma)\Omega_2(u(\sigma) + |x'(\sigma)|)d\sigma) d\tau, \quad t \in I,$$

or

$$v(t) \leq \frac{\rho(0)|\eta|}{R} + \frac{1}{R} \int_0^t p(\tau)\Omega_3(u(\tau) + v(\tau) + L_1 \int_0^\tau m_1(\sigma)\Omega_1(u(\sigma) + v(\sigma))d\sigma + L_2 \int_0^T m_2(\sigma)\Omega_2(u(\sigma) + v(\sigma))d\sigma) d\tau, \quad t \in I.$$

Denoting by $z(t)$ the right hand side of the above inequality we have:

$$z(0) = \frac{\rho(0)|\eta|}{R}, \quad v(t) \leq z(t), \quad t \in I$$

and

$$\begin{aligned} z'(t) &= \frac{1}{R} p(t)\Omega_3(u(t) + v(t) + L_1 \int_0^t m_1(\sigma)\Omega_1(u(\sigma) + v(\sigma))d\sigma + L_2 \int_0^T m_2(\sigma)\Omega_2(u(\sigma) + v(\sigma))d\sigma) d\tau, \\ &\leq \frac{1}{R} p(t)\Omega_3(u(t) + z(t) + L_1 \int_0^t m_1(\sigma)\Omega_1(u(\sigma) + z(\sigma))d\sigma + L_2 \int_0^T m_2(\sigma)\Omega_2(u(\sigma) + z(\sigma))d\sigma) d\tau, \quad t \in I. \end{aligned}$$

Since $v(t) \leq z(t)$ we have

$$\begin{aligned} u'(t) &\leq \frac{1}{R} \int_0^t p(\tau)\Omega_3(u(\tau) + z(\tau) + L_1 \int_0^\tau m_1(\sigma)\Omega_1(u(\sigma) + z(\sigma))d\sigma + L_2 \int_0^T m_2(\sigma)\Omega_2(u(\sigma) + z(\sigma))d\sigma) d\tau, \quad t \in I. \end{aligned}$$

Let

$$\begin{aligned} w(t) &= u(t) + z(t) + L_1 \int_0^t m_1(\sigma)\Omega_1(u(\sigma) + z(\sigma))d\sigma + L_2 \int_0^T m_2(\sigma)\Omega_2(u(\sigma) + z(\sigma))d\sigma, \quad t \in I. \end{aligned}$$

Then

$$w(0) = u(0) + z(0) = c, \quad u(t) + z(t) \leq w(t), \quad t \in I,$$

and

$$\begin{aligned}
w'(t) &= u'(t) + z'(t) + L_1 m_1(t) \Omega_1(u(t) + z(t)) + L_2 m_2(t) \Omega_2(u(t) + z(t)) \\
&\leq \frac{1}{R} \int_0^t p(\tau) \Omega_3(w(\tau)) d\tau + \frac{1}{R} p(t) \Omega_3(w(t)) + L_1 m_1(t) \Omega_1(u(t) + z(t)) \\
&\quad + L_2 m_2(t) \Omega_2(u(t) + z(t)) \\
&\leq \frac{1}{R} \Omega_3(w(t)) \int_0^t p(\tau) d\tau + \frac{1}{R} p(t) \Omega_3(w(t)) + L_1 m_1(t) \Omega_1(u(t) + z(t)) \\
&\quad + L_2 m_2(t) \Omega_2(u(t) + z(t)) \\
&\leq \widehat{m}(t) [2\Omega_3(w(t)) + \Omega_1(w(t)) + \Omega_2(w(t))], \quad t \in I.
\end{aligned}$$

This implies

$$\int_{w(0)}^{w(t)} \frac{ds}{2\Omega_3(s) + \Omega_1(s) + \Omega_2(s)} \leq \int_0^T \widehat{m}(\tau) d\tau < \int_c^{+\infty} \frac{ds}{2\Omega_3(s) + \Omega_1(s) + \Omega_2(s)}, \quad t \in I.$$

This inequality implies that there is a constant K such that $w(t) \leq K$, $t \in I$. Then

$$\begin{aligned}
|x(t)| &\leq \mu(t) \leq u(t), \quad t \in I \\
|x'(t)| &\leq v(t) \leq z(t), \quad t \in I
\end{aligned}$$

and hence

$$\|x^*\| \leq \|x\|_r + \|x\|_1 \leq K.$$

In the second step we rewrite the initial value problem (1.1)-(1.4) as an integral operator and will prove that this operator is completely continuous. The proof will be given for the first case where $\phi(0) = 0$.

Consider the space E of all functions $x \in \Omega^1$ endowed with the norm $\|x\|_T$.

Then the set

$$E_0 = \{x \in E : x(0) = 0\}$$

is a subspace of E .

Now define an operator $Q : E_0 \rightarrow E$ by

$$\begin{aligned}
Qx(t) &= \rho(0) \int_0^t \frac{ds}{\rho(s)} \eta + \int_0^t \frac{1}{\rho(s)} \int_0^s f(\tau, x_\tau, x'(\sigma), \int_0^\tau a(\tau, \sigma) g(\sigma, x_\sigma, x'(\sigma)) d\sigma, \\
&\quad \int_0^T b(\tau, \sigma) h(\sigma, x_\sigma, x'(\sigma)) d\sigma) d\tau ds \\
&\quad + \sum_{0 < t_k < t} \left[I_k(x(t_k^-)) + J_k(x(t_k^-)) \int_{t_k}^t \frac{\rho(t_k)}{\rho(s)} ds \right], \quad t \in I.
\end{aligned}$$

where

$$x_\tau(\theta) = \begin{cases} x(\tau + \theta), & \tau + \theta \geq 0, \\ \phi(\tau + \theta), & \tau + \theta < 0. \end{cases}$$

Obviously $Q(E_0) \subseteq E_0$. It will now be shown that Q is completely continuous.

The continuity of Q follows easily from that of f . For the rest let B be a bounded subset of E_0 . Then there exists $b \geq 0$ such that $\|x\|_T \leq b$, $x \in B$.

Following exactly the same arguments as in the proof of Theorem 3.1 in [17] we can prove that there exists a compact subset D_1 of D such that $\widehat{B} \subseteq D_1$ where $\widehat{B} = \{x_t : x \in B, t \in I\}$.

Let now a bounded sequence $\{x_v\}$ in E_0 . Then the sequence $\{x_v\}$, $t \in I$ is bounded in D and, moreover, there exists a compact subset D_1 in D such that $x_{vt} \in D_1$ for every v and $t \in I$. Thus, if b_1 is the bound of $\{x_v\}$, it is obvious that the set $X = [0, T] \times B \times \overline{B}(0, b_1) \times \overline{B}(0, b_1) \times \overline{B}(0, b_1)$ ($\overline{B}(0, b_1)$ is the closed ball in \mathbb{R}^n with centre 0 and radius b_1) is compact in $[0, T] \times D \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$. Then we can prove that

$$\|Qh_v\|_0 \leq \widehat{K} \quad \text{and} \quad \|(Qh_v)'\|_1 \leq \widehat{K},$$

where

$$K = \max \left\{ \rho(0)|\eta| \int_0^T \frac{ds}{\rho(s)} + M \int_0^T \frac{s}{\rho(s)} ds, \frac{\rho(0)|\eta| + TM}{R} \right\},$$

and

$$M = \max\{|f(t, u, v, w, x)| : (t, u, v, w, x) \in X\}.$$

Also the sequence $\{Qh_v\}$ is equicontinuous. This follows easily from the relations

$$|Qh_v(t_1) - Qh_v(t_2)| = \left| \int_{t_1}^{t_2} (Qh_v)'(s) ds \right| \leq \widehat{K} |t_1 - t_2|$$

and

$$|(Qh_v(t_1))' - (Qh_v(t_2))'| \leq |\eta| \rho(0) \left| \frac{1}{\rho(t_1)} - \frac{1}{\rho(t_2)} \right| + \frac{M}{R} |t_1 - t_2|$$

and from the uniform continuity of the function $\frac{1}{\rho}$ on I .

Thus, by the Arzela-Ascoli theorem the operator Q is completely continuous.

For the proof in the general case when $\phi(0) \neq 0$, we note simply that the transformation

$$y = x - \phi(0)$$

reduces the initial value problem (1.1)-(1.4) into the following

$$\begin{aligned}
 (\rho(t)y'(t))' &= f\left(t, y_t + \phi(0), y'(t), \int_0^t a(t, s)g(s, y_s + \phi(0), y'(s))ds, \right. \\
 &\quad \left. \int_0^T b(t, s)h(s, y_s + \phi(0), y'(s))ds\right), \quad t \in I = [0, T] \setminus \{t_1, \dots, t_m\}, \\
 \Delta y|_{t=t_k} &= I_k(y(t_k^-)), \quad k = 1, \dots, m, \\
 \Delta y'|_{t=t_k} &= J_k(y(t_k^-)), \quad k = 1, \dots, m, \quad \text{and} \\
 y_0 &= \phi - \phi(0) = \widehat{\phi}, \quad y'(0) = \eta,
 \end{aligned}$$

for which $\widehat{\phi}(0) = 0$.

Finally, the set $\Phi(Q) = \{x \in E_0 : x = \lambda Qx, \lambda \in (0, 1)\}$ is bounded, as we proved in the first part. Hence by Lemma 2.1, the operator Q has a fixed point in E_0 . Then it is clear that the function

$$z(t) = \begin{cases} x(t), & t \in [0, T] \\ \phi(t), & t \in [-r, 0] \end{cases}$$

is a solution of the initial value problem (1.1)-(1.4).

Hence the proof of the theorem is complete. \square

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