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LIGHTLIKE HYPERSURFACES OF A SEMI-RIEMANNIAN MANIFOLD OF QUASI-CONSTANT CURVATURE

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Abstract. In this paper, we study the geometry lightlike hypersurfaces $(M, g, S(TM))$ of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of quasi-constant curvature subject to the conditions: (1) The curvature vector field of $\widetilde{M}$ is tangent to M, and (2) the screen distribution $S(TM)$ is either totally geodesic in M or totally umbilical in $\widetilde{M}$.

1. Introduction

B. Y. Chen and K. Yano [2] introduced the notion of a Riemannian manifold of quasi-constant curvature as a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ with the curvature tensor $\widetilde{R}$ satisfying the condition

$$\begin{aligned} \widetilde{g}(\widetilde{R}(X, Y)Z, W) = \alpha\{&\widetilde{g}(Y, Z)g(X, W) - \widetilde{g}(X, Z)g(Y, W)\} \\ + \beta\{&\widetilde{g}(X, W)\theta(Y)\theta(Z) - \widetilde{g}(X, Z)\theta(Y)\theta(W) \\ +\ &\widetilde{g}(Y, Z)\theta(X)\theta(W) - \widetilde{g}(Y, W)\theta(X)\theta(Z)\}, \end{aligned} \tag{1.1}$$

where α, β are scalar functions and θ is a 1-form defined by

$$\theta(X) = \widetilde{g}(X, \zeta), \tag{1.2}$$

and ζ is a unit vector field on $\widetilde{M}$ which called the *curvature vector field* of $\widetilde{M}$. It is well known that if the curvature tensor $\widetilde{R}$ is of the form (1.1), then $\widetilde{M}$ is conformally flat. If $\beta = 0$, then $\widetilde{M}$ is a space of constant curvature.

A non-flat Riemannian manifold $\widetilde{M}$ of dimension $n(> 2)$ is called a quasi-Einstein manifold [1] if its Ricci tensor $\widetilde{Ric}$ satisfies the condition

$$\widetilde{Ric}(X, Y) = a\, \widetilde{g}(X, Y) + b\, \phi(X)\phi(Y),$$

where a, b are scalar functions such that $b \neq 0$ and ϕ is a non-vanishing 1-form such that $\widetilde{g}(X, U) = \phi(X)$ for any vector field X, where U is a unit vector

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field. If $b = 0$, then $\widetilde{M}$ is an Einstein manifold. It can be easily seen that every Riemannian manifold of quasi-constant curvature is a quasi-Einstein manifold.

The purpose of this paper is to study lightlike hypersurfaces of a semi-Riemannian manifold of quasi-constant curvature. We prove two characterization theorems for lightlike hypersurfaces $(M, g, S(TM))$ of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of quasi-constant curvature:

• If $S(TM)$ is totally geodesic in M and the curvature vector field ζ of $\widetilde{M}$ is tangent to M, then $\widetilde{M}$ and M are flat manifolds (Theorem 3.3).

• If $S(TM)$ is totally umbilical in $\widetilde{M}$ and ζ is tangent to M, then $\widetilde{M}$ is a space of non-zero constant curvature α and M is an Einstein manifold (Theorem 3.4).

2. Lightlike hypersurface

It is well known that the normal bundle $TM^{\perp}$ of the lightlike hypersurfaces M of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ is a subbundle of TM of rank 1. A complementary vector bundle $S(TM)$ of $TM^{\perp}$ in TM is non-degenerate distribution on M, called a *screen distribution* on M, and

$$TM = TM^{\perp} \oplus_{orth} S(TM), \tag{2.1}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. It is well-known [4] that, for any null section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ in $S(TM)^{\perp}$ satisfying

$$\widetilde{g}(\xi, N) = 1, \quad \widetilde{g}(N, N) = \widetilde{g}(N, X) = 0, \ \forall X \in \Gamma\left(S(TM)|_{\mathcal{U}}\right). \tag{2.2}$$

Then the tangent bundle $T\widetilde{M}$ of $\widetilde{M}$ is decomposed as follows;

$$T\widetilde{M} = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus_{orth} S(TM). \tag{2.3}$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to $S(TM)$ respectively.

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{M}$ and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2.1). For any vector fields $X, Y \in \Gamma(TM)$, the local Gauss and Weingartan formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.4}$$

$$\widetilde{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.5}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.6}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.7}$$

where ∇ and ∇^* are the linear connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively and τ is a 1-form on TM. Since $\widetilde{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric. From the fact that $B(X, Y) = g(\widetilde{\nabla}_X Y, \xi)$, we know that B is independent of the choice of a screen distribution and satisfies

$$B(X, \xi) = 0, \quad \forall\, X \in \Gamma(TM). \tag{2.8}$$

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\,\eta(Z) + B(X, Z)\,\eta(Y), \tag{2.9}$$

for any $X, Y, Z \in \Gamma(TM)$, where η is a 1-form such that

$$\eta(X) = \widetilde{g}(X, N), \quad \forall\, X \in \Gamma(TM). \tag{2.10}$$

But the connection ∇^* on $S(TM)$ is metric. The above two local second fundamental forms of M and on $S(TM)$ are related to their shape operators by

$$B(X, Y) = g(A^*_\xi X, Y), \qquad \widetilde{g}(A^*_\xi X, N) = 0, \tag{2.11}$$

$$C(X, PY) = g(A_N X, PY), \quad \widetilde{g}(A_N X, N) = 0. \tag{2.12}$$

From (2.11), A^*_ξ is $S(TM)$-valued and self-adjoint on TM such that

$$A^*_\xi \xi = 0. \tag{2.13}$$

We denote by $\widetilde{R}$, R and R^* the curvature tensors of the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{M}$, the induced connection ∇ of M and the connection ∇^* on $S(TM)$, respectively. Using the Gauss-Weingarten equations for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$ such that

$$\widetilde{g}(\widetilde{R}(X,Y)Z, PW) = g(R(X,Y)Z, PW) + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW), \tag{2.14}$$

$$\widetilde{g}(\widetilde{R}(X,Y)Z, \xi) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(Y,Z)\tau(X) - B(X,Z)\tau(Y), \tag{2.15}$$

$$\widetilde{g}(\widetilde{R}(X,Y)Z, N) = \widetilde{g}(R(X,Y)Z, N), \tag{2.16}$$

$$\widetilde{g}(\widetilde{R}(X,Y)\xi, N) = g(A^*_\xi X, A_N Y) - g(A^*_\xi Y, A_N X) - 2d\tau(X,Y), \tag{2.17}$$

$$g(R(X,Y)PZ, PW) = g(R^*(X,Y)PZ, PW) + C(X,PZ)B(Y,PW) - C(Y,PZ)B(X,PW), \tag{2.18}$$

$$\widetilde{g}(R(X,Y)PZ, N) = (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X) \tag{2.19}$$

for any $X, Y, Z, W \in \Gamma(TM)$.

The *Ricci tensor*, denoted by $\widetilde{Ric}$, of $\widetilde{M}$ is defined by

$$\widetilde{Ric}(X,Y) = trace\{Z \to \widetilde{R}(Z,X)Y\}$$

for any $X,\, Y \in \Gamma(T\widetilde{M})$. Let $\dim \widetilde{M} = m+2$. Locally, $\widetilde{Ric}$ is given by

$$\widetilde{Ric}(X,Y) = \sum_{i=1}^{m+2} \epsilon_i\, \widetilde{g}(\widetilde{R}(E_i, X)Y,\ E_i), \tag{2.20}$$

where $\{E_1,\, \ldots,\, E_{m+2}\}$ is an orthonormal frame field of $T\widetilde{M}$. If

$$\widetilde{Ric} = \widetilde{\kappa}\widetilde{g}, \quad \widetilde{\kappa} \text{ is a constant}, \tag{2.21}$$

then $\widetilde{M}$ is an *Einstein manifold.* The *scalar curvature* $\widetilde{r}$ is defined by

$$\widetilde{r} = \sum_{i=1}^{m+2} \epsilon_i\, \widetilde{Ric}(E_i,\, E_i). \tag{2.22}$$

Putting (2.21) in (2.22) implies that $\widetilde{M}$ is Einstein if and only if

$$\widetilde{Ric} = \frac{\widetilde{r}}{m+2}\, \widetilde{g}.$$

3. Tangential curvature vector field

Let M be a lightlike hypersurface of a semi-Riemannian manifold $\widetilde{M}$ of quasi-constant curvature. We may assume that the curvature vector field ζ of $\widetilde{M}$ is a unit spacelike vector field and $\dim \widetilde{M} > 3$. Consider an induced quasi-orthonormal frame field $\{\xi; W_a\}$ on M, where $TM^{\perp} = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$ and let $E = \{\xi, N, W_a\}$ be the corresponding frame field on $\widetilde{M}$. By using (2.20), we get

$$\begin{aligned} \widetilde{Ric}(X,Y) = &\sum_{a=1}^{m} \epsilon_a\, \widetilde{g}(\widetilde{R}(W_a, X)Y,\, W_a) \\ &+ \widetilde{g}(\widetilde{R}(\xi, X)Y,\, N) \;+\; \widetilde{g}(\widetilde{R}(N, X)Y,\, \xi), \end{aligned} \tag{3.1}$$

where ϵ_a denotes the causal character (± 1) of respective vector field W_a. Let $R^{(0,\,2)}$ denote the induced Ricci type tensor of type (0, 2) on M given by

$$R^{(0,\,2)}(X,Y) = trace\{Z \to R(Z,X)Y\}\,, \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.2}$$

Using the induced quasi-orthonormal frame field $\{\xi; W_a\}$ on M, we obtain

$$R^{(0,\,2)}(X,Y) = \sum_{a=1}^{m} \epsilon_a\, g(R(W_a,X)Y,\, W_a) + \widetilde{g}(R(\xi,X)Y,\, N). \tag{3.3}$$

Substituting (2.15) and (2.18) in (3.1) an using (2.12) and (2.13), we obtain

$$\begin{aligned} R^{(0,\,2)}(X,Y) = &\widetilde{Ric}(X,Y) + B(X,Y)trA_N - g(A_N X, A^*_\xi Y) \\ &- \widetilde{g}(R(\xi,Y)X,\, N), \quad \forall\, X,\, Y \in \Gamma(TM). \end{aligned} \tag{3.4}$$

This shows that $R^{(0,2)}$ is not symmetric. The tensor field $R^{(0,2)}$, defined by (3.2), is called its *induced Ricci tensor* [5], denoted by *Ric*, if it is symmetric. Using (2.17), (3.4) and the first Bianchi's identity, we obtain

$$R^{(0,2)}(X, Y) - R^{(0,2)}(Y, X) = 2d\tau(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

Theorem 3.1 ([4, 5])**.** *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\widetilde{M}$. Then the Ricci type tensor $R^{(0,2)}$ is symmetric if and only if the 1-form τ is closed, i.e., $d\tau = 0$, on any coordinate neighborhood $\mathcal{U} \subset M$.*

In the sequel, we assume that the curvature vector field ζ of $\widetilde{M}$ is tangent to M and let $e = \theta(N)$. From (1.1), (2.15) and (2.16), we have

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = B(X, Z)\tau(Y) - B(Y, Z)\tau(X), \tag{3.5}$$

$$\begin{aligned} \widetilde{g}(R(X, Y)Z, N) = &\{\alpha\eta(X) + e\beta\theta(X)\}g(Y, Z) \\ &- \{\alpha\eta(Y) + e\beta\theta(Y)\}g(X, Z) \\ &+\beta\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(Z) \end{aligned} \tag{3.6}$$

for all $X, Y, Z \in \Gamma(TM)$. Using (1.1), (2.20) and (3.6), we have

$$\widetilde{Ric}(X, Y) = \{(m + 1)\alpha + \beta\}g(X, Y) + m\beta\,\theta(X)\theta(Y), \tag{3.7}$$

$$\widetilde{g}(R(\xi, Y)X, N) = \alpha g(X, Y) + \beta\,\theta(X)\theta(Y), \ \forall X, Y \in \Gamma(TM). \tag{3.8}$$

Substituting this two equations into (3.4), we have

$$\begin{aligned} R^{(0,2)}(X, Y) = &\{m\alpha + \beta\}g(X, Y) + \beta(m - 1)\theta(X)\theta(Y) \\ &+ \ B(X, Y)trA_N - g(A_N X, A^*_\xi Y), \ \forall X, Y \in \Gamma(TM). \end{aligned} \tag{3.9}$$

Definition 1. We say that the screen distribution $S(TM)$ of M is *totally umbilical* [4] in M if, on any coordinate neighborhood $\mathcal{U} \subset M$, there is a smooth function γ such that $A_N X = \gamma PX$ for any $X \in \Gamma(TM)$, or equivalently,

$$C(X, PY) = \gamma\, g(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{3.10}$$

In case $\gamma = 0$ on $\mathcal{U}$, we say that $S(TM)$ is *totally geodesic* in M.

Using (3.9) and the fact A^*_ξ is self-adjoint, we have:

Theorem 3.2. *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\widetilde{M}$ of quasi-constant curvature. If $S(TM)$ is totally umbilical in M and ζ is tangent to M, then $R^{(0,2)}$ is an induced Ricci tensor Ric of M.*

Theorem 3.3. *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\widetilde{M}$ of quasi constant curvature. If $S(TM)$ is totally geodesic in M and ζ is tangent to M, then the functions α and β, defined by* (1.1)*, vanish identically. Furthermore, $\widetilde{M}$ and M are flat manifolds.*

Proof. As $C = 0$, we have $\widetilde{g}(R(X, Y)PZ, N) = 0$ due to (2.19). From Theorem 3.1 and Theorem 3.2, we show that $d\tau = 0$ on TM. Thus we also have $\widetilde{g}(R(X, Y)\xi, N) = 0$ due to (2.17). From this two results we get

$$\widetilde{g}(R(X, Y)Z, N) = 0, \quad \forall X, Y, Z \in \Gamma(TM). \tag{3.11}$$

From this equation and the equation (3.8), we have

$$\beta\,\theta(X)\theta(Y) = -\,\alpha\, g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.12}$$

Substituting (3.12) into (1.1) and using (2.14), (2.15) and (3.5), we have

$$R(X,Y)Z = \alpha\{g(X,Z)Y - g(Y,Z)X\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM). \tag{3.13}$$

Thus M is a space of constant curvature $-\,\alpha$. Taking $X = Y = \zeta$ to (3.12), we have $\alpha + \beta = 0$. Substituting (3.12) and $A_N = 0$ into (3.9), we have

$$Ric(X,Y) = 0, \quad \forall\, X,\, Y \in \Gamma(TM).$$

Substituting (3.13) and $g(R(\xi,Y)X,\, N) = 0$ into (3.3), we also have

$$Ric(X,Y) = -\,(m-1)\alpha\, g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

From the last two equations, we obtain $\alpha = 0$ as $m > 1$. Thus $\beta = 0$ and $\widetilde{M}$ and M are flat manifolds, by (1.1) and (3.13). □

Corollary 1. *There exist no non-flat lightlike hypersurfaces M of semi-Riemannian manifold $\widetilde{M}$ of quasi-constant curvature such that $S(TM)$ is totally geodesic in M and the curvature vector field ζ of $\widetilde{M}$ is tangent to M.*

Theorem 3.4. Theorem 3.4. *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\widetilde{M}$ of quasi constant curvature. If $S(TM)$ is totally umbilical in $\widetilde{M}$ and ζ is tangent to M, then the scaler function β vanishes identically. Furthermore, $\widetilde{M}$ is a space of constant curvature α and M is an Einstein manifold such that $Ric = (r/m)\, g$, where r is the scalar curvature of M.*

Proof. Assume $S(TM)$ is totally umbilical in $\widetilde{M}$. Then we have (3.10) and

$$B(X,Y) = \rho\, g(X,Y),\ \forall\, X,\, Y \in \Gamma(TM). \tag{3.14}$$

Applying ∇_Z to (3.10) and using (2.9), we have

$$(\nabla_X C)(Y,PZ) = (X\gamma)g(Y,PZ) + \gamma B(X,PZ)\eta(Y)$$

for all $X,\, Y,\, Z \in \Gamma(TM)$. Substituting this into (2.19) and using (3.6), (3.14) and the fact $\theta(\xi) = 0$, we obtain

$$\begin{aligned} &\{X[\gamma] - \gamma\tau(X) - \rho\gamma\eta(X) - \alpha\eta(X) - e\beta\theta(X)\}g(Y,Z) \\ &= \{Y[\gamma] - \gamma\tau(Y) - \rho\gamma\eta(Y) - \alpha\eta(Y) - e\beta\theta(Y)\}g(X,Z) \\ &\quad +\ \beta\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(Z), \quad \forall\, X,\, Y,\, Z \in \Gamma(TM). \end{aligned}$$

Replacing Y by ξ to this equation and using the fact $\theta(\xi) = 0$, we have

$$\{\xi[\gamma] - \gamma\tau(\xi) - \rho\gamma - \alpha\}g(X,Y) = \beta\theta(X)\theta(Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

Taking $X = Y = \zeta$ to this equation, we have $\beta = \xi[\gamma] - \gamma\tau(\xi) - \rho\gamma - \alpha$ and

$$g(X,Y) = \theta(X)\theta(Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.15}$$

Substituting (3.15) into (1.1) and using (2.14), (2.15) and (3.14), we have

$$g(R(X,Y)Z,\,W) = (\alpha+2\beta+\rho\gamma)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \tag{3.16}$$

for all $X,\,Y,\,Z,\,W \in \Gamma(TM)$. Substituting (3.8) and (3.16) into (3.3), we have

$$Ric(X,Y) = \{m\alpha+(2m-1)\beta+(m-1)\rho\gamma)g(X,Y),\ \forall\, X,\,Y \in \Gamma(TM). \tag{3.17}$$

On the other hand, substituting (3.15) into (3.9) and using the facts $trA_N = m\gamma$ and $g(A^*_\xi X, A_N Y) = \rho\gamma g(X,Y)$, we have

$$Ric(X,Y) = \{m(\alpha+\beta)+(m-1)\rho\gamma\}g(X,Y),\ \forall\, X,\,Y \in \Gamma(TM). \tag{3.18}$$

Comparing (3.17) and (3.18), we obtain

$$(m-1)\beta = 0.$$

As $m > 1$, we have $\beta = 0$. Thus M is a space of constant curvature α.

Let $\kappa = m\alpha + (m-1)\rho\gamma$. Then the equations (3.17) and (3.18) reduce to

$$Ric(X,Y) = \kappa g(X,Y),\ \forall\, X,\,Y \in \Gamma(TM). \tag{3.19}$$

Thus M is an Einstein manifold. The scalar quantity r [3] of M, obtained from $R^{(0,\,2)}$ by the method of (2.22) is given by

$$r = R^{(0,\,2)}(\xi,\xi) + \sum_{a=1}^{m} \epsilon_a\, R^{(0,\,2)}(W_a, W_a).$$

Since M is an Einstein manifold satisfying (3.19), we obtain

$$r = \kappa g(\xi,\,\xi) + \kappa \sum_{a=1}^{m} \epsilon_a\, g(W_a,\,W_a) = \kappa\, m.$$

Thus we have

$$Ric(X,Y) = (r/m)g(X,Y)$$

which provides a geometric interpretation of lightlike Einstein hypersurfaces (same as in Riemannian case) as we have shown that the constant $\kappa = r/m$. □

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