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SOME REMARKS ON EXTREMAL PROBLEMS IN WEIGHTED BERGMAN SPACES OF ANALYTIC FUNCTIONS

Romi F. Shamoyan and Miloš Arsenović

Abstract. We prove some sharp extremal distance results for functions in weighted Bergman spaces on the upper halfplane. We also prove new analogous results in the context of bounded strictly pseudoconvex domains with smooth boundary.

1. Introduction

If Y is a normed space and $X \subset Y$, then we set $\mathrm{dist}_Y(f, X) = \inf_{g\in X} \|f - g\|_Y$. If the space Y is clear from the context, we write simply $\mathrm{dist}(f, X)$. In the problems we are going to consider, X itself is going to be a (quasi)-Banach space.

We denote by $H(\Omega)$ the space of all holomorphic functions on an open set $\Omega \subset \mathbb{C}^n$. In this paper we consider distance problems in weighted Bergman spaces over the upper half-plane $\mathbb{C}_+ = \{x + iy : y > 0\}$ and over a bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary.

The weighted Bergman space $A^p_\alpha(\mathbb{C}_+)$ consists of all functions $f \in H(\mathbb{C}_+)$ such that

$$\|f\|_{p,\alpha} = \left(\int_0^\infty \int_{-\infty}^\infty |f(x+iy)|^p y^\alpha dxdy\right)^{\frac{1}{p}} < \infty,$$

where $\alpha > -1$ and $0 < p < \infty$ (see [5] and [6]). The above spaces are Banach spaces for $p \geq 1$ and complete metric spaces for $0 < p < 1$. It is natural to consider the space $A^\infty_\nu = A^\infty_\nu(\mathbb{C}_+)$ of all holomorphic functions F on $\mathbb{C}_+$ satisfying

$$\|F\|_{A^\infty_\nu} = \sup_{y>0} \sup_{x\in\mathbb{R}} |F(x+iy)| y^\nu < \infty,$$

where $\nu > 0$, this space is also a Banach space.

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Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and let $\delta(z)$ denote the distance from $z \in D$ to the boundary of D with respect to some Riemannian metric, see [1]. We define

$$L^p_s(D) = L^p(D, \delta(z)^{s-\frac{n+1}{p}} dV(z)),$$

where $sp > n$, $0 < p < \infty$ and $dV(z)$ is the volume element on D, (see [1]). Set

$$A^p_s(D) = L^p_s(D) \cap H(D), \quad sp > n, \;\; 0 < p < \infty,$$

$$A^\infty_s(D) = \{f \in H(D) : \sup_{z\in D} |f(z)|\delta^s(z) < \infty\}, \;\; s \geq 0.$$

These spaces are Banach spaces for $1 \leq p \leq \infty$ and complete metric spaces for $0 < p < 1$.

In this paper we investigate the following two problems: estimate the distance, in $A^\infty_{\frac{\nu+2}{q}}(\mathbb{C}_+)$ norm, from $f \in A^\infty_{\frac{\nu+2}{q}}(\mathbb{C}_+)$ to $A^q_\nu(\mathbb{C}_+)$ and estimate the distance, in $A^\infty_s(D)$ norm, from $f \in A^\infty_s(D)$ to A^q_s. In both cases we give sharp results. The next section deals with the upper half-plane case, the last one deals with strongly pseudoconvex domains. Techniques used to obtain our results in these two different settings are similar, and the same ideas were used to study analogous problems for analytic Besov spaces in the unit ball and polydisc (see [10] and [11]). The literature on the extremal problems in spaces of analytic functions is extensive, even in the case of the unit disk, a classical exposition of these problems treated by duality methods developed by S. Havinson, W. Rogosinski and H. Shapiro can be found in [8].

2. Distance problems in $A^p_{\frac{\nu+2}{p}}(\mathbb{C}_+)$ spaces

The main tool in our investigation is Bergman representation formula. We first collect the results needed in the proofs of our theorems. The following result is contained in [5]:

Lemma 1. *If $f \in A^p_\alpha(\mathbb{C}_+)$, $0 < p < \infty$ and $\alpha > -1$, then*

$$f(z) = \frac{\beta+1}{\pi} \int_0^\infty \int_{-\infty}^\infty \frac{f(w)(\Im w)^\beta}{(\overline{w}-z)^{2+\beta}} dm_2(w),$$

where $0 < p \leq 1$, $\beta \geq \frac{2+\alpha}{p} - 2$ or $1 \leq p < \infty$, $\beta \geq \frac{1+\alpha}{p} - 1$.

Here and in the following m_2 denotes two dimensional Lebesgue measure. The next two results are contained in [6].

Lemma 2. *Let $f \in H(\mathbb{C}_+)$. Assume $\mathbb{C}_+$ is covered by dyadic squares: $\mathbb{C}_+ = \cup_{k=1}^\infty \Delta_k$ and let $(\Delta^\star_k)$ be the corresponding family of enlarged squares (see [6]). Then $\Delta^\star_k$ is a finitely overlapping covering of $\mathbb{C}_+$ and*

$$\sup_{z\in\Delta_k} |f(z)|^p (\Im z)^\alpha \leq \frac{C}{|\Delta^\star_k|} \int_{\Delta^\star_k} |f(z)|^p (\Im w)^\alpha dm_2(w)$$

for $0 < p < \infty$ and $\alpha > -1$.

Lemma 3. *Let Δ_k and $\Delta_k^\star$ be as in the previous lemma, let w_k be the center of the diadic square Δ_k. Then we have*:

$$(\Im w_k)^2 \asymp |\Delta_k| = m_2(\Delta_k) \asymp m_2(\Delta_k^\star),$$

$$|\overline{w} - z| \asymp |\overline{w}_k - z|, \quad w \in \Delta_k \quad , z \in \mathbb{C}_+,$$

$$\Im w \asymp \Im w_k, \quad w \in \Delta_k$$

and the following integral estimate:

$$\int_{\mathbb{C}_+} \frac{(\Im z)^\alpha dm_2(z)}{|\overline{w} - z|^{(2+\beta)p}} \leq C(\Im w)^{\alpha+2-(\beta+2)p}, \quad w \in \mathbb{C}_+$$

valid for all β satisfying $(\beta + 2)p - 2 > \alpha$, $\alpha > -1$.

Theorem 1 (see [3] and [4])**.** *Let $0 < p < \infty$, $\nu > 0$. Then there is a constant $C = C(p, \nu) > 0$ such that for all $x + iy \in \mathbb{C}_+$ and all $F \in A^p_\nu(\mathbb{C}_+)$ we have*

$$|F(x + iy)| \leq Cy^{-\frac{\nu+2}{p}} \|F\|_{A^p_\nu(\mathbb{C}_+)}.$$

Also, there is a constant $C = C(p, \nu) > 0$ such that for all $y > 0$ and all $F \in A^p_\nu(\mathbb{C}_+)$ we have

$$\left(\int_{-\infty}^{\infty} |F(x + iy)|^p dx\right)^{1/p} \leq Cy^{-\frac{\nu+1}{p}} \|F\|_{A^p_\nu(\mathbb{C}_+)}.$$

We clearly have

$$\|F\|_{A^\infty_{\frac{\nu+2}{p}}} \leq C\|F\|_{A^p_\nu}$$

for $0 < p < \infty$, $\nu > 0$ and $F \in A^p_\nu(\mathbb{C}_+)$.

The next two theorems show that $\operatorname{dist}_{A^q_{\frac{\nu+2}{q}}}(f, A^q_\nu)$ can be explicitly given in the case of the upper halfplane $\mathbb{C}_+$. We treat separately cases $0 < q \leq 1$ and $q > 1$. The distance we are looking for is described using the following sets:

$$V_{\epsilon,t}(f) = \{z = x + iy \in \mathbb{C}_+ : |f(x + iy)|y^t \geq \epsilon\}.$$

Theorem 2. *Let $q > 1$, $\nu > -1$, $t = \frac{\nu+2}{q}$, $\beta > \max(\frac{\nu}{q}, \frac{\nu+2}{q} - 1)$ and $f \in A^\infty_{\frac{\nu+2}{q}}(\mathbb{C}_+)$. Then $l_1 = l_2$ where*

$$l_1 = \operatorname{dist}_{A^\infty_{\frac{\nu+2}{q}}}(f, A^q_\nu),$$

$$(1) \quad l_2 = \inf\left\{\epsilon > 0 : \int_{\mathbb{C}_+} \left(\int_{V_{\epsilon,t}(f)} \frac{(\Im w)^{\beta-t} dm_2(w)}{|\overline{w} - z|^{\beta+2}}\right)^q (\Im z)^\nu dm_2(z) < \infty\right\}.$$

Theorem 3. *Let $0 < q \leq 1$, $\nu > -1$, $t = \frac{\nu+2}{q}$, $\beta > \frac{\nu+2}{q} - 2$ and $f \in A^\infty_{\frac{\nu+2}{q}}(\mathbb{C}_+)$. Then $t_1 = t_2$ where*

$$t_1 = \operatorname{dist}_{A^\infty_{\frac{\nu+2}{q}}}(f, A^q_\nu),$$

$$(2)\quad t_2=\inf\left\{\epsilon>0:\int_{\mathbb{C}_+}\left(\int_{V_{\epsilon,t}(f)}\frac{(\Im w)^{\beta-t}dm_2(w)}{|\overline{w}-z|^{\beta+2}}\right)^q(\Im z)^\nu dm_2(z)<\infty\right\}.$$

We present now the proofs of the above theorems, since they significantly overlap a unified presentation is possible. Assume $l_1<l_2$ or $t_1<t_2$. Then just using classical textbook definition of inf we will get immediately there are $\epsilon>\epsilon_1>0$ and $f_{\epsilon_1}\in A^q_\nu(\mathbb{C}_+)$ such that $\|f-f_{\epsilon_1}\|_{A^\infty_{\frac{\nu+2}{q}}}\le\epsilon_1$ and

$$\int_{\mathbb{C}_+}\left(\int_{V_{\epsilon,t}(f)}\frac{(\Im w)^{\beta-t}dm_2(w)}{|\overline{w}-z|^{\beta+2}}\right)^q(\Im z)^\nu dm_2(z)=+\infty.$$

Since $\|f-f_{\epsilon_1}\|_{A^\infty_{\frac{\nu+2}{q}}}\le\epsilon_1$, from the definition of the set $V_{\epsilon,t}(f)$ and making very simple transformation with the norm estimate we just wrote above we conclude immediately that

$$(\epsilon-\epsilon_1)\chi_{V_{\epsilon,t}(f)}(z)(\Im z)^{-\frac{\nu+2}{q}}\le|f_{\epsilon_1}(z)|$$

indeed will the fixed point from the unit disk belongs to mentioned $V_{\epsilon,t}(f)$ set or not we will arrive to correct estimate in any case as it can be easily checked and therefore after multiplication of both sides of the very last estimate above by appropriate expressions we will easily now have the following

$$\begin{aligned}+\infty&=\int_{\mathbb{C}_+}\left(\int_{\mathbb{C}_+}\frac{\chi_{V_{\epsilon,t}(f)}(w)(\Im w)^{\beta-t}dm_2(w)}{|\overline{w}-z|^{\beta+2}}\right)^q(\Im z)^\nu dm_2(z)\\&\le\int_{\mathbb{C}_+}\left(\int_{\mathbb{C}_+}\frac{|f_{\epsilon_1}(w)|(\Im w)^{\beta}}{|\overline{w}-z|^{2+\beta}}dm_2(w)\right)^q(\Im z)^\nu dm_2(z)=M.\end{aligned}$$

Note that this estimate is valid for $0<q<\infty$ and therefore works for both of the above theorems. In both cases we are going to arrive at the contradiction by proving $M<+\infty$. Let us first consider the case $q>1$. Using Hölder's inequality and integral estimate of Lemma 3 (with $\alpha=0$) we obtain

$$\begin{aligned}I(z)&=\left(\int_{\mathbb{C}_+}\frac{|f_{\epsilon_1}(w)|(\Im w)^{\beta}}{|\overline{w}-z|^{2+\beta}}dm_2(w)\right)^q\\&\le\int_{\mathbb{C}_+}\frac{|f_{\epsilon_1}(w)|^q(\Im w)^{\beta q}}{|\overline{w}-z|^{\beta q-\epsilon q+2}}dm_2(w)\cdot\left(\int_{\mathbb{C}_+}\frac{dm_2(w)}{|\overline{w}-z|^{\epsilon p+2}}\right)^{q/p}\\&\le C\int_{\mathbb{C}_+}\frac{|f_{\epsilon_1}(w)|^q(\Im w)^{\beta q}}{|\overline{w}-z|^{\beta q-\epsilon q+2}}dm_2(w)(\Im z)^{-\epsilon q}\end{aligned}$$

for $w \in \mathbb{C}_+$, $\epsilon > 0$. Using this estimate, Fubini's theorem and again integral estimate from Lemma 3 we obtain

$$\begin{aligned} M &\leq C \int_{\mathbb{C}_+} |f_{\epsilon_1}(w)|^q (\Im w)^{\beta q} \left(\int_{\mathbb{C}_+} \frac{(\Im z)^{\nu - \epsilon q}}{|\overline{w} - z|^{\beta q - \epsilon q + 2}} dm_2(z) \right) dm_2(w) \\ &\leq C \int_{\mathbb{C}_+} |f_{\epsilon_1}(w)|^q (\Im w)^{\nu} dm_2(w) < \infty, \end{aligned}$$

where $\epsilon > 0$ is small enough so that $\nu - \epsilon q > -1$.

Now let us turn to the case $q \leq 1$. We have, using several times Lemma 2 and Lemma 3,

$$\begin{aligned} I(z) &= \left(\int_{\mathbb{C}_+} \frac{|f_{\epsilon_1}(w)|(\Im w)^{\beta}}{|\overline{w} - z|^{2+\beta}} dm_2(w) \right)^q \\ &= \left(\sum_k \int_{\Delta_k} \frac{|f_{\epsilon_1}(w)|(\Im w)^{\beta}}{|\overline{w} - z|^{2+\beta}} dm_2(w) \right)^q \\ &\leq C \sum_{k=1}^{\infty} \max_{w \in \Delta_k} |f_{\epsilon_1}(w)|^q (m_2(\Delta_k))^q \frac{(\Im w_k)^{\beta q}}{|\overline{w} - z|^{(2+\beta)q}} \\ &\leq C \int_{\mathbb{C}_+} \frac{|f_{\epsilon_1}(w)|^q (\Im w)^{\beta q + 2q - 2} dm_2(w)}{|\overline{w} - z|^{(2+\beta)q}}, \end{aligned}$$

in the first estimate in chain of estimates above we used first the well-known fact that such a family of dyadic cubes from Lemma 2 exists, then we used the well-known classical estimate for sums for $q < 1$ and (non)integral estimates from Lemma 3 then at last in the last estimate we used first integral estimate from Lemma 2 and then finite overlapping property of the family of enlarged cubes which also stated there . Now we get $M < \infty$ copying arguments from the case $q > 1$, namely by applying Funini's theorem and integral estimate from Lemma 3.

The reverse inequalities $l_1 \leq l_2$ and $t_1 \leq t_2$ can be proved simultaneously. We fix $\epsilon > 0$ such that the integrals in (1), respectively (2), are finite (if there are no such $\epsilon > 0$, the inequality is trivial) and use integral representation from Lemma 1. This integral representation is valid for $p = \infty$, $\alpha > 0$ and β sufficiently large.

$$\begin{aligned} f(z) &= \frac{\beta + 1}{\pi} \left(\int_{\mathbb{C}_+ \setminus V_{\epsilon,t}(f)} \frac{f(w)(\Im w)^{\beta} dm_2(w)}{(\overline{w} - z)^{2+\beta}} + \int_{V_{\epsilon,t}(f)} \frac{f(w)(\Im w)^{\beta} dm_2(w)}{(\overline{w} - z)^{2+\beta}} \right) \\ &= \frac{\beta + 1}{\pi} (f_1(z) + f_2(z)), \end{aligned}$$

where β is sufficiently large. The estimate for $f_1(z)$ is immediate, using Lemma 3:

$$|f_1(z)| \leq \epsilon \int_{\mathbb{C}_+} \frac{(\Im w)^{\beta - t} dm_2(w)}{|\overline{w} - z|^{2+\beta}} \leq C\epsilon(\Im z)^{-t},$$

and this means $f_1 \in A^{\infty}_{\frac{\nu+2}{q}}$ with $\|f_1\|_{A^\infty_t} \le C\epsilon$. Next, by the choice of $\epsilon > 0$ we have:

$$\begin{aligned}\|f_2\|_{A^q_\nu} &\le C\|f\|_{A^\infty_t} \int_{\mathbb{C}_+} \left(\int_{V_{\epsilon,t}(f)} \frac{(\Im w)^{\beta-t} dm_2(w)}{|\overline{w}-z|^{2+\beta}} \right)^q (\Im z)^\nu dm_2(z) \\ &\le C\|f\|_{A^\infty_t}.\end{aligned}$$

This completes the proofs of both theorems.

We end this section with an observation that the upper halfplane is the simplest tube domain. Analysis on tube domains over symmetric cones is an active area of research (see [2], [3] and [4]) and availability of integral representations points to possibility of extending our results to this more general setting.

3. Distance problems in $A^q_s(\Omega)$ spaces

We now consider Bergman type spaces in $D \subset \mathbb{C}^n$, where D is a smoothly bounded relatively compact strictly pseudoconvex domain, providing also sharp results in this case. Our proofs are heavily based on the estimates from [1], where more general situation was considered.

Since $|f(z)|^p$ is subharmonic (even plurisubharmonic) for a holomorphic f, we have $A^p_s(D) \subset A^\infty_t(D)$ for $0 < p < \infty$, $sp > n$ and $t = s$. Also, $A^p_s(D) \subset A^1_s(D)$ for $0 < p \le 1$ and $A^p_s(D) \subset A^1_t(D)$ for $p > 1$ and t sufficiently large. Therefore we have an integral representation

$$f(z) = \int_D f(\xi) K(z,\xi) \delta^t(\xi) dV(\xi), \tag{3}$$

where $K(z,\xi)$ is a kernel of type $n+t+1$, that is a measurable function on $D \times D$ such that $|K(z,\xi)| \le C|\tilde{\Phi}(z,\xi)|^{-(n+1+t)}$, where $\tilde{\Phi}(z,\xi)$ is so called Henkin-Ramirez function for D. From now on we work with a fixed Henkin-Ramirez function $\tilde{\Phi}$ and a fixed kerenel K of type $n+t+1$. We are going to use the following results from [1].

Lemma 4 ([1, Corollary 5.3])**.** *If $r > 0$, $0 < p \le 1$, $s > -1$, $p(s+n+1) > n$ and $f \in H(D)$, then we have*

$$\begin{aligned}&\left(\int_D |f(\xi)| |\tilde{\Phi}(z,\xi)|^r \delta^s(\xi) dV(\xi) \right)^p \\ &\le C \int_D |f(\xi)|^p |\tilde{\Phi}(z,\xi)|^{rp} \delta^{p(s+n+1)-(n+1)}(\xi) dV(\xi).\end{aligned}$$

Lemma 5 ([1, Corollary 3.9])**.** *Assume $T(z,\xi)$ is a kernel of type β, and $\sigma > 0$ satisfies $\sigma + n - \beta < 0$. Then we have*

$$\int_D T(z,\xi) \delta^{\sigma-1}(z) dV(z) \le C\delta^{\sigma+n-\beta}(\xi).$$

A natural problem is to estimate $\mathrm{dist}_{A_s^\infty(D)}(f, A_s^q(D))$ where $0 < q < \infty$, $sq > n$ and $f \in A_s^\infty(D)$. We give sharp estimates below, treating cases $0 < q \le 1$ and $q > 1$ separately.

Theorem 4. *Let $0 < q \le 1$, $sq > n$, $f \in A_s^\infty(D)$ and $t > s$ is sufficiently large. Then $\omega_1 = \omega_2$ where*

$$\omega_1 = \mathrm{dist}_{A_s^\infty(D)}(f, A_s^q(D)),$$

$$\omega_2 = \inf\left\{\epsilon > 0 : \int_D \left(\int_{\Omega_{\epsilon,s}} |K(z,\xi)|\delta^{t-s}(\xi)dV(\xi)\right)^q \delta^{sq-n-1}(z)dV(z) < \infty\right\},$$

where $K(z,\xi)$ is the above kerenel of type $n+t+1$ and

$$\Omega_{\epsilon,s} = \{z \in D : |f(z)|\delta^s(z) \ge \epsilon\}.$$

Proof. Let us prove that $\omega_1 \le \omega_2$. We fix $\epsilon > 0$ such that the above integral is finite and use (3):

$$f(z) = \int_{D\setminus\Omega_{\epsilon,s}} f(\xi)K(z,\xi)dV(\xi) + \int_{\Omega_{\epsilon,s}} f(\xi)K(z,\xi)dV(\xi) = f_1(z) + f_2(z).$$

We estimate f_1:

$$\begin{aligned}|f_1(z)| &\le C\epsilon \int_D |K(z,\xi)|\delta^{t-s}dV(\xi)\\ &\le C\epsilon \int_D \frac{\delta^{t-s}(\xi)dV(\xi)}{|\tilde{\Phi}(z,\xi)|^{n+t+1}}\\ &\le C\epsilon\delta^{-s}(z),\end{aligned}$$

where the last estimate is contained in [1] (see p. 375). Next,

$$\begin{aligned}\|f_2\|^q_{A_s^q} = &\int_D |f_2(z)|^q\delta^{sq-n-1}(z)dV(z)\\ &\le C\int_D \left(\int_{\Omega_{\epsilon,s}} |f(\xi)|K(z,\xi)\delta^t(\xi)dV(\xi)\right)^q \delta^{sq-n-1}(z)dV(z)\\ &\le C'\|f\|^q_{A_s^\infty}.\end{aligned}$$

Now we have

$$\mathrm{dist}_{A_s^\infty(D)}(f, A_s^q(D)) \le \|f - f_2\|_{A_s^\infty(D)} = \|f_1\|_{A_s^\infty(D)} \le C\epsilon.$$

Now assume that $\omega_1 < \omega_2$. Then there are $\epsilon > \epsilon_1 > 0$ and $f_{\epsilon_1} \in A_s^q(D)$ such that $\|f - f_{\epsilon_1}\|_{A_s^\infty} \le \epsilon_1$ and

$$I = \int_D \left(\int_{\Omega_{\epsilon,s}} |K(z,\xi)|\delta^{t-s}(\xi)dV(\xi)\right)^q \delta^{sq-n-1}(z)dV(z) = \infty.$$

As in the case of the upper half-plane one uses $\|f - f_{\epsilon_1}\|_{A_s^\infty} \le \epsilon_1$ to obtain

$$(\epsilon - \epsilon_1)\chi_{\Omega_{\epsilon,s}}(z)\delta^{-s}(z) \le C|f_{\epsilon_1}(z)|.$$

Now the following chain of estimates leads to a contradiction:

$$\begin{aligned} I &= \int_D \left(\int_D \chi_{\Omega_{\epsilon,s}}(\xi) \delta^{t-s}(\xi) K(z,\xi) dV(\xi) \right)^q \delta^{sq-n-1}(z) dV(z) \\ &\le C \int_D \left(\int_D |f_{\epsilon_1}(\xi)| \delta^t(\xi) K(z,\xi) dV(\xi) \right)^q \delta^{sq-n-1}(z) dV(z) \\ &\le C \int_D \left(\int_D |f_{\epsilon_1}(\xi)| \delta^t(\xi) \frac{dV(\xi)}{|\tilde{\Phi}(z,\xi)|^{n+t+1}} \right)^q \delta^{sq-n-1}(z) dV(z) \\ &\le C \int_D \int_D |f_{\epsilon_1}(\xi)|^q \frac{\delta^{sq-n-1}(z) \delta^{q(t+n+1)-(n+1)}(\xi)}{|\tilde{\Phi}(z,\xi)|^{q(n+t+1)}} dV(z) dV(\xi) \\ &\le C \int_D |f_{\epsilon_1}(\xi)|^q \delta^{sq-n-1}(\xi) dV(\xi) < \infty, \end{aligned}$$

where we used Lemma 4 and Lemma 5 with $\beta = q(n+t+1)$, $\sigma = sq - n$. □

Next theorem deals with the case $1 < q < \infty$.

Theorem 5. *Let $q > 1$, $sq > n$, $t > s$, $t > \frac{s+n+1}{q}$ and $f \in A^\infty_s(D)$. Then $\omega_1 = \omega_2$ where*

$$\omega_1 = \mathrm{dist}_{A^\infty_s(D)}(f, A^q_s(D)),$$

$$\omega_2 = \inf \left\{ \epsilon > 0 : \int_D \left(\int_{\Omega_{\epsilon,s}} |K(z,\xi)| \delta^{t-s}(\xi) dV(\xi) \right)^q \delta^{sq-n-1}(z) dV(z) < \infty \right\}.$$

Proof. An inspection of the proof of the previous theorem shows that it extends to this case also, provided one can prove the estimate:

$$\begin{aligned} J &= \int_D \left(\int_D |f_{\epsilon_1}(\xi)| \delta^t(\xi) K(z,\xi) dV(\xi) \right)^q \delta^{sq-n-1}(z) dV(z) \\ &\le C \int_D |f_{\epsilon_1}(\xi)|^q \delta^{sq-n-1}(\xi) dV(\xi) < \infty, \end{aligned}$$

where $q > 1$. Using Hölder's inequality and Lemma 5, with $\sigma = 1$ and $\beta = n + 1 + p\epsilon$, we obtain

$$\begin{aligned} I(z) &= \left(\int_D |f_{\epsilon_1}(\xi)| \delta^t(\xi) K(z,\xi) dV(\xi) \right)^q \\ &\le \int_D \frac{|f_{\epsilon_1}(\xi)|^q \delta^{tq}(\xi) dV(\xi)}{|\tilde{\Phi}(z,\xi)|^{n+1+tq-\epsilon q}} \cdot \left(\int_D \frac{dV(\xi)}{|\tilde{\Phi}(z,\xi)|^{n+1+p\epsilon}} \right)^{q/p} \\ &\le C \int_D \frac{|f_{\epsilon_1}(\xi)|^q \delta^{tq}(\xi) dV(\xi)}{|\tilde{\Phi}(z,\xi)|^{n+1+tq-\epsilon q}} \delta^{-q\epsilon}(z), \end{aligned}$$

and this gives

$$J \leq C \int_D \int_D \frac{|f_{\epsilon_1}(\xi)|^q \delta^{tq}(\xi) \delta^{-q\epsilon+sq-n-1}(z)}{|\tilde{\Phi}(z,\xi)|^{n+1+tq-\epsilon q}} dV(z)\, dV(\xi)$$
$$\leq C \int_D |f_{\epsilon_1}(\xi)|^q \delta^{sq-n-1}(\xi) dV(\xi) < \infty,$$

where we again used Lemma 5, with $\beta = n+1+tq-\epsilon q$ and $\sigma = q(s-\epsilon)-n > 0$. □

Remark. We note that most results of this paper and the previous one ([11]) on distances can be extended to bounded symmetric domains $\Omega \subset \mathbb{C}^n$. Indeed, the methods of proofs are based on Bergman representation formula, asymptotic properties of the Bergman kernel and Forelli-Rudin type estimates for integrals

$$\int_\Omega K^\alpha(w,w)|K(z,w)|^\beta dV(w), \quad \alpha > 0,\ \beta > 0\ \ z \in \Omega,$$

where $K(z,w)$ is a Bergman reproducing kernel for the weighted Bergman space $A^2_\alpha(\Omega)$. The relevant estimates can be found in [2] and [9].

References

[1] F. Beatrous, Jr. *L^p estimates for extensions of holomorphic functions*, Michigan Math. J. **32** (1985), no. 3, 361–380.
[2] D. Bekolle, C. Berger, L. Coburn, and K. Zhu, *BMO in the Bergman metric on bounded symmetric domains*, J. Funct. Anal. **93** (1990), no. 2, 310–350.
[3] D. Bekolle and A. Bonami, *Estimates for the Bergman and Szego projections in two symmetric domains*, Colloq. Math. **68** (1995), no. 1, 81–100.
[4] D. Bekolle, A. Bonami, G. Garrigos, C. Nana, M. Peloso, and F. Ricci *Lecture notes on Bergman projections on tube domains over cones: an analytic and geometric viewpoint*, preprint 2002.
[5] M. Djrbashian and A. Djrbashian, *Integral representations for some classes of analytic functions in the half-plane*, Dokl Acad NAuk **285** (1985), 547–550.
[6] A. Djrbashian and K. Karapetyan, *Integral inequalities between conjugate pluriharmonic functions in multidimensional domains*, Izv NAts Akad Nauk Armenii (1988), 216–236.
[7] M. Djrbashian and F. Shamoian, *Topics in the theory of A^p_α classes*, Teubner Texte zur Mathematik, 1988, v 105.
[8] P. Duren, *Theory of H^p Spaces*, Academic Press, 1970.
[9] J. Faraut and A. Koranyi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Funct. Anal. **88** (1990), no. 1, 64–89.
[10] R. Shamoyan and O. Mihić, *On new estimates for distances in analytic function spaces in higher dimension*, Sib. Èlektron. Mat. Izv. **6** (2009), 514–517.
[11] ———, *On new estimates for distances in analytic function spaces in the unit disk, polydisk and unit ball*, to appear in Bol. Asoc. MAt. Venez.

Romi F. Shamoyan
Bryansk University
Bryansk 241036, Russia
E-mail address: rshamoyan@yahoo.com

Miloš Arsenović
Faculty of Mathematics
University of Belgrade
Studentski Trg 16
11000 Belgrade, Serbia
E-mail address: arsenovic@matf.bg.ac.rs