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SEMI-RIEMANNIAN SUBMANIFOLDS OF A SEMI-RIEMANNIAN MANIFOLD WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. We study some properties of a semi-Riemannian submanifold of a semi-Riemannian manifold with a semi-symmetric non-metric connection. Then, we prove that the Ricci tensor of a semi-Riemannian submanifold of a semi-Riemannian space form admitting a semi-symmetric non-metric connection is symmetric but is not parallel. Last, we give the conditions under which a totally umbilical semi-Riemannian submanifold with a semi-symmetric non-metric connection is projectively flat.

1. Introduction

The notion of a semi-symmetric linear connection on a differentiable manifold was initiated by Friedmann and Schouten [5] in 1924. In 1992, Agashe and Chafle [1] defined a semi-symmetric non-metric connection on a Riemannian manifold and studied the Weyl projective curvature tensor with respect this connection. Moreover, in 1994 they considered in [2] a submanifold admitting a semi-symmetric non-metric connection and studied some of its properties when the ambient manifold is a space form admitting a semi-symmetric non-metric connection. In 1995, the properties of hypersurfaces of a Riemannian manifold with a semi-symmetric non-metric connection were studied by De and Kamilya [4]. In 2000, Sengupta, De and Binh [9] defined a semi-symmetric non-metric connection which generalized the notion of the semi-symmetric non-metric connection introduced by Agashe and Chafle. Later, they derived the curvature tensor and the Weyl projective curvature tensor with respect to the semi-symmetric non-metric connection. Prasad and Verma [8], in 2004, got the necessary and sufficient condition in order that the Weyl projective curvature tensor of a semi-symmetric non-metric connection is equal to the Weyl projective curvature of the Riemannian connection. Moreover, they showed that if the curvature tensor with respect to the semi-symmetric non-metric connection

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vanishes, then the Riemannian manifold is projectively flat. Yücesan and Yaşar [11] studied non-degenerate hypersurfaces of a semi-Riemannian manifold with a semi-symmetric non-metric connection and got the conditions under which a non-degenerate hypersurface with a semi-symmetric non-metric connection is projectively flat.

This paper is organized as follows: In Section 2, we consider a semi-Riemannian submanifold immersed in an ambient semi-Riemannian manifold. Then we determine the semi-symmetric non-metric connection, and give the equations of Gauss and Weingarten for a semi-Riemannian submanifold of a semi-Riemannian manifold with a semi-symmetric non-metric connection. Furthermore, we show that on a semi-Riemannian submanifold the connection induced from the semi-symmetric non-metric connection is also a semi-symmetric non-metric connection. In Section 3, by using the equations stated above, we derive Gauss curvature and Codazzi-Mainardi equations with respect to the semi-symmetric non-metric connection. In Section 4, we show that the Ricci tensor of a semi-Riemannian submanifold of a semi-Riemannian space form admitting a semi-symmetric non-metric connection is symmetric, but is not parallel. In the last section, we prove that a totally umbilical semi-Riemannian submanifold in a projectively flat semi-Riemannian manifold with a semi-symmetric non-metric connection is projectively flat.

2. Semi-symmetric non-metric connection

We suppose that M is an n-dimensional semi-Riemannian manifold of an $(n+p)$-dimensional semi-Riemannian manifold $\widetilde{M}$ with semi-Riemannian metric $\widetilde{g}$ of index $0 \leq \nu \leq n+p$. Let us denote by g the induced semi-Riemannian metric tensor on M from $\widetilde{g}$ on $\widetilde{M}$. As M has codimension p, we can locally choose p cross sections ξ_α, $1 \leq \alpha \leq p$, of the normal bundle $TM^\perp$ of M in $\widetilde{M}$ which are orthonormal at each point of M. The index of $\widetilde{g}$ restricted to $TM^\perp$ is called the co-index of M in $\widetilde{M}$ and $ind\widetilde{M} = \nu = indM + coindM$ (see [7]).

A linear connection $\widetilde{\bigtriangledown}$ on $\widetilde{M}$ is called a *semi-symmetric non-metric connection* if its torsion tensor $\widetilde{T}$ satisfies

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \widetilde{\pi}(\widetilde{Y})\widetilde{X} - \widetilde{\pi}(\widetilde{X})\widetilde{Y}$$

and

$$(\widetilde{\bigtriangledown}_{\widetilde{X}}\widetilde{g})(\widetilde{Y},\widetilde{Z}) = -\widetilde{\pi}(\widetilde{Y})\widetilde{g}(\widetilde{X},\widetilde{Z}) - \widetilde{\pi}(\widetilde{Z})\widetilde{g}(\widetilde{X},\widetilde{Y})$$

for $\widetilde{X}$, $\widetilde{Y} \in \chi(\widetilde{M})$, where $\widetilde{\pi}$ is a 1-form on $\widetilde{M}$ (see [1]).

We define a linear connection $\widetilde{\bigtriangledown}$ on $\widetilde{M}$ given by

$$\widetilde{\bigtriangledown}_{\widetilde{X}}\widetilde{Y} = \overset{\circ}{\widetilde{\bigtriangledown}}_{\widetilde{X}}\widetilde{Y} + \widetilde{\pi}(\widetilde{Y})\widetilde{X} \tag{2.1}$$

for $\widetilde{X}$, $\widetilde{Y} \in \chi(\widetilde{M})$, where $\overset{\circ}{\widetilde{\bigtriangledown}}$ denotes the Levi-Civita connection with respect to $\widetilde{g}$ and $\widetilde{\pi}$ is a 1-form associated to a vector field $\widetilde{Q}$ by $\widetilde{g}(\widetilde{Q},\widetilde{X}) = \widetilde{\pi}(\widetilde{X})$ for

$\widetilde{X} \in \chi(\widetilde{M})$. Then $\widetilde{\nabla}$ is a semi-symmetric non-metric connection on $\widetilde{M}$. On M we define a vector field Q and real valued functions μ_α, $1 \leq \alpha \leq p$, by decomposing $\widetilde{Q}$ into its unique tangential and normal components, thus

$$\widetilde{Q} = Q + \sum_{\alpha=1}^{p} \mu_\alpha \xi_\alpha. \tag{2.2}$$

If we denote by $\overset{\circ}{\nabla}$ the induced Levi-Civita connection on M from $\overset{\circ}{\widetilde{\nabla}}$ on $\widetilde{M}$, then we have the Gauss equation with respect to $\overset{\circ}{\nabla}$ given by

$$\overset{\circ}{\widetilde{\nabla}}_X Y = \overset{\circ}{\nabla}_X Y + \sum_{\alpha=1}^{p} \overset{\circ}{h}_\alpha(X,Y)\xi_\alpha \tag{2.3}$$

for X, $Y \in \chi(M)$, where $\overset{\circ}{h}_\alpha$, $1 \leq \alpha \leq p$, are the second fundamental forms on M [7]. Let ∇ on M be induced connection from the semi-symmetric non-metric connection $\widetilde{\nabla}$ on $\widetilde{M}$. Thus, the equation given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sum_{\alpha=1}^{p} h_\alpha(X,Y)\xi_\alpha, \tag{2.4}$$

will be called the *Gauss equation* with respect to $\widetilde{\nabla}$ for X, $Y \in \chi(M)$, where h_α, $1 \leq \alpha \leq p$, are tensors of type $(0, 2)$ on M.

Substituting (2.3) and (2.4) into (2.1), we see that

$$\nabla_X Y + \sum_{\alpha=1}^{p} h_\alpha(X,Y)\xi_\alpha = \overset{\circ}{\nabla}_X Y + \sum_{\alpha=1}^{p} \overset{\circ}{h}_\alpha(X,Y)\xi_\alpha + \pi(Y)X$$

from which we get

$$\nabla_X Y = \overset{\circ}{\nabla}_X Y + \pi(Y)X, \tag{2.5}$$

where

$$\pi(Y) = g(Y,Q),$$

and we obtain

$$h_\alpha = \overset{\circ}{h}_\alpha, \ \ 1 \leq \alpha \leq p, \tag{2.6}$$

for X, $Y \in \chi(M)$. By using (2.5), we deduce that

$$(\nabla_X g)(Y,Z) = -\pi(Y)g(X,Z) - \pi(Z)g(X,Y) \tag{2.7}$$

for X, Y, $Z \in \chi(M)$.

Also, from (2.5) the torsion tensor of the connection ∇, denoted by T, can be obtained as

$$T(X,Y) = \pi(Y)X - \pi(X)Y. \tag{2.8}$$

Then from (2.7) and (2.8) we have the following theorem:

Theorem 1. *The induced connection on a semi-Riemannian submanifold of a semi-Riemannian manifold with a semi-symmetric non-metric connection is also a semi-symmetric non-metric connection.*

The Weingarten equation with respect to $\overset{\circ}{\widetilde{\nabla}}$ is given by

$$\overset{\circ}{\widetilde{\nabla}}_X \xi_\alpha = -\overset{\circ}{A}_{\xi_\alpha}(X) + D_X\xi_\alpha, \; 1 \leq \alpha \leq p, \tag{2.9}$$

for $X \in \chi(M)$, where D is a metric connection on the normal bundle $TM^\perp$ with respect to the fibre metric induced from $\widetilde{g}$, and the $(1,1)$ tensor fields $\overset{\circ}{A}_{\xi_\alpha}$,$1 \leq \alpha \leq p$, on M such that

$$h_\alpha(X,Y) = \varepsilon_\alpha g(\overset{\circ}{A}_{\xi_\alpha}(X), Y) \tag{2.10}$$

are called the shape operators of $M \subset \widetilde{M}$ (see [7]).

By virtue of (2.1) and (2.2), we get

$$\widetilde{\nabla}_X \xi_\alpha = \overset{\circ}{\widetilde{\nabla}}_X \xi_\alpha + \varepsilon_\alpha \mu_\alpha X, \; 1 \leq \alpha \leq p.$$

From the above and (2.9) it follows that

$$\widetilde{\nabla}_X \xi_\alpha = -(\overset{\circ}{A}_{\xi_\alpha} - \varepsilon_\alpha \mu_\alpha I)(X) + D_X\xi_\alpha \;, 1 \leq \alpha \leq p, \tag{2.11}$$

where I is the identity tensor and

$$\varepsilon_\alpha = \begin{cases} -1, & \xi_\alpha \text{ is timelike,} \\ +1, & \xi_\alpha \text{ is spacelike.} \end{cases}$$

Let the shape operators A_{ξ_α}, $1 \leq \alpha \leq p$, of type $(1,1)$ on M be denoted by

$$A_{\xi_\alpha} = \overset{\circ}{A}_{\xi_\alpha} - \varepsilon_\alpha \mu_\alpha I \;, \; 1 \leq \alpha \leq p. \tag{2.12}$$

So, equation (2.11), called the *Weingarten equation* with respect to $\widetilde{\nabla}$, can be rewritten as

$$\widetilde{\nabla}_X \xi_\alpha = -A_{\xi_\alpha}(X) + D_X\xi_\alpha, \; 1 \leq \alpha \leq p, \tag{2.13}$$

for $X \in \chi(M)$.

By using (2.6), (2.10) and (2.12), we have

$$\varepsilon_\alpha h_\alpha(X,Y) = g(A_{\xi_\alpha}X, Y) + \varepsilon_\alpha \mu_\alpha g(X,Y). \tag{2.14}$$

Let $\xi = \sum_{\alpha=1}^{p} a_\alpha \xi_\alpha$, $\eta = \sum_{\alpha=1}^{p} b_\alpha \xi_\alpha$ be two normal vector fields on M. Then from (2.12), we see that

$$A_\xi A_\eta = \overset{\circ}{A}_\xi \overset{\circ}{A}_\eta - \varepsilon_\alpha a_\alpha \mu_\alpha \overset{\circ}{A}_\eta - \varepsilon_\alpha b_\alpha \mu_\alpha \overset{\circ}{A}_\xi + a_\alpha b_\alpha \mu_\alpha^2 I.$$

Thus,

$$[A_\xi, A_\eta] = [\overset{\circ}{A}_\xi, \overset{\circ}{A}_\eta],$$

and

$$g([\overset{\circ}{A}_\xi, \overset{\circ}{A}_\eta]X, Y) = g([A_\xi, A_\eta]X, Y)$$

for all X, $Y \in \chi(M)$. Hence we have:

Theorem 2. *Let M be a semi-Riemannian submanifold of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection. Then the second fundamental tensors with respect to the semi-symmetric non-metric connection are simultaneously diagonalizable if and only if second fundamental tensors with respect to the Levi-Civita connection are simultaneously diagonalizable.*

Let $E_1, \ldots, E_\nu, \ldots, E_n$ be the principal vector fields on M corresponding to the unit normal section $\xi = \sum_{\alpha=1}^{p} a_\alpha \xi_\alpha$ with respect to $\overset{\circ}{\widetilde{\nabla}}$. Then by using (2.12), we have

$$A_\xi(E_i) = (\overset{\circ}{k}_i - \varepsilon_\alpha a_\alpha \mu_\alpha)E_i,\ 1 \leq i \leq n, \tag{2.15}$$

where $\overset{\circ}{k}_i$, $1 \leq i \leq n$, are the principal curvatures corresponding to the unit normal section ξ with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$. Taking

$$k_i = \overset{\circ}{k}_i - \varepsilon_\alpha a_\alpha \mu_\alpha,\ 1 \leq i \leq n,\ 1 \leq \alpha \leq p. \tag{2.16}$$

So, by (2.16), equation (2.15) is reformed as

$$A_\xi(E_i) = k_i E_i,\ 1 \leq i \leq n, \tag{2.17}$$

where k_i, $1 \leq i \leq n$, are the principal curvatures of the unit normal section ξ with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$.

From (2.15), (2.16) and (2.17), we assert the following:

Theorem 3. *The principal directions of the unit normal direction ξ with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ and the semi-symmetric non-metric connection $\widetilde{\nabla}$ coincides and corresponding principal curvatures are equal if and only if ξ is orthogonal to $\widetilde{Q}$.*

The mean curvature vector field of M with respect to $\overset{\circ}{\nabla}$ is given by

$$\overset{\circ}{H} = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i \sum_{\alpha=1}^{p} \overset{\circ}{h}_\alpha(E_i, E_i)\xi_\alpha, \tag{2.18}$$

where

$$\varepsilon_i = \begin{cases} -1, & E_i \text{ is timelike,} \\ +1, & E_i \text{ is spacelike} \end{cases}$$

(see [7]). We define similarly the *mean curvature vector field* of M with respect to ∇ by

$$H = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i \sum_{\alpha=1}^{p} h_\alpha(E_i, E_i)\xi_\alpha. \tag{2.19}$$

From (2.6), (2.18) and (2.19), $H = \overset{\circ}{H}$. Hence we have:

Lemma 4. *A semi-Riemannian submanifold M of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection is totally geodesic with respect to the semi-symmetric non-metric connection if and only if it is totally geodesic with respect to the Levi-Civita connection.*

Lemma 5. *A semi-Riemannian submanifold M of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection is totally umbilical with respect to the semi-symmetric non-metric connection if and only if it is totally umbilical with respect to the Levi-Civita connection.*

3. The Gauss curvature and Codazzi-Mainardi equations

We denote by

$$\overset{\circ}{\widetilde{R}}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\overset{\circ}{\widetilde{\nabla}}_{\widetilde{Y}}\widetilde{Z} - \overset{\circ}{\widetilde{\nabla}}_{\widetilde{Y}}\overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Z} - \overset{\circ}{\widetilde{\nabla}}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}$$

and

$$\overset{\circ}{R}(X,Y)Z = \overset{\circ}{\nabla}_X \overset{\circ}{\nabla}_Y Z - \overset{\circ}{\nabla}_Y \overset{\circ}{\nabla}_X Z - \overset{\circ}{\nabla}_{[X,Y]}Z,$$

the curvature tensors of $\widetilde{M}$ and M with respect to $\overset{\circ}{\widetilde{\nabla}}$ and $\overset{\circ}{\nabla}$, respectively, where $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z} \in \chi(\widetilde{M})$ and X, Y, $Z \in \chi(M)$. Then the Gauss curvature and Codazzi-Mainardi equations with respect to $\overset{\circ}{\widetilde{\nabla}}$ and $\overset{\circ}{\nabla}$, respectively, are given by

$$\begin{aligned}\overset{\circ}{\widetilde{R}}(X,Y,Z,W) &= \overset{\circ}{R}(X,Y,Z,W) \\ &+ \sum_{\alpha=1}^{p} \varepsilon_\alpha \{\overset{\circ}{h}_\alpha(X,Z)\overset{\circ}{h}_\alpha(Y,W) - \overset{\circ}{h}_\alpha(Y,Z)\overset{\circ}{h}_\alpha(X,W)\},\end{aligned}$$

and

$$\begin{aligned}\overset{\circ}{\widetilde{R}}(X,Y,Z,\xi_\alpha) &= \varepsilon_\alpha\{(\overset{\circ}{\nabla}_X \overset{\circ}{h}_\alpha)(Y,Z) - (\overset{\circ}{\nabla}_Y \overset{\circ}{h}_\alpha)(X,Z)\} \\ &+ \sum_{\beta=1}^{p} \widetilde{g}(\overset{\circ}{h}_\beta(Y,Z)D_X\xi_\beta - \overset{\circ}{h}_\beta(X,Z)D_Y\xi_\beta, \xi_\alpha)\end{aligned}$$

for X, Y, $Z \in \chi(M)$ (see [7]), where

$$\overset{\circ}{\widetilde{R}}(X,Y,Z,W) = \widetilde{g}(\overset{\circ}{\widetilde{R}}(X,Y)Z,W),\ \overset{\circ}{R}(X,Y,Z,W) = g(\overset{\circ}{R}(X,Y)Z,W).$$

Now we shall find the Gauss curvature and the Codazzi-Mainardi equations with respect to the semi-symmetric non-metric connections $\widetilde{\nabla}$ and ∇. The curvature tensors of $\widetilde{M}$ and M with respect to $\widetilde{\nabla}$ and ∇, respectively, are defined by

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z$$

and

$$R(X,Y)Z = \bigtriangledown_X \bigtriangledown_Y Z - \bigtriangledown_Y \bigtriangledown_X Z - \bigtriangledown_{[X,Y]} Z$$

for X, Y, $Z \in \chi(M)$.

By using (2.4) and (2.13), we have the curvature tensor of the semi-symmetric non-metric connection $\widetilde{\nabla}$ given by

(3.1)

$$\begin{aligned}\widetilde{R}(X,Y)Z = R(X,Y)Z + \sum_{\alpha=1}^{p} \{ & h_\alpha(X,Z)A_{\xi_\alpha}Y - h_\alpha(Y,Z)A_{\xi_\alpha}X \\ & +(\nabla_X h_\alpha)(Y,Z)\xi_\alpha - (\nabla_Y h_\alpha)(X,Z)\xi_\alpha + h_\alpha(\pi(Y)X - \pi(X)Y, Z)\xi_\alpha \\ & + h_\alpha(Y,Z)D_X\xi_\alpha - h_\alpha(X,Z)D_Y\xi_\alpha \}.\end{aligned}$$

From (3.1), the *Gauss curvature equation and the Codazzi-Mainardi equation* with respect to $\widetilde{\bigtriangledown}$ and $\bigtriangledown$, respectively, are obtained as:

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + \sum_{\alpha=1}^{p} \varepsilon_\alpha \{ & h_\alpha(X,Z)h_\alpha(Y,W) \\ & - h_\alpha(Y,Z)h_\alpha(X,W) + \mu_\alpha h_\alpha(Y,Z)g(X,W) \\ & - \mu_\alpha h_\alpha(X,Z)g(Y,W)\},\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\widetilde{R}(X,Y,Z,\xi_\alpha) = \varepsilon_\alpha \{ & (\bigtriangledown_X h_\alpha)(Y,Z) - (\bigtriangledown_Y h_\alpha)(X,Z) \\ & + h_\alpha(\pi(Y)X - \pi(X)Y, Z)\} + \sum_{\beta=1}^{p} \widetilde{g}(h_\beta(Y,Z)D_X\xi_\beta \\ & - h_\beta(X,Z)D_Y\xi_\beta, \xi_\alpha)\end{aligned}$$

for X, Y, $Z \in \chi(M)$.

From (2.14) and (3.2), we get

$$\begin{aligned}\widetilde{R}(X,Y,X,Y) = R(X,Y,X,Y) + \sum_{\alpha=1}^{p} \{ & \varepsilon_\alpha g(A_{\xi_\alpha}(X),X)g(A_{\xi_\alpha}(Y),Y) \\ & - g(A_{\xi_\alpha}(X),Y)^2\} + \sum_{\alpha=1}^{p}\{\mu_\alpha g(A_{\xi_\alpha}(Y),Y)g(X,X) \\ & - \mu_\alpha g(A_{\xi_\alpha}(X),Y)g(X,Y)\}\end{aligned}$$

for X, $Y \in \chi(M)$. Therefore we have the following theorem:

Theorem 6. *Let $\mathcal{P}$ be a 2-dimensional non-degenerate subspace of T_xM, and let $\widetilde{K}(\mathcal{P})$ and $K(\mathcal{P})$ be the sectional curvatures of $\mathcal{P}$ in $\widetilde{M}$ and M with respect to the semi-symmetric non-metric connections $\widetilde{\bigtriangledown}$ and $\bigtriangledown$, respectively. If X*

and Y form an orthonormal base of $\mathcal{P}$, then

$$\widetilde{K}(\mathcal{P}) = K(\mathcal{P}) + \frac{1}{g(X,X)g(Y,Y)} \sum_{\alpha=1}^{p} \{g(A_{\xi_\alpha}(X),X)g(A_{\xi_\alpha}(Y),Y) - g(A_{\xi_\alpha}(X),Y)^2 + \mu_\alpha g(A_{\xi_\alpha}(Y),Y)g(X,X)\}.$$

As an immediate consequences of Theorem 6 we obtain:

Corollary 7. *If $\widetilde{M}$ is a 3-dimensional flat Lorentz manifold and M is a spacelike or timelike surface in $\widetilde{M}$, then there exists a semi-symmetric non-metric connection ∇ on M for which $\det A_\xi$ is an intrinsic invariant of M, and when $\widetilde{Q}$ is tangent to M, $\det A_\xi (= K(\mathcal{P}))$ is equal to $\det \overset{\circ}{A}_\xi$ which is the Gauss curvature of M.*

4. The equation of Ricci with respect to a semi-symmetric non-metric connection

Let ξ be a normal vector field on M. We get

$$\widetilde{R}(X,Y)\xi = \widetilde{\nabla}_X \widetilde{\nabla}_Y \xi - \widetilde{\nabla}_Y \widetilde{\nabla}_X \xi - \widetilde{\nabla}_{[X,Y]}\xi, \tag{4.1}$$

where X, $Y \in \chi(M)$. Using (2.12) and (4.1), we have

$$\begin{aligned} \widetilde{R}(X,Y)\xi = R^N(X,Y)\xi + \sum_{\alpha=1}^{p} \{h_\alpha(A_\xi X,Y) - h_\alpha(A_\xi Y,X)\}\xi_\alpha \\ + A_{D_X \xi}Y - A_{D_Y \xi}X - Tor_{A_\xi}(X,Y), \end{aligned} \tag{4.2}$$

where R^N is the curvature tensor of the normal connection. Using (2.13) and (4.2), we obtain

$$\widetilde{R}(X,Y,\xi,\eta) = R^N(X,Y,\xi,\eta) - g([A_\xi, A_\eta]X,Y), \tag{4.3}$$

where η is a normal vector field on M. Equation (4.3) is called the *equation of Ricci* with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$.

A relation between the curvature tensor of the semi-symmetric non-metric connection $\widetilde{\nabla}$ and the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ is given by

$$\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \overset{\circ}{\widetilde{R}}(\widetilde{X},\widetilde{Y})\widetilde{Z} + \widetilde{\alpha}(\widetilde{X},\widetilde{Z})\widetilde{Y} - \widetilde{\alpha}(\widetilde{Y},\widetilde{Z})\widetilde{X}, \tag{4.4}$$

where $\widetilde{\alpha}$ is a tensor of type $(0,2)$ defined by

$$\begin{aligned} \widetilde{\alpha}(\widetilde{X},\widetilde{Y}) &= (\overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{\pi})\widetilde{Y} - \widetilde{\pi}(\widetilde{X})\widetilde{\pi}(\widetilde{Y}) \\ &= (\widetilde{\nabla}_{\widetilde{X}}\widetilde{\pi})\widetilde{Y} \end{aligned} \tag{4.5}$$

for any vector fields $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$ on $\widetilde{M}$.

Presently, we consider the semi-Riemannian manifold $\widetilde{M}$ with constant curvature k. Then we have (see [7])

$$\overset{\circ}{\widetilde{R}}(\widetilde{X},\widetilde{Y})\widetilde{Z} = k\{\widetilde{g}(\widetilde{Y},\widetilde{Z})\widetilde{X} - \widetilde{g}(\widetilde{X},\widetilde{Z})\widetilde{Y}\}. \tag{4.6}$$

From (4.4), (4.5) and (4.6), we have

$$\widetilde{R}(X,Y)\xi = ((\widetilde{\nabla}_X\widetilde{\pi})\xi)Y - ((\widetilde{\nabla}_Y\widetilde{\pi})\xi)X$$

for any vector fields X, Y and a normal vector field ξ on M. Thus, $\widetilde{R}(X,Y)\xi$ is tangent to M and hence equation (4.3) reduces to

$$R^N(X,Y,\xi,\eta) = g([A_\xi, A_\eta]X,Y).$$

The normal connection D in the normal bundle $TM^\perp$ is said to be flat if

$$R^N(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}$$

vanishes identically on M.

Hence we have:

Corollary 8. *Let M be a semi-Riemannian submanifold of a semi-Riemannian manifold $\widetilde{M}$ with constant curvature admitting a semi-symmetric non-metric connection. Then the normal connection D in the normal bundle $TM^\perp$ is flat if and only if all the second fundamental tensors with respect to the semi-symmetric non-metric connection are simultaneously diagonalizable.*

Theorem 9. *The Ricci tensor of a semi-Riemannian submanifold M with respect to the semi-symmetric non-metric connection is symmetric if and only if π is closed.*

Proof. The Ricci tensor of a semi-Riemannian submanifold M with respect to the semi-symmetric non-metric connection is given by

$$Ric(X,Y) = \sum_{i=1}^{n} \varepsilon_i g(R(E_i,X)Y,E_i) \tag{4.7}$$

for $\forall X,Y \in \chi(M)$. Then using (4.4) in (4.7), we obtain

$$Ric(X,Y) = \overset{\circ}{Ric}(X,Y) - (n-1)\alpha(X,Y),$$

where $\overset{\circ}{Ric}$ denotes the Ricci tensor of M with respect to the Levi-Civita connection and

$$\alpha(X,Y) = (\nabla_X\pi)Y.$$

From above it follows that

$$\begin{aligned} Ric(X,Y) - Ric(Y,X) &= (n-1)(\alpha(Y,X) - \alpha(X,Y)) \\ &= 2(n-1)d\pi(Y,X) \end{aligned}$$

which completes the proof. □

Theorem 10. *Let M be a semi-Riemannian submanifold of a semi-Riemannian manifold $\widetilde{M}$. If $\widetilde{Ric}$ and Ric are the Ricci tensor of $\widetilde{M}$ and M with respect to the semi-symmetric non-metric connection, respectively, then for $\forall X, Y \in \chi(M)$*

$$\begin{aligned}\widetilde{Ric}(X,Y) = Ric(X,Y) - \sum_{\alpha=1}^{p} \varepsilon_\alpha f_\alpha h_\alpha(X,Y) + h_\alpha(A_{\xi_\alpha}X, Y) \\ + n\varepsilon_\alpha \mu_\alpha h_\alpha(X,Y) + \varepsilon_\alpha \widetilde{g}(\widetilde{R}(\xi_\alpha, X)Y, \xi_\alpha),\end{aligned} \tag{4.8}$$

where if ξ_α is spacelike, $\varepsilon = +1$ or if ξ_α is timelike, $\varepsilon = -1$ and $f_\alpha = \sum\limits_{i=1}^{n} \varepsilon_i h_\alpha(E_i, E_i)$.

Proof. Let $\{E_1, \ldots, E_\nu, E_{\nu+1}, \ldots, E_n, \xi_1, \ldots, \xi_p\}$ be an orthonormal basis of $\chi(\widetilde{M})$. Then the Ricci curvature of $\widetilde{M}$ with respect to the semi-symmetric non-metric connection is given by

$$\widetilde{Ric}(X,Y) = \sum_{i=1}^{n} \varepsilon_i \widetilde{g}(\widetilde{R}(E_i, X)Y, E_i) + \sum_{\alpha=1}^{p} \varepsilon_\alpha \widetilde{g}(\widetilde{R}(\xi_\alpha, X)Y, \xi_\alpha) \tag{4.9}$$

for $\forall X, Y \in \chi(M)$. By taking account of (4.9), (3.2), (2.14) and considering the symmetry of shape operators we get (4.8). □

Theorem 11. *Let M be a semi-Riemannian submanifold of a semi-Riemannian manifold $\widetilde{M}$. If $\widetilde{\rho}$ and ρ are the scalar curvatures of $\widetilde{M}$ and M with respect to the semi-symmetric non-metric connection, respectively, then*

$$\widetilde{\rho} = \rho - \sum_{\alpha=1}^{p} \varepsilon_\alpha f_\alpha^2 + n\varepsilon_\alpha \mu_\alpha f_\alpha + f_\alpha^* + 2\varepsilon_\alpha \widetilde{Ric}(\xi_\alpha, \xi_\alpha), \tag{4.10}$$

where $f_\alpha^* = \sum\limits_{i=1}^{n} \varepsilon_i h_\alpha(A_{\xi_\alpha} E_i, E_i)$.

Proof. Assume that $\{E_1, \ldots, E_\nu, E_{\nu+1}, \ldots, E_n, \xi_1, \ldots, \xi_p\}$ is an orthonormal basis of $\chi(\widetilde{M})$, then the scalar curvature of $\widetilde{M}$ with respect to the semi-symmetric non-metric connection is

$$\widetilde{\rho} = \sum_{i=1}^{n} \varepsilon_i \widetilde{Ric}(E_i, E_i) + \sum_{\alpha=1}^{p} \varepsilon_\alpha \widetilde{Ric}(\xi_\alpha, \xi_\alpha). \tag{4.11}$$

By virtue of (4.8), (4.11), we obtain (4.10). □

We now assume that the 1-form π is closed. Then we can define the sectional curvature for a section with respect to the semi-symmetric non-metric connection (see [1]).

Suppose that the semi-symmetric non-metric connection $\widetilde{\nabla}$ is of constant sectional curvature, then $\widetilde{R}(X,Y)Z$ should be of the form

$$\widetilde{R}(X,Y)Z = c\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\} \tag{4.12}$$

c being a certain scalar. Thus $\widetilde{M}$ is a semi-Riemannian manifold of constant curvature c with respect to semi-symmetric non-metric connection and denote it by $\widetilde{M}(c)$.

Theorem 12. *Let M be a semi-Riemannian submanifold of a semi-Riemannian space form $\widetilde{M}(c)$ with a semi-symmetric non-metric connection. Then we have*

$$\begin{aligned} Ric(X,Y) = c(n-1)g(X,Y) + \sum_{\alpha=1}^{p} \{ &\varepsilon_\alpha f_\alpha h_\alpha(X,Y) \\ &- h_\alpha(A_{\xi_\alpha}X,Y) - \varepsilon_\alpha n\mu_\alpha h_\alpha(X,Y)\} \end{aligned} \tag{4.13}$$

for $\forall X,Y \in \chi(M)$, *where* $\varepsilon_i = g(E_i,E_i)$, $\varepsilon_i = 1$, *if* E_i *is spacelike or* $\varepsilon_i = -1$, *if* E_i *is timelike, and* $f_\alpha = \sum\limits_{i=1}^{n} \varepsilon_i h_\alpha(E_i,E_i)$.

Proof. Taking into account of (4.8) and (4.12), we have (4.13). □

From (4.13), the following corollary can be stated as:

Corollary 13. *Let M be a semi-Riemannian submanifold of a semi-Riemannian space form $\widetilde{M}(c)$ with a semi-symmetric non-metric connection. Then the Ricci tensor of M is symmetric.*

Corollary 14. *Let M be a semi-Riemannian submanifold of a semi-Riemannian space form $\widetilde{M}(c)$ with a semi-symmetric non-metric connection. Then the Ricci tensor of M is not parallel.*

5. Projective curvature tensor of a semi-Riemannian submanifold with a semi-symmetric non-metric connection

We denote by

$$\overset{\circ}{\widetilde{P}}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \overset{\circ}{\widetilde{R}}(\widetilde{X},\widetilde{Y})\widetilde{Z} - \frac{1}{n+p-1}\{\overset{\circ}{\widetilde{Ric}}(\widetilde{Y},\widetilde{Z})\widetilde{X} - \overset{\circ}{\widetilde{Ric}}(\widetilde{X},\widetilde{Z})\widetilde{Y}\},$$

the Weyl projective curvature tensor of an $(n+p)$-dimensional semi-Riemannian manifold $\widetilde{M}$ with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ for $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z} \in \chi(\widetilde{M})$, where $\overset{\circ}{\widetilde{Ric}}$ is Ricci tensor of $\widetilde{M}$ with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ (see [3] and [10]).

Analogous to this definition, the Weyl projective curvature tensor of $\widetilde{M}$ with respect to the semi-symmetric non-metric connection can be defined as

$$\widetilde{P}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z} - \frac{1}{n+p-1}\{\widetilde{Ric}(\widetilde{Y},\widetilde{Z})\widetilde{X} - \widetilde{Ric}(\widetilde{X},\widetilde{Z})\widetilde{Y}\} \tag{5.1}$$

for any $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z} \in \chi(\widetilde{M})$, where $\widetilde{Ric}$ is the Ricci tensor $\widetilde{M}$ with respect to the connection $\widetilde{\nabla}$. Thus, from (5.1), the Weyl projective curvature tensors with respect to the semi-symmetric non-metric connection $\widetilde{\widetilde{\nabla}}$ and induced connection ∇, respectively, are given by

$$\begin{aligned}\widetilde{P}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) = \widetilde{R}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) - \frac{1}{n+p-1}\{&\widetilde{Ric}(\widetilde{Y},\widetilde{Z})\widetilde{g}(\widetilde{X},\widetilde{U}) \\ &- \widetilde{Ric}(\widetilde{X},\widetilde{Z})\widetilde{g}(\widetilde{Y},\widetilde{U})\}\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}P(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{n-1}\{&Ric(Y,Z)g(X,U) \\ &- Ric(X,Z)g(Y,U)\}\end{aligned} \tag{5.3}$$

for $\forall X,\ Y,\ Z \in \chi(M)$, where

$$\widetilde{P}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) = \widetilde{g}(\widetilde{P}(\widetilde{X},\widetilde{Y})\widetilde{Z},\widetilde{U}),\ P(X,Y,Z,U) = g(P(X,Y)Z,U)$$

and Ric is the Ricci tensor of M with respect to induced connection ∇.

From (5.2), we obtain

$$\widetilde{P}(\xi_\alpha,Y,Z,\xi_\alpha) = \widetilde{R}(\xi_\alpha,Y,Z,\xi_\alpha) - \frac{\varepsilon_\alpha}{n+p-1}\widetilde{Ric}(Y,Z). \tag{5.4}$$

Applying (4.8) to (5.4), we have

$$\begin{aligned}Ric(Y,Z) = \frac{n+p-2}{n+p-1}\widetilde{Ric}(Y,Z) - \sum_{\alpha=1}^{p}\{&\varepsilon_\alpha\widetilde{P}(\xi_\alpha,Y,Z,\xi_\alpha) \\ &+ f_\alpha\varepsilon_\alpha h_\alpha(Y,Z) - n\mu_\alpha\varepsilon_\alpha h_\alpha(Y,Z) - h_\alpha(A_{\xi_\alpha}Y,Z)\}.\end{aligned} \tag{5.5}$$

Then, using (5.2), (5.5) and (3.2) into (5.3) we obtain

$$\begin{aligned}&P(X,Y,Z,U) \\ &= \widetilde{P}(X,Y,Z,U) - \sum_{\alpha=1}^{p}\varepsilon_\alpha\{h_\alpha(X,Z)h_\alpha(Y,U) - h_\alpha(Y,Z)h_\alpha(X,U) \\ &\quad + \mu_\alpha h_\alpha(Y,Z)g(X,U) - \mu_\alpha h_\alpha(X,Z)g(Y,U)\} \\ &\quad + \frac{1}{n-1}\sum_{\alpha=1}^{p}\varepsilon_\alpha\{\widetilde{P}(\xi_\alpha,Y,Z,\xi_\alpha)g(X,U) - \widetilde{P}(\xi_\alpha,X,Z,\xi_\alpha)g(Y,U)\} \\ &\quad + \frac{p-1}{(n-1)(n+p-1)}(\widetilde{Ric}(X,Z) - \widetilde{Ric}(Y,Z)) \\ &\quad + \frac{1}{n-1}g(Y,U)\left\{\sum_{\alpha=1}^{p}\varepsilon_\alpha f_\alpha h_\alpha(X,Z) - n\varepsilon_\alpha\mu_\alpha h_\alpha(X,Z) - h_\alpha(A_{\xi_\alpha}X,Z)\right\} \\ &\quad - \frac{1}{n-1}g(X,U)\left\{\sum_{\alpha=1}^{p}\varepsilon_\alpha f_\alpha h_\alpha(Y,Z) - n\varepsilon_\alpha\mu_\alpha h_\alpha(Y,Z) - h_\alpha(A_{\xi_\alpha}Y,Z)\right\}.\end{aligned} \tag{5.6}$$

From (5.6), we have the following theorem:

Theorem 15. *A totally umbilical semi-Riemannian submanifold in a projectively flat semi-Riemannian manifold with a semi-symmetric non-metric connection is projectively flat.*

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