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ON GRAPHS WITH EQUAL CHROMATIC TRANSVERSAL DOMINATION AND CONNECTED DOMINATION NUMBERS

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Abstract. Let $G = (V, E)$ be a graph with chromatic number $\chi(G)$. A dominating set D of G is called a chromatic transversal dominating set (ctd-set) if D intersects every color class of every χ-partition of G. The minimum cardinality of a ctd-set of G is called the chromatic transversal domination number of G and is denoted by $\gamma_{ct}(G)$. In this paper we characterize the class of trees, unicyclic graphs and cubic graphs for which the chromatic transversal domination number is equal to the connected domination number.

1. Introduction

All the graphs considered in this paper unless otherwise specifically stated are finite, connected and simple and are consistent with the terminology used in Harary [4]. Let $G = (V, E)$ be a simple graph of order p. For a subset S of V, $N(S)$ denotes the set of all vertices adjacent to some vertex in S and $N[S] = N(S) \cup S$.

A vertex v of G is called a *support* if it is adjacent to a pendant vertex. Any vertex of degree greater than one is called an *internal vertex*. A graph G is called a *unicyclic graph*, if G contains exactly one cycle.

A subset $D \subseteq V$ is a *dominating set*, if every $v \in V - D$ is adjacent to some $u \in D$. The *domination number* $\gamma = \gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set D is called a *connected dominating set* if the induced subgraph $\langle D \rangle$ is connected. The minimum cardinality of a connected dominating set is called the *connected domination number* and is denoted by $\gamma_c(G)$ or simply γ_c.

The minimum number of colors required to color the vertices of G such that no two adjacent vertices receive the same color is called the *chromatic number* of G and is denoted by $\chi(G)$. By a χ-partition of G, we mean the partition

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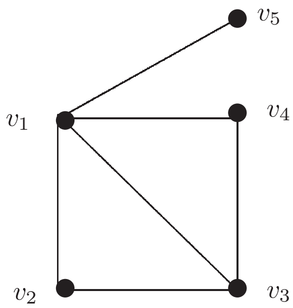
$\{V_1, V_2, \ldots, V_{\chi(G)}\}$ of $V(G)$ where each V_i is the color class representing the color i for $i = 1, 2, \ldots, \chi(G)$.

A dominating set D is said to be a *chromatic transversal dominating set* (ctd-set) if D intersects every color class of every χ-partition of G. The cardinality of a minimum ctd-set is called the *chromatic transversal domination number* of G and is denoted by $\gamma_{ct}(G)$.

Example 1.



The two χ-partitions are $\{\{v_1\}, \{v_3, v_5\}, \{v_2, v_4\}\}$ and $\{\{v_1\}, \{v_3\}, \{v_2, v_4, v_5\}\}$. Therefore, the possible ctd-sets are $D_1 = \{v_1, v_2, v_3\}$ and $D_2 = \{v_1, v_3, v_4\}$. Hence, $\gamma_{ct}(G) = 3$.

If a graph G has a critical vertex, say u, then $\{u\}$ is a color class of some color partition of G and consequently u will be in every ctd-set of G. For example, if G is a cycle of odd length, say n, then $\gamma_{ct} = n$ itself. The parameter γ_{ct} for a few well known graphs was computed by L. Benedict et al. [8]. S. K. Ayyaswamy et al. [2] characterized graphs for which $\gamma_{ct} = 2$.

Theorem 1 ([7]). *Let G be a connected bipartite graph of order $p \geq 3$ with partition (V_1, V_2) of V, where $|V_1| \leq |V_2|$. Then $\gamma_{ct}(G) = \gamma(G) + 1$ if and only if every vertex in V_1 has at least two pendant neighbors.*

Theorem 2 ([1]). *Let G be a unicyclic graph with a cycle C of length $n \geq 5$ and let X be the set of all vertices of degree 2 in C. Then $\gamma(G) = \gamma_c(G)$ if and only if the following conditions hold good*:

(a) *Every vertex of degree at least 2 in $V - N[X]$ is a support.*
(b) *$\langle X \rangle$ is connected and $|X| \leq 3$.*
(c) *If $\langle X \rangle = P_1$ or P_3, both vertices in $N(X)$ of degree greater than 2 are supports and if $\langle X \rangle = P_2$, at least one vertex in $N(X)$ of degree greater than 2 is a support.*

Theorem 3 ([1]). *For a tree T of order $p \geq 3$, $\gamma(T) = \gamma_c(T)$ if and only if every internal vertex of T is a support.*

Theorem 4 ([7]). *For a tree T, $\gamma_{ct}(T) = \gamma(T) + 1$ if and only if either T is K_2 or T satisfies the condition that whenever v is a support vertex, then each* *vertex w with $d(v, w)$ even is also a support vertex and each support vertex has at least two pendant neighbors. Otherwise $\gamma_{ct}(G) = \gamma(G)$.*

2. Main results

In this section we characterize trees, unicyclic graphs and cubic graphs for which $\gamma_{ct} = \gamma_c$. We now characterize trees.

2.1. Trees

By a *double star* we mean a tree obtained by attaching the centres of two given stars by an edge.

Theorem 5. *For a tree T of order $p \geq 3$, $\gamma_{ct}(T) = \gamma_c(T)$ if and only if either every internal vertex of T is a support or T is a double star.*

Proof. Suppose $\gamma_{ct}(T) = \gamma_c(T)$.

If $\gamma_{ct}(T) = \gamma(T)$, then $\gamma(T) = \gamma_c(T)$ and so every internal vertex of T is a support by Theorem 3.

If $\gamma_{ct}(T) = \gamma(T)+1$, then by Theorem 4, T is K_2 or it satisfies the condition that whenever v is a support then each vertex w with $d(v, w)$ even is also a support and each support has at least two pendant neighbors.

Claim 1. T has exactly two supports.

Suppose there are three support vertices, say u, v and w such that $d(u, v) = 2r_1$ and $d(v, w) = 2r_2$ for $r_1, r_2 \geq 1$.

Case 1. Let $r_1 = r_2 = 1$. Then $\gamma_{ct}(T) = 4$ whereas $\gamma_c(T) = 5$.

Case 2. Let $r_1 \neq 1$ and $r_2 \neq 1$. Leaving the neighbors of u, v and w, there are two paths P_l and P_m where $l = 2r_1 - 3$ and $m = 2r_2 - 3$ which are dominated by $\left\lceil \frac{2r_1-3}{3} \right\rceil$ and $\left\lceil \frac{2r_2-3}{3} \right\rceil$ vertices respectively. Therefore $\gamma_{ct}(T) = 3 + \left\lceil \frac{2r_1-3}{3} \right\rceil + \left\lceil \frac{2r_2-3}{3} \right\rceil$ but $\gamma_c(T) = 2r_1 + 2r_2 + 1$.

Case 3. Let $r_1 = 1$ and $r_2 \neq 1$. Then $\gamma_{ct}(T) = 3 + \left\lceil \frac{2r_2-3}{3} \right\rceil$ and $\gamma_c(T) = 3+2r_2$.

Thus in all cases $\gamma_{ct}(T) < \gamma_c(T)$. Therefore T has exactly two support vertices.

Claim 2. $d(u, v) = 2$, where u and v are the support vertices of T.

If not, let $d(u, v) = 2r$, $r \geq 2$. Then $\gamma_{ct}(T) = \left\lceil \frac{2r-3}{3} \right\rceil + 2$, whereas $\gamma_c(T) = 2r + 1$. Therefore $\gamma_{ct}(T) < \gamma_c(T)$ for all $r \geq 2$. Thus $d(u, v) = 2$ and consequently T is a double star.

The converse is obvious. □

2.2. Unicyclic graphs

We now characterize unicyclic graphs for which $\gamma_{ct} = \gamma_c$.

Theorem 6. *Let $G = (V, E)$ be a connected unicyclic graph with an even cycle C of length n and let (V_1, V_2) be the χ-partition of V such that $|V_1| \leq |V_2|$. Then $\gamma_{ct}(G) = \gamma_c(G)$ if and only if either* (i) *or* (ii) *holds good:*

(i) $|V_1| = 2$ *and both vertices of* V_1 *have at least two pendant neighbors.*

(ii) (a) *Every vertex of degree at least two in* $V - N[X]$ *is a support, where* X *is the set of all vertices of degree* 2 *in* C.
(b) $\langle X \rangle$ *is connected and* $|X| \leq 3$.
(c) $\langle X \rangle = P_1$ *or* P_3*, both vertices in* $N(X)$ *of degree greater than 2 are supports and if* $\langle X \rangle = P_2$*, at least one vertex in* $N(X)$ *of degree greater than* 2 *is a support.*

Proof. Let us assume that $\gamma_{ct}(G) = \gamma_c(G)$. As G is bipartite by Thoerem 1, $\gamma_{ct}(G)$ is either $\gamma(G)$ or $\gamma(G) + 1$.

Case 1. Let $\gamma_{ct}(G) = \gamma(G) + 1$. Then by Theorem 1, every vertex in V_1 has at least two pendant neighbors and $\gamma_{ct}(G) = |V_1| + 1$. But for a connected bipartite graph, $\gamma_c = 2|V_1| - 1$. This implies $|V_1| = 2$.

Case 2. Let $\gamma_{ct}(G) = \gamma(G)$. Then by Theorem 2, the conditions (a), (b) and (c) in (ii) hold good.

The converse is obvious. □

Theorem 7. *Let* G *be a unicyclic graph with an odd cycle* C *of length, say* m. *Let* X *be the set of all vertices of degree* 2 *in* C*,* F *be the set of all internal vertices in* $V(G) - V(C)$ *and* S *be the set of all vertices in* F *which are not supports of leaves in* G*. Then* $\gamma_{ct}(G) = \gamma_c(G)$ *if and only if one of the following conditions hold*:

(i) *If* $|X| = 0$*,* $S = \phi$.

(ii) *If* $|X| = 1$ *or* $|X| \geq 2$ *such that no two vertices are adjacent, then* $|S| = 1$.

(iii) *If* $|X| \geq 2$ *and at least two vertices of* X *are adjacent, then* $|S| = 2$ *or* $S = \{v_1, v_2, v_3\}$ *such that*
($*$) $\langle S \rangle$ *is the path* $v_1v_2v_3$ *and* $\deg(v_2) = 2$ *in* G.

Proof. As we know every vertex v in an odd cycle C is a χ-critical vertex, $\{v\}$ forms a color class of some χ-partition of G. This implies every ctd-set contains all the vertices of C. Also every ctd-set of G contains all supports of leaves of G.

Every γ_c-set of G contains F and $m - r$ vertices of C where $r = 0, 1, 2$ and this r depends on the nature of the set X. For example, if $\langle X \rangle$ is connected and $|X| = 2$, then $\gamma_c(G) = m - 2 + |F|$.

Assume that $\gamma_{ct}(G) = \gamma_c(G)$.

Case 1. Let $|X| = 0$. Then $\gamma_c(G) = m + |F|$. We claim that $S = \emptyset$; otherwise $|V(C)|$ and $|F| - 1$ number of vertices in $V(G) - V(C)$ will form a ctd-set of G so that $\gamma_{ct}(G) \leq m + |F| - 1 < \gamma_c(G)$, a contradiction.

Case 2. Let $|X| = 1$ or $|X| \geq 2$ and no two vertices in X are adjacent. Then $\gamma_c(G) = m - 1 + |F|$.

Claim: $|S| = 1$. Clearly $S \neq \emptyset$. Otherwise $\gamma_{ct}(G) = m + |F| > \gamma_c(G)$. Suppose $|S| \geq 2$. Then every γ_{ct}-set of G will contain at most $|F| - 2$ vertices in $V(G) - V(C)$ and so $\gamma_{ct}(G) \leq m - 2 + |F| < \gamma_c(G)$, a contradiction.

Case 3. Let $|X| \geq 2$ and at least two vertices in X are adjacent. Then $\gamma_c(G) = m - 2 + |F|$.

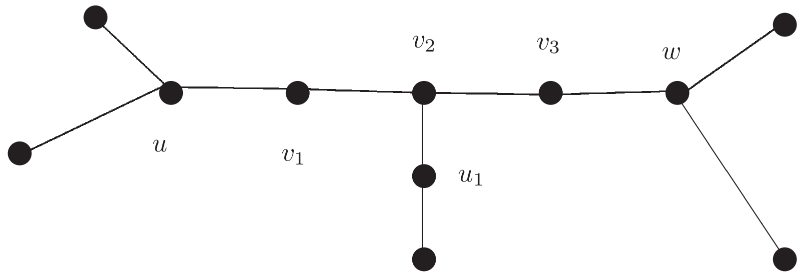
Claim: $|S| = 2$ or $S = \{v_1, v_2, v_3\}$ such that $v_1v_2v_3$ is a path and $\deg(v_2) = 2$. If $|S| = 0$ or 1, then clearly $\gamma_{ct}(G) = m + |F|$ and $m + |F| - 1$ respectively. So, assume that $|S| \geq 4$. Then at least $|F| - 3$ vertices of $V(G) - V(C)$ will be in every ctd-set of G and so $\gamma_{ct}(G) \leq m - 3 + |F| < \gamma_c(G)$, which is a contradiction.

Next, suppose that $|S| = 3$ and S does not satisfy the condition $(*)$. Then there are two possibilities:

(i) either v_1 and v_2 are not adjacent or v_2 and v_3 are not adjacent.
(ii) $\deg(v_2) > 2$.

(i) implies there is a support vertex between v_1 and v_2 or between v_2 and v_3. For example, if there is a support vertex u between v_1 and v_2, then u will be in every γ_{ct}-set of G dominating v_1 and v_2. As $|S| = 3$, clearly v_3 will be adjacent to a support vertex of a leaf. Therefore, $\gamma_{ct}(G) \leq m - 3 + |F| < \gamma_c(G)$.

Now, come to the case $\deg(v_2) > 2$. Then v_2 has neighbors other than v_1 and v_2 which are support vertices of leaves, say $u_1 \in N(v_2) - \{v_1, v_3\}$.



In this case v_1, v_2 and v_3 are dominated by u, u_1 and w, respectively and so $\gamma_{ct}(G) \leq m - 3 + |F| < \gamma_c(G)$. Thus $\gamma_{ct}(G) = \gamma_c(G)$ and $|S| = 3$ implies condition $(*)$.

Thus if $|S| = 3$, then $\langle S \rangle$ is a path $v_1v_2v_3$ with $\deg(v_2) = 2$.

The converse is obvious. □

2.3. Cubic graphs

Theorem 8. *For a cubic bipartite graph G, $\gamma_{ct}(G) = \gamma_c(G)$ if and only if $G = K_{3,3}$.*

Proof. Let us assume that $\gamma_{ct}(G) = \gamma_c(G)$. As G has no supports with at least two pendant vertices, $\gamma_{ct}(G) = \gamma(G)$. Therefore $\gamma_c(G) = \gamma(G)$ and so G is $K_{3,3}$. □

Let G be a connected cubic graph with at least one odd cycle. We know $\chi(G) = 3$. If X is a transversal of a χ-partition of G, then clearly X will have an odd cycle as an induced subgraph. Otherwise there exists a χ-partition of G in which the vertices of X can be colored with two colors. Therefore, every ctd-set contains at least one odd cycle and hence a γ_{ct}-set being a minimum ctd-set will contain an odd cycle C of smallest length, say m and a γ-set S of $G^{'} = G - N[C]$ where by a γ-set of $G^{'}$ we mean a least subset S of G which dominates all the vertices of $G^{'}$. This implies $\gamma_{ct}(G) = |V(C)| + |S| = m + |S|$.

Let D be a γ_c-set of G. Then we define $T = D \cap (V(G^{'}) \cup (N(C) - C)) - S$ and $l = m - |D \cap V(C)|$.

Theorem 9. *Let G be a connected cubic graph with at least one odd cycle and let C be an odd cycle of smallest length, say m. Then $\gamma_{ct}(G) = \gamma_c(G)$ if and only if there exists a γ_c-set D of G such that $|T| = l$.*

Proof. Let us assume that $\gamma_{ct}(G) = \gamma_c(G)$.

Suppose there exists no γ_c-set D such that $|T| = l$. Therefore $\gamma_c(G) = |D| = m + |T| - l + |S|$. If $|T| - l > 0$, then $\gamma_c(G) > \gamma_{ct}(G)$. On the other hand, if $|T| - l < 0$, then $\gamma_c(G) < \gamma_{ct}(G)$. Thus, in both cases $\gamma_{ct}(G) \neq \gamma_c(G)$.

Conversely, assume that there exists a γ_c-set D of G such that D contains a γ-set S of $G^{'}$ and $m - l$ number of vertices of C. If $T = l$, then $\gamma_c(G) = |D| = m - l + |T| + |S| = m + |S| = \gamma_{ct}(G)$. □

Open problems

1. Can we improve Theorem 9 in terms of graph structure?
2. Characterize block graphs and cactus graphs for which $\gamma_{ct} = \gamma_c$.

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