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ON A CLASS OF THREE-DIMENSIONAL TRANS-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study 3-dimensional trans-Sasakian manifolds with conservative curvature tensor and also 3-dimensional conformally flat trans-Sasakian manifolds. Next we consider compact connected η-Einstein 3-dimensional trans-Sasakian manifolds. Finally, an example of a 3-dimensional trans-Sasakian manifold is given, which verifies our results.

1. Introduction

Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures by D. Chinea and C. Gonzales [6] and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds. Again in the Gray-Hervella classification of almost Hermite manifolds [10], there appears a class W_4 of Hermitian manifolds which are closely related to locally conformally Kähler manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [17] if the product manifold $M \times \mathbb{R}$ belongs to the class W_4. The class $C_6 \bigoplus C_5$ ([15], [16]) coincides with the class of trans-Sasakian structures of type (α,β). In [16], the local nature of the two subclasses C_5 and C_6 of trans-Sasakian structures is characterized completely. In [7], some curvature identities and sectional curvatures for C_5, C_6 and trans-Sasakian manifolds are obtained. It is known that [12] trans-Sasakian structures of type (0,0), $(0,\beta)$, and $(\alpha,0)$ are cosymplectic, β-Kenmotsu and α-Sasakian, respectively, where α, $\beta \in \mathbb{R}$.

The local structure of trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by J. C. Marrero [15]. He proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α-Sasakian or β-Kenmotsu manifold. Three-dimensional trans-Sasakian manifolds have been studied by De and Tripathi [9], De and Sarkar [8], Kim, Prasad and Tripathi [14], Bagewadi and Venkatesha [1], Shukla and Singh [18] and many others. In

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[13] Jun and Kim studied 3-dimensional almost contact metric manifolds. The curvature tensor R in a Riemanian manifold is said to be conservative [11], that is, $divR = 0$ if and only if $(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)$ where S is the Ricci tensor of the manifold. Moreover, Boyer and Galicki [5] proved that if M is a compact η-Einstein K-contact manifold with Ricci tensor $S = ag + b\eta \otimes \eta$, and if $a \geq -2$, then M is Sasakian. Motivated by these works in this paper we study some curvature conditions in a 3-dimensional trans-Sasakian manifold.

The paper is organized as follows. In Section 2, some preliminary results are recalled. After preliminaries in Section 3, we give an example of a 3-dimensional trans-Sasakian manifold of type (α, β). Then we study 3-dimensional connected trans-Sasakian manifold with conservative curvature tensor. In the next section, we study 3-dimensional conformally flat connected trans-Sasakian manifold. In Section 6, we prove that if a compact connected 3-dimensional trans-Sasakian manifold is η-Einstein with constant coefficients, then it is either α-Sasakian or β-Kenmotsu. Finally, we construct an example of a 3-dimensional trans-Sasakian manifold with constant function α, β on M.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ,ξ,η,g), that is, ϕ is an (1,1) tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta\phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y),\ g(X, \xi) = \eta(X) \tag{2.3}$$

for all X and Y tangent to M ([2], [3]).

The fundamental 2-form Φ of the manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y) \tag{2.4}$$

for all X and Y tangent to M.

An almost contact metric structure (ϕ,ξ,η,g) on a connected manifold M is called trans-Sasakian structure [17] if $(M \times \mathbb{R}, J, G)$ belongs to the class W_4 [10], where J is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J\left(X, f\frac{d}{df}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

for any vector fields X on M, f is a smooth function on $M \times \mathbb{R}$ and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [4]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.5}$$

for smooth functions α and β on M. Hence we say that the trans-Sasakian structure is of type (α,β). From (2.5) it follows that

$$\nabla_X \xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi), \tag{2.6}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.7}$$

An explicit example of a 3-dimensional proper trans-Sasakian manifold is constructed in [15]. In [9], Ricci tensor and curvature tensor for 3-dimensional trans-Sasakian manifolds are studied and their explicit formulae are given.

From [9] we know that for a 3-dimensional trans-Sasakian manifold

$$2\alpha\beta + \xi\alpha = 0, \tag{2.8}$$

$$S(X,\xi) = (2(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - X\beta - (\phi X)\alpha, \tag{2.9}$$

$$\begin{aligned} S(X,Y) = {} & \Big(\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\Big)\, g(X,Y) \\ & - \Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\, \eta(X)\eta(Y) \\ & - (Y\beta + (\phi Y)\alpha)\eta(X) - (X\beta + (\phi X)\alpha)\eta(Y), \end{aligned} \tag{2.10}$$

$$\begin{aligned} R(X,Y)\xi = {} & (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) \\ & - \eta(Y)(X\beta)\xi + \phi(X)\alpha\xi \\ & + \eta(X)(Y\beta)\xi + \phi(Y)\alpha\xi \\ & - (Y\beta)X + (X\beta)Y - (\phi(Y)\alpha)X + (\phi(X)\alpha)Y, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} R(X,Y)Z = {} & \Big(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2)\Big)\, (g(Y,Z)X - g(X,Z)Y \\ & - g(Y,Z)\Big[\, \Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\, \eta(X)\xi \\ & - \eta(X)(\phi grad\alpha - grad\beta) + (X\beta + (\phi X)\alpha)\xi\Big] \\ & + g(X,Z)\Big[\, \Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\, \eta(Y)\xi \\ & - \eta(Y)(\phi grad\alpha - grad\beta) + (Y\beta + (\phi Y)\alpha)\xi\Big] \\ & - [(Z\beta + (\phi Z)\alpha)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(Z) \\ & + \frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\eta(Z)]X \\ & + \Big[(Z\beta + (\phi Z)\alpha)\eta(X) + (X\beta + (\phi X)\alpha)\eta(Z) \\ & + \frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\eta(Z)\Big]Y, \end{aligned} \tag{2.12}$$

where S is the Ricci tensor of type $(0,2)$ and R is the curvature tensor of type $(1,3)$ and r is the scalar curvature of the manifold M.

3. Example of a 3-dimensional trans-Sasakian manifold of type (α, β)

We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are standard co-ordinate of $\mathbb{R}^3$.

The vector fields

$$e_1 = z\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right), \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = 1,$$

$$\phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$, the set of all smooth vector fields on M.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to metric g and R be the curvature tensor of g. Then we have

$$[e_1, e_2] = ye_2 - z^2 e_3, \ [e_1, e_3] = -\frac{1}{z}e_1 \text{ and } [e_2, e_3] = -\frac{1}{z}e_2.$$

Taking $e_3 = \xi$ and using Koszul formula for the Riemannian metric g, we can easily calculate

$$\nabla_{e_1} e_3 = -\frac{1}{z}e_1 + \frac{1}{z^2}e_2, \quad \nabla_{e_1} e_2 = -\frac{1}{2}z^2 e_3,$$

$$\nabla_{e_1} e_1 = \frac{1}{z}e_3, \nabla_{e_2} e_3 = -\frac{1}{z}e_2 - \frac{1}{2}z^2 e_1,$$

$$\nabla_{e_2} e_2 = ye_1 + \frac{1}{z}e_3, \quad \nabla_{e_2} e_1 = \frac{1}{2}z^2 e_3 - ye_2,$$

$$\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = -\frac{1}{2}z^2 e_1, \quad \nabla_{e_3} e_1 = \frac{1}{2}z^2 e_2.$$

From the above it can be easily seen that (ϕ, ξ, η, g) is a trans-Sasakian structure on M. Consequently $M^3(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold with $\alpha = -\frac{1}{2}z^2 \neq 0$ and $\beta = -\frac{1}{z} \neq 0$.

4. 3-Dimensional connected trans-Sasakian manifolds with conservative curvature tensor

Let M be a 3-dimensional connected trans-Sasakian manifold with conservative curvature tensor [11], that is, $divR = 0$. Then its Ricci tensor is given by $(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$. From this we obtain $r =$ constant. We know that

$$(\nabla_X S)(Y,Z) = \nabla_X S(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{4.1}$$

Using (2.10) we have

$$\begin{aligned}
&(\nabla_X S)(Y,Z)\\
&= \left[\frac{dr(X)}{2} + \nabla_X(\xi\beta) - 2\alpha d\alpha(X) + 2\beta d\beta(X)\right] g(Y,Z)\\
&+ \left(\frac{r}{2} + \xi\beta - (\alpha^2-\beta^2)\right) \nabla_X g(Y,Z)\\
&- \left[\frac{dr(X)}{2} + \nabla_X(\xi\beta) - 6\alpha d\alpha(X) + 6\beta d\beta(X)\right] \eta(Y)\eta(Z)\\
&- \left(\frac{r}{2} + \xi\beta - 3(\alpha^2-\beta^2)\right) [\nabla_X\eta(Y)\eta(Z) + \eta(Y)\nabla_X\eta(Z)]\\
&- (\nabla_X(Z\beta + (\phi Z)\alpha))\eta(Y) - (Z\beta + (\phi Z)\alpha)\nabla_X\eta(Y)\\
&- (\nabla_X(Y\beta + (\phi Y)\alpha))\eta(Z) - (Y\beta + (\phi Y)\alpha)\nabla_X\eta(Z)\\
&- \left(\frac{r}{2} + \xi\beta - (\alpha^2-\beta^2)\right) g(\nabla_X Y, Z)\\
&+ \left(\frac{r}{2} + \xi\beta - 3(\alpha^2-\beta^2)\right) \eta(\nabla_X Y)\eta(Z)\\
&+ (Z\beta + (\phi Z)\alpha)\eta(\nabla_X Y) + ((\nabla_X Y)\beta + (\phi(\nabla_X Y))\alpha)\eta(Z)\\
&- \left(\frac{r}{2} + \xi\beta - (\alpha^2-\beta^2)\right) g(Y, \nabla_X Z)\\
&+ \left(\frac{r}{2} + \xi\beta - 3(\alpha^2-\beta^2)\right) \eta(Y)\eta(\nabla_X Z)\\
&+ ((\nabla_X Z)\beta + (\phi(\nabla_X Z))\alpha)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(\nabla_X Z)r.
\end{aligned} \tag{4.2}$$

The above relation can be written as

$$\begin{aligned}
&(\nabla_X S)(Y,Z)\\
&= \left[\frac{dr(X)}{2} + \nabla_X(\xi\beta) - 2\alpha d\alpha(X) + 2\beta d\beta(X)\right] g(Y,Z)\\
&- \left[\frac{dr(X)}{2} + \nabla_X(\xi\beta) - 6\alpha d\alpha(X) + 6\beta d\beta(X)\right] \eta(Y)\eta(Z)\\
&- \left(\frac{r}{2} + \xi\beta - 3(\alpha^2-\beta^2)\right) [(\nabla_X\eta)(Y)\eta(Z) + \eta(Y)(\nabla_X\eta)(Z)]
\end{aligned} \tag{4.3}$$

$$
\begin{aligned}
&- (\nabla_X(Z\beta + (\phi Z)\alpha))\eta(Y) - (Z\beta + (\phi Z)\alpha)(\nabla_X\eta)(Y)\\
&- (\nabla_X(Y\beta + (\phi Y)\alpha))\eta(Z) - (Y\beta + (\phi Y)\alpha)(\nabla_X\eta)(Z)\\
&+ ((\phi(\nabla_X Y))\alpha)\eta(Z) + ((\phi(\nabla_X Z))\alpha)\eta(Y).
\end{aligned}
$$

Now from (4.3) we have

$$
\begin{aligned}
&(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\\
= &\left[\frac{dr(X)}{2} + \nabla_X(\xi\beta) - 2\alpha d\alpha(X) + 2\beta d\beta(X)\right] g(Y,Z)\\
&- \left[\frac{dr(Y)}{2} + \nabla_Y(\xi\beta) - 2\alpha d\alpha(Y) + 2\beta d\beta(Y)\right] g(X,Z)\\
&- \left[\frac{dr(X)}{2} + \nabla_X(\xi\beta) - 6\alpha d\alpha(X) + 6\beta d\beta(X)\right] \eta(Y)\eta(Z)\\
&+ \left[\frac{dr(Y)}{2} + \nabla_Y(\xi\beta) - 6\alpha d\alpha(Y) + 6\beta d\beta(Y)\right] \eta(X)\eta(Z)\\
&- \Big(\frac{r}{2} + \xi\beta - 3(\alpha^2-\beta^2)\Big) [(\nabla_X\eta)(Y)\eta(Z) + \eta(Y)(\nabla_X\eta)(Z)\\
&- (\nabla_Y\eta)(X)\eta(Z) - \eta(X)(\nabla_Y\eta)(Z)]\\
&- (\nabla_X(Z\beta + (\phi Z)\alpha))\eta(Y) + (\nabla_Y(Z\beta + (\phi Z)\alpha))\eta(X)\\
&- (\nabla_X(Y\beta + (\phi Y)\alpha))\eta(Z) + (\nabla_Y(X\beta + (\phi X)\alpha))\eta(Z)\\
&- (Z\beta + (\phi Z)\alpha)[(\nabla_X\eta)(Y) - (\nabla_Y\eta)(X)]\\
&- (Y\beta + (\phi Y)\alpha)(\nabla_X\eta)(Z)\\
&+ (X\beta + (\phi X)\alpha)(\nabla_Y\eta)(Z)\\
&+ ((\phi(\nabla_X Y))\alpha)\eta(Z) - ((\phi(\nabla_Y X))\alpha)\eta(Z)\\
&+ ((\phi(\nabla_X Z))\alpha)\eta(Y) - ((\phi(\nabla_Y Z))\alpha)\eta(X).
\end{aligned}
\tag{4.4}
$$

Suppose $divR = 0$ and α, β are constants. Then using (2.7) in (4.4) and using $r =$ constant, we obtain

$$
\begin{aligned}
&\Big(\frac{r}{2} - 3(\alpha^2-\beta^2)\Big) [-\alpha g(\phi X, Y)\eta(Z)\\
&- \alpha g(\phi X, Z)\eta(Y) + \alpha g(\phi Y, X)\eta(Z)\\
&+ \alpha g(\phi Y, Z)\eta(X) + \beta g(\phi X, \phi Z)\eta(Y) - \beta g(\phi Y, \phi Z)\eta(X)] = 0.
\end{aligned}
\tag{4.5}
$$

Let $\{e_0, e_1, e_2\}$ be a local ϕ-basis, that is, an orthonormal frame such that $e_0 = \xi$ and $e_2 = \phi e_1$. In (4.5) putting $X = e_1$, $Y = e_2$, we get

$$
2\alpha\left[\Big(\frac{r}{2} - 3(\alpha^2-\beta^2)\Big)\right]\eta(Z) = 0. \tag{4.6}
$$

This implies either $\alpha = 0$ or $r = 6(\alpha^2-\beta^2)$, or both holds. If $r = 6(\alpha^2-\beta^2)$, then from (2.10) it follows that

$$
S(X,Y) = 2(\alpha^2-\beta^2)g(X,Y). \tag{4.7}
$$

This implies that the manifold is an Einstein manifold. This leads to the following theorem:

Theorem 4.1. *If a 3-dimensional connected trans-Sasakian manifold is of conservative curvature tensor, then the manifold is either a β-Kenmotsu manifold or an Einstein manifold or both holds provided $\alpha, \beta =$ constant.*

If the manifold is an Einstein manifold, then the manifold is of conservative curvature tensor. Hence we obtain the following:

Corollary 1. *A 3-dimensional connected trans-Sasakian manifold which is not a β-Kenmotsu manifold is of conservative curvature tensor if and only if the manifold is an Einstein manifold provided $\alpha, \beta =$ constant.*

5. 3-Dimensional conformally flat connected trans-Sasakian manifolds

Let M be a 3-dimensional conformally flat connected trans-Sasakian manifold. At first we prove the following:

Lemma 5.1. *Let M be a 3-dimensional connected trans-Sasakian manifold with $\alpha, \beta =$ constant. If there exist functions L and N on M such that*

$$(\nabla_X Q)Y - (\nabla_Y Q)X = LX + NY, \; X, \; Y \in \chi(M), \tag{5.1}$$

then either $\alpha = 0$ or

$$QX = 2(\alpha^2 - \beta^2)X. \tag{5.2}$$

Proof. We have from (2.10),

$$QX = aX + b\eta(X)\xi, \tag{5.3}$$

where $a = (\frac{r}{2} - (\alpha^2 - \beta^2))$ and $b = -(\frac{r}{2} - 3(\alpha^2 - \beta^2))$ and thus using (5.3) we have

(5.4)
$$\begin{aligned} &(\nabla_X Q)Y - (\nabla_Y Q)X \\ &= (Xa)Y - (Ya)X + (Xb)\eta(Y)\xi - (Yb)\eta(X)\xi \\ &\quad + b\alpha(\eta(x)\phi Y - \eta(Y)\phi(X)) + b\beta(\eta(Y)X - \eta(X)Y) - 2\alpha b g(\phi X, Y)\xi. \end{aligned}$$

Replacing X by ϕX and Y by ϕY in (5.3) we get

$$(\nabla_{\phi X} Q)\phi Y - (\nabla_{\phi Y} Q)\phi X = (\phi X a)\phi Y - (\phi Y a)\phi X - 2\alpha b g(\phi^2 X, \phi Y)\xi. \tag{5.5}$$

From (5.1) and (5.5), we obtain

$$(L + (\phi Y)a)\phi X + (N - (\phi X)a)\phi Y = 2\alpha b g(\phi^2 X, \phi Y)\xi. \tag{5.6}$$

Using (2.1) in (5.6) yields

$$2\alpha b g(X, \phi Y) = 0, \tag{5.7}$$

which implies either $\alpha = 0$ or $b = 0$. Thus from the definition of η-Einstein manifold, we get $QX = aX$ and hence $QX = 2(\alpha^2 - \beta^2)X$. □

It is classical that on a 3-dimensional conformally flat Riemannian manifold [19], we have

$$(\nabla_X Q)Y - (\nabla_Y Q)X = \frac{1}{4}(dr(X)Y - dr(Y)X). \tag{5.8}$$

Then by Lemma 5.1 we get either $\alpha = 0$ or $QX = 2(\alpha^2 - \beta^2)X$. This leads to the following theorem:

Theorem 5.1. *A 3-dimensional conformally flat connected trans-Sasakian manifold is either a β-Kenmotsu manifold or an Einstein manifold.*

Since an Einstein manifold is of conservative curvature tensor, hence we obtain the following:

Corollary 2. *In a 3-dimensional conformally flat connected trans-Sasakian manifold which is not a β-Kenmotsu manifold, the curvature tensor is conservative.*

6. Compact connected η-Einstein manifolds

Let M be a 3-dimensional compact connected trans-Sasakian manifold. If the manifold is η-Einstein, then the Ricci tensor S of type $(0,2)$ of the manifold is given by

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{6.1}$$

where a, b are smooth functions on M. Here we suppose that a and b are constants. Putting $Y = \xi$ in (6.1) and using (2.9), we get

$$X\beta + (\phi X)\alpha + [(a+b) - 2(\alpha^2 - \beta^2) + \xi\beta]\eta(X) = 0. \tag{6.2}$$

For $X = \xi$, (6.2) yields

$$\xi\beta = (\alpha^2 - \beta^2) - \frac{(a+b)}{2}. \tag{6.3}$$

By virtue of (6.2) and (6.3), it follows that

$$X\beta + (\phi X)\alpha + \left[\frac{(a+b)}{2} - \alpha^2 + \beta^2\right]\eta(X) = 0. \tag{6.4}$$

The gradient of the function β is related to the exterior derivative $d\beta$ by the formula

$$d\beta(X) = g(grad\beta, X). \tag{6.5}$$

Using (6.5) in (6.4) we obtain

$$d\beta(X) + g(grad\alpha, \phi X) + \left[\frac{(a+b)}{2} - \alpha^2 + \beta^2\right]\eta(X) = 0. \tag{6.6}$$

Differentiating (6.6) covariantly with respect to Y we get

$$\begin{aligned}(6.7)\quad &(\nabla_Y d\beta)(X) + g(\nabla_Y grad\alpha, \phi X) + g(grad\alpha, (\nabla_Y \phi)X) \\ &+ Y(\beta^2 - \alpha^2)\eta(X) + \left[\frac{(a+b)}{2} - \alpha^2 + \beta^2\right](\nabla_Y \eta)(X) = 0.\end{aligned}$$

Interchanging X and Y in (6.7), we get

$$\begin{aligned}(6.8)\quad &(\nabla_X d\beta)(Y) + g(\nabla_X grad\alpha, \phi Y) + g(grad\alpha, (\nabla_X \phi)Y) \\ &+ X(\beta^2 - \alpha^2)\eta(Y) + \left[\frac{(a+b)}{2} - \alpha^2 + \beta^2\right](\nabla_X \eta)(Y) = 0.\end{aligned}$$

Subtracting (6.7) from (6.8)we get

$$\begin{aligned}(6.9)\quad &g(\nabla_X grad\alpha, \phi Y) - g(\nabla_Y grad\alpha, \phi X) + [(\nabla_X \phi)Y) - (\nabla_Y \phi)X)]\alpha \\ &+ [X(\beta^2 - \alpha^2)\eta(Y) - Y(\beta^2 - \alpha^2)\eta(X)] \\ &+ \left[\frac{(a+b)}{2} - \alpha^2 + \beta^2\right][(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X)] = 0.\end{aligned}$$

From (2.7) and (2.4) we get

$$(6.10)\quad (\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 2\alpha\Phi(X, Y).$$

Using (6.10) in (6.9) we have

$$\begin{aligned}(6.11)\quad &g(\nabla_X grad\alpha, \phi Y) - g(\nabla_Y grad\alpha, \phi X) + [(\nabla_X \phi)Y) - (\nabla_Y \phi)X)]\alpha \\ &+ [X(\beta^2 - \alpha^2)\eta(Y) - Y(\beta^2 - \alpha^2)\eta(X)] \\ &+ 2\left[\frac{(a+b)}{2} - \alpha^2 + \beta^2\right]\Phi(X, Y) = 0.\end{aligned}$$

Let $\{e_0, e_1, e_2\}$ be a local ϕ-basis, that is, an orthonormal frame such that $e_0 = \xi$ and $e_2 = \phi e_1$. In (2.5) putting $X = e_1$, $Y = e_2$, we get

$$\begin{aligned}(6.12)\quad (\nabla_{e_1}\phi)e_2 &= \alpha(g(e_1, e_2)\xi - \eta(e_2)e_1) + \beta(g(\phi e_1, e_2)\xi - \eta(e_2)\phi e_1) \\ &= \beta g(\phi e_1, e_2)\xi = \beta\xi.\end{aligned}$$

Similarly,

$$(6.13)\quad (\nabla_{e_2}\phi)e_1 = -\beta\xi.$$

Now,

$$(6.14)\quad \Phi(e_1, e_2) = g(e_1, \phi e_2) = g(e_1, \phi^2 e_1) = -1.$$

In (6.11) putting $X = e_1$ and $Y = e_2$ and using (6.12), (6.13) and (6.14) we obtain

(6.15)

$$g(\nabla_{e_1} grad\alpha, e_1) + g(\nabla_{e_2} grad\alpha, e_2) = 2\beta\xi\alpha - 2\alpha\left[\frac{(a+b)}{2} - \alpha^2 + \beta^2\right].$$

Also (2.8) can be written as

$$(6.16)\quad g(grad\alpha, \xi) = -2\alpha\beta.$$

Differentiating (6.16) covariantly with respect to ξ we get

$$g(\nabla_\xi grad\alpha, \xi) + g(grad\alpha, \nabla_\xi \xi) = -2\beta(\xi\alpha) - 2\alpha(\xi\beta). \tag{6.17}$$

In view of (6.3) we can write the above relation as

$$g(\nabla_\xi grad\alpha, \xi) = -2\beta(\xi\alpha) + 2\alpha\left[\frac{(a+b)}{2} - \alpha^2 + \beta^2\right]. \tag{6.18}$$

From (6.15) and (6.18), we get $\Delta\alpha = 0$, where Δ is the Laplacian defined by $\Delta\alpha = \sum_{i=0}^{2} g(\nabla_{e_i} grad\alpha, e_i)$.

Since M is compact, we get α is constant.

Now let us consider the following two cases:

Case i): In this case we suppose that α is a non-zero constant. Then by (2.8), $\beta = 0$ everywhere on M.

Case ii): In this case let $\alpha = 0$. Then from (6.4) it follows

$$X\beta + \left[\frac{(a+b)}{2} + \beta^2\right]\eta(X) = 0,$$

that is,

$$g(grad\beta, X) + \left[\frac{(a+b)}{2} + \beta^2\right] g(X, \xi) = 0.$$

Therefore,

$$grad\beta + \left[\frac{(a+b)}{2} + \beta^2\right]\xi = 0. \tag{6.19}$$

Differentiating (6.19) covariantly with respect to X we have

$$\nabla_X grad\beta + (X\beta^2)\xi + \left[\frac{(a+b)}{2} + \beta^2\right]\nabla_X \xi = 0.$$

Using (2.6) we get from above

$$\nabla_X grad\beta + (X\beta^2)\xi + \left[\frac{(a+b)}{2} + \beta^2\right](-\alpha\phi X + \beta(X - \eta(X)\xi)) = 0.$$

Now taking inner product with X, we have

$$\begin{aligned} g(\nabla_X grad\beta, X) = & -g((X\beta^2)\xi, X) - \left[\frac{(a+b)}{2} + \beta^2\right](g(-\alpha\phi X, X) \\ & + \beta g(X - \eta(X)\xi, X)). \end{aligned} \tag{6.20}$$

Therefore putting $X = e_i$ and taking summation over i, $i = 0, 1, 2$, we get from above

$$\Delta\beta = -2\beta\left(\xi\beta + \frac{(a+b)}{2} + \beta^2\right). \tag{6.21}$$

For $\alpha = 0$, (6.3) yields $\xi\beta = -(\frac{(a+b)}{2} + \beta^2)$, which in view of (6.21) gives $\Delta\beta = 0$. Hence β=constant, M being compact. This leads to the following:

Theorem 6.1. *If a compact 3-dimensional trans-Sasakian manifold is an η-Einstein manifold with constant coefficients, then it is either α-Sasakian or β-Kenmotsu.*

7. Example of a 3-dimensional trans-Sasakian manifold

We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are standard co-ordinate of $\mathbb{R}^3$.

The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = z\frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$$

that is, the form of the metric becomes

$$g = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = 1,$$

$$\phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W)$$

for any $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to metric g. Then we have

$$\begin{aligned}
[e_1, e_3] &= e_1 e_3 - e_3 e_1 \\
&= z\frac{\partial}{\partial x}\left(z\frac{\partial}{\partial z}\right) - z\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial x}\right) \\
&= z^2\frac{\partial^2}{\partial x \partial z} - z^2\frac{\partial^2}{\partial z \partial x} - z\frac{\partial}{\partial x} \\
&= -e_1.
\end{aligned}$$

Similarly,

$$[e_1, e_2] = 0 \quad \text{and} \quad [e_2, e_3] = -e_2.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned}2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]),\end{aligned} \tag{7.1}$$

which known as Koszul's formula.

Using (7.1) we have

$$\begin{aligned}2g(\nabla_{e_1} e_3, e_1) &= -2g(e_1,e_1) \\ &= 2g(-e_1,e_1).\end{aligned} \tag{7.2}$$

Again by (7.1)

$$2g(\nabla_{e_1} e_3, e_2) = 0 = 2g(-e_1, e_2) \tag{7.3}$$

and

$$2g(\nabla_{e_1} e_3, e_3) = 0 = 2g(-e_1, e_3). \tag{7.4}$$

From (7.2), (7.3) and (7.4) we obtain

$$2g(\nabla_{e_1} e_3, X) = 2g(-e_1, X)$$

for all $X \in \chi(M)$.

Thus

$$\nabla_{e_1} e_3 = -e_1.$$

Therefore, (7.1) further yields

$$\begin{aligned}&\nabla_{e_1} e_3 = -e_1, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_1 = e_3, \\ &\nabla_{e_2} e_3 = -e_2, \quad \nabla_{e_2} e_2 = e_3, \quad \nabla_{e_2} e_1 = 0, \\ &\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0.\end{aligned} \tag{7.5}$$

(7.5) tells us that the manifold satisfies (2.6) for $\alpha = 0$ and $\beta = -1$ and $\xi = e_3$. Hence the manifold is a trans-Sasakian manifold of type $(0, -1)$.

It is known that

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{7.6}$$

With the help of the above results and using (7.6) it can be easily verified that

$$\begin{aligned}&R(e_1,e_2)e_3 = 0, \quad R(e_2,e_3)e_3 = -e_2, \quad R(e_1,e_3)e_3 = -e_1, \\ &R(e_1,e_2)e_2 = -e_1, \quad R(e_2,e_3)e_2 = e_3, \quad R(e_1,e_3)e_2 = 0, \\ &R(e_1,e_2)e_1 = e_2, \quad R(e_2,e_3)e_1 = 0, \quad R(e_1,e_3)e_1 = e_3.\end{aligned}$$

From the above expressions of the curvature tensor we obtain

$$\begin{aligned}S(e_1,e_1) &= g(R(e_1,e_2)e_2,e_1) + g(R(e_1,e_3)e_3,e_1) \\ &= -2.\end{aligned}$$

Similarly we have

$$S(e_2,e_2) = S(e_3,e_3) = -2.$$

Therefore,

$$r = S(e_1,e_1) + S(e_2,e_2) + S(e_3,e_3) = -6.$$

We note that here α, β and r are all constants. $\beta \neq 0$ implies that the manifold is a β-Kenmotsu manifold. From the expressions of the Ricci tensor it follows that the manifold is an Einstein manifold. Therefore Theorem 4.1 is verified.

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