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THE INTEGRAL EXPRESSION INVOLVING THE FAMILY OF LAGUERRE POLYNOMIALS AND BESSEL FUNCTION

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Abstract. The principal aim of the paper is to investigate new integral expression

$$\int_0^\infty x^{s+1} e^{-\sigma x^2} L_m^{(\gamma,\delta)}\left(\zeta;\sigma x^2\right) L_n^{(\alpha,\beta)}\left(\xi;\sigma x^2\right) J_s\left(xy\right) dx,$$

where y is a positive real number; σ, ζ and ξ are complex numbers with positive real parts; s, α, β, γ and δ are complex numbers whose real parts are greater than -1; $J_n(x)$ is Bessel function and $L_n^{(\alpha,\beta)}(\gamma;x)$ is generalized Laguerre polynomials. Some integral formulas have been obtained. The Maple implementation has also been examined.

1. Introduction and definition

Laguerre polynomials occur in many fields of research in science, engineering and numerical mathematics such as, in quantum mechanics [5], communication theory [1] and numerical inverse Laplace transform [6]. Explicit evaluation of integrals involving Laguerre polynomials is very often required in these and other applied areas of research.

A Hankel transform integral of a product of a power, an exponential function and two Laguerre polynomials are given in several tables [2, 3, 8] with error. In 1996, that integral $\int_0^\infty x^{s+1} e^{-\sigma x^2} L_m^{s-\beta}\left(\sigma x^2\right) L_n^{\beta}\left(\sigma x^2\right) J_s\left(xy\right) dx$ was corrected by Kölbig and Scherb [4].

This paper is devoted to the extension of aforesaid formula.

We used the following notations throughout this paper:

$\mathbb{N}$: The set of natural numbers/the set of nonnegative integers.

$\mathbb{I}$: The set of positive integers.

$\mathbb{R}$: The set of real numbers.

$\mathbb{R}^+$: The set of positive real numbers.

$\mathbb{C}$: The set of complex numbers.

$\mathbb{C}^+ = \{a + ib \mid a \in \mathbb{R}^+, b \in \mathbb{R}\}$.

$\mathbb{C}^+_{-1} = \{a + ib \mid a, b \in \mathbb{R} \wedge a > -1\}$.

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The Bessel function (Rainville [9]) is defined as

$$J_n(x) = \sum_{r=0}^{n} \frac{(-1)^r}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}.$$

Prabhakar and Suman [7] defined the polynomials $L_n^{(\alpha,\beta)}(x)$ as

$$L_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha n+\beta+1)}{\Gamma(n+1)} \sum_{k=0}^{n} \frac{(-n)_k\, x^k}{k!\Gamma(\alpha k+\beta+1)}, \tag{1.1}$$

where $\alpha \in \mathbb{C}^+$, $\beta \in \mathbb{C}_{-1}^+$ and $n \in \mathbb{N}$.

If $\alpha = 1$, then (1.1) reduces as:

$$L_n^{(1,\beta)}(x) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)} \sum_{k=0}^{n} \frac{(-n)_k\, x^k}{k!\Gamma(k+\beta+1)} = L_n^{\beta}(x), \tag{1.2}$$

where $L_n^{\beta}(x)$ is well-known generalized Laguerre polynomials (Rainville [9]).

The Konhauser polynomial of second kind (Srivastava [12]) is defined as

$$Z_n^{\beta}(x;k) = \frac{\Gamma(kn+\beta+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj+\beta+1)}, \tag{1.3}$$

where $\beta \in \mathbb{C}_{-1}^+$, $n \in \mathbb{N}$ and $k \in \mathbb{I}$.

It can easily be verifed that,

$$L_n^{k,\beta}\left(x^k\right) = Z_n^{\beta}(x;k), \tag{1.4}$$

$$Z_n^{\beta}(x;1) = L_n^{\beta}(x). \tag{1.5}$$

The polynomial $Z_n^{\alpha,\beta}(x;k)$is defined [10] as,

$$Z_n^{\alpha,\beta}(x;k) = \sum_{j=0}^{n} \frac{\Gamma(kn+\beta+1)\,(-1)^j\, x^{kj}}{j!\;\Gamma(kj+\beta+1)\Gamma(\alpha n-\alpha j+1)}, \tag{1.6}$$

where $\alpha \in \mathbb{C}^+$, $\beta \in \mathbb{C}_{-1}^+$, $n \in \mathbb{N}$ and $k \in \mathbb{I}$.

From (1.3) and (1.6), we get

$$Z_n^{1,\beta}(x;k) = Z_n^{\beta}(x;k). \tag{1.7}$$

If $\alpha \in \mathbb{N}$, then (1.6) can be written in the following form

$$Z_n^{\alpha,\beta}(x\;;k) = \frac{\Gamma(kn+\beta+1)}{\Gamma(\alpha n+1)} \sum_{m=0}^{n} \frac{(-\alpha n)_{\alpha m}\, x^{km}}{m!\Gamma(km+\beta+1)\;(-1)^{(\alpha-1)m}}. \tag{1.8}$$

The set of polynomials $L_n^{\alpha,\beta}(\gamma;x)$ is defined [10] as,

$$L_n^{\alpha,\beta}(\gamma;x) = \sum_{r=0}^{n} \frac{\Gamma(\alpha n+\beta+1)\,(-1)^r\, x^r}{r!\Gamma(\alpha r+\beta+1)\Gamma(\gamma n-\gamma r+1)}, \tag{1.9}$$

where $\alpha, \gamma \in \mathbb{C}^+$, $\beta \in \mathbb{C}_{-1}^+$, $n \in \mathbb{N}$.

From (1.9) and (1.1), we have

$$L_n^{\alpha,\beta}(1\,;x) = L_n^{\alpha,\beta}(x)\,. \tag{1.10}$$

One can easily verify that

$$L_n^{k,\beta}\left(\alpha\,;x^k\right) = Z_n^{\alpha,\beta}(x\,;k)\,; \tag{1.11}$$

$$Z_n^{1,\beta}(x\,;1) = L_n^{\beta}(\alpha\,;x)\,; \tag{1.12}$$

$$Z_n^{1,\beta}(x\,;1) = Z_n^{\beta}(x\,;1) = L_n^{\beta}(x)\,; \tag{1.13}$$

$$L_n^{1,\beta}(1\,;x) = L_n^{1,\beta}(x) = L_n^{\beta}(x)\,. \tag{1.14}$$

The following integral has been evaluated in the next section,

$$\int_0^\infty x^{s+1}e^{-\sigma x^2}L_m^{(\gamma,\delta)}\left(\zeta;\sigma x^2\right)L_n^{(\alpha,\beta)}\left(\xi;\sigma x^2\right)J_s(xy)\,dx. \tag{1.15}$$

The following result is given in Gradshteyn and Ryzhik ([3]),

$$\int_0^\infty x^{2h+s+1}e^{-\sigma x^2}J_s(xy)\,dx = \frac{h!y^s\sigma^{-h}}{(2\sigma)^{s+1}}\exp\left(-\frac{y^2}{4\sigma}\right)L_h^s\left(\frac{y^2}{4\sigma}\right), \tag{1.16}$$

where $y>0$, $h\in\mathbb{N}$, $h+s\in\mathbb{C}_{-1}^{+}$ and $\sigma\in\mathbb{C}^{+}$.

Kölbig and Scherb [4] proved the following formula:

$$\begin{aligned}&(-1)^{m+n}L_m^{\beta-m+n}(x)\,L_n^{s-\beta+m-n}(x)\\ &= \sum_{h=0}^{m+n}\sum_{k=0}^{h}(-1)^h\binom{h}{k}\binom{m+s-\beta}{m-k}\binom{n+\beta}{n-h+k}L_h^s(x)\,,\end{aligned} \tag{1.17}$$

where $y>0$ and $\beta, s\in\mathbb{C}_{-1}^{+}$.

Some facts are listed below (see Spanier and Oldham [11]),

$$(-x)_n = (-1)^n(x-n+1)_n\,, \tag{1.18}$$

$$(x+y)_n = \sum_{j=0}^{n}\binom{n}{j}(x)_j\,(y)_{n-j}\,, \tag{1.19}$$

$$(x)_{n+m} = (x)_n\,(x+n)_m \ \text{ and} \tag{1.20}$$

$$\binom{x}{n} = \frac{(-1)^n}{n!}(-x)_n\,. \tag{1.21}$$

2. The evaluation of the integral

Theorem 2.1. (a) *If* $y \in \mathbb{R}^+; \sigma, \xi, \zeta \in \mathbb{C}^+$ *and* $s, \alpha, \beta, \gamma, \delta \in \mathbb{C}^+_{-1}$, *then*

$$\int_0^\infty x^{s+1} e^{-\sigma x^2} L_m^{(\gamma,\delta)}\left(\zeta; \sigma x^2\right) L_n^{(\alpha,\beta)}\left(\xi; \sigma x^2\right) J_s(xy)\, dx = \frac{y^s}{(2\sigma)^{s+1}} \exp\left(-\frac{y^2}{4\sigma}\right) \sum_{h=0}^{m+n} \Delta_{\alpha,\beta;\gamma,\delta}^{n,m,\xi.\zeta}(h)\, L_h^s\left(\frac{y^2}{4\sigma}\right), \tag{2.1}$$

where $\Delta_{\alpha,\beta;\gamma,\delta}^{n,m,\xi.\zeta}(h)$ *is defined as*

$$\sum_{k=0}^{h} \left[\binom{h}{k} \frac{(-1)^h \Gamma(\alpha n+\beta+1)\Gamma(\gamma m+\delta+1)}{\Gamma(\alpha k+\beta+1)\Gamma(\gamma(h-k)+\delta+1)\Gamma(\zeta(m-h+k)+1)\Gamma(\xi(n-k)+1)} \right]. \tag{2.2}$$

(b) *If* $y \in \mathbb{R}^+; \sigma \in \mathbb{C}^+$; $s, \alpha, \beta, \gamma, \delta \in \mathbb{C}^+_{-1}$ *and* $\xi, \zeta \in \mathbb{N}$, *then*

$$\int_0^\infty x^{s+1} e^{-\sigma x^2} L_m^{(\gamma,\delta)}\left(\zeta; \sigma x^2\right) L_n^{(\alpha,\beta)}\left(\xi; \sigma x^2\right) J_s(xy)\, dx = \frac{y^s}{(2\sigma)^{s+1}} \exp\left(-\frac{y^2}{4\sigma}\right) \sum_{h=0}^{m+n} \nabla_{\alpha,\beta;\gamma,\delta}^{n,m,\xi.\zeta}(h)\, L_h^s\left(\frac{y^2}{4\sigma}\right), \tag{2.3}$$

where $\nabla_{\alpha,\beta;\gamma,\delta}^{n,m,\xi.\zeta}(h)$ *is defined as*

$$\frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma m+\delta+1)}{\Gamma(\zeta m+1)\Gamma(\xi n+1)} \sum_{k=0}^{h} \left[\binom{h}{k} \frac{(-1)^{h-\zeta(h-k)-\xi k} (-\zeta m)_{\zeta(h-k)} (-\xi n)_{\xi k}}{\Gamma(\alpha k+\beta+1)\Gamma(\gamma(h-k)+\delta+1)} \right]. \tag{2.4}$$

Proof. (a) Using (1.9), we have

$$L_n^{(\alpha,\beta)}(\xi; x) = \sum_{k=0}^{n} \frac{\Gamma(\alpha n+\beta+1)}{\Gamma(\alpha k+\beta+1)\Gamma(\xi(n-k)+1)} \left(\frac{(-x)^k}{k!}\right). \tag{2.5}$$

By using (2.5), we get

$$L_n^{(\alpha,\beta)}(\xi; x) L_m^{(\gamma,\delta)}(\zeta; x) = \sum_{k=0}^{n}\sum_{r=0}^{m} \frac{1}{\Gamma(\xi(n-k)+1)\Gamma(\zeta(m-r)+1)} \times \frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma m+\delta+1)}{\Gamma(\alpha k+\beta+1)\Gamma(\gamma r+\delta+1)} \left(\frac{(-x)^{r+k}}{k!r!}\right).$$

Denoting the left-hand side by$L$, we have

$$L = L_n^{(\alpha,\beta)}(\xi; x) L_m^{(\gamma,\delta)}(\zeta; x).$$

Taking $r+k=h$, we have

$$L=\sum_{k=0}^{n}\sum_{h=k}^{m+k}\frac{\Gamma\left(\alpha n+\beta+1\right)\Gamma\left(\gamma m+\delta+1\right)}{\Gamma\left(h-k+1\right)\Gamma\left(\zeta\left(m-h+k\right)+1\right)\Gamma\left(k+1\right)\Gamma\left(\xi\left(n-k\right)+1\right)}$$
$$\times\frac{\left(-1\right)^{h}x^{h}}{\Gamma\left(\alpha k+\beta+1\right)\Gamma\left(\gamma\left(h-k\right)+\delta+1\right)}.$$

Since $\frac{1}{\Gamma(\xi(n-k)+1)}=0$ for $k>n$ and $\frac{1}{\Gamma(\zeta(m-h+k)+1)}=0$ for h $>m+k$

$$L=\sum_{k=0}^{m+n}\sum_{h=k}^{m+n}\frac{\Gamma\left(\alpha n+\beta+1\right)\Gamma\left(\gamma m+\delta+1\right)}{\Gamma\left(h-k+1\right)\Gamma\left(\zeta\left(m-h+k\right)+1\right)\Gamma\left(k+1\right)\Gamma\left(\xi\left(n-k\right)+1\right)}$$
$$\times\frac{\left(-1\right)^{h}x^{h}}{\Gamma\left(\alpha k+\beta+1\right)\Gamma\left(\gamma\left(h-k\right)+\delta+1\right)}.$$

Further simplification gives,

(2.6)

$$L=\sum_{h=0}^{m+n}\sum_{k=0}^{h}\frac{\Gamma\left(\alpha n+\beta+1\right)\Gamma\left(\gamma m+\delta+1\right)}{\Gamma\left(h-k+1\right)\Gamma\left(\zeta\left(m-h+k\right)+1\right)\Gamma\left(k+1\right)\Gamma\left(\xi\left(n-k\right)+1\right)}$$
$$\times\frac{\left(-1\right)^{h}x^{h}}{\Gamma\left(\alpha k+\beta+1\right)\Gamma\left(\gamma\left(h-k\right)+\delta+1\right)}.$$

Now, denoting a new integral expression by I, and

$$I=\int_{0}^{\infty}x^{s+1}e^{-\sigma x^{2}}L_{m}^{(\gamma,\delta)}\left(\zeta;\sigma x^{2}\right)L_{n}^{(\alpha,\beta)}\left(\xi;\sigma x^{2}\right)J_{s}\left(xy\right)dx,$$

which, upon using (2.6), yields

$$I=\int_{0}^{\infty}x^{s+1}e^{-\sigma x^{2}}\left[\begin{array}{l}\sum_{h=0}^{m+n}\sum_{k=0}^{h}\frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma m+\delta+1)}{\Gamma(h-k+1)\Gamma(\zeta(m-h+k)+1)\Gamma(k+1)}\\ \times\frac{(-1)^{h}\left(\sigma x^{2}\right)^{h}}{\Gamma(\xi(n-k)+1)\Gamma(\alpha k+\beta+1)\Gamma(\gamma(h-k)+\delta+1)}\end{array}\right]J_{s}\left(xy\right)dx.$$

Interchanging the order of the integration and summation, we have

$$I=\sum_{h=0}^{m+n}\sum_{k=0}^{h}\left[\begin{array}{l}\frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma m+\delta+1)}{\Gamma(h-k+1)\Gamma(\zeta(m-h+k)+1)\Gamma(k+1)}\\ \times\frac{(-1)^{h}\sigma^{h}}{\Gamma(\xi(n-k)+1)\Gamma(\alpha k+\beta+1)\Gamma(\gamma(h-k)+\delta+1)}\end{array}\right]$$
$$\times\int_{0}^{\infty}x^{2h+s+1}e^{-\sigma x^{2}}J_{s}\left(xy\right)dx.$$

and by making use of (1.16), gives

$$I=\sum_{h=0}^{m+n}\sum_{k=0}^{h}\left[\begin{array}{l}\frac{\Gamma(\alpha n+\beta+1)\Gamma(\gamma m+\delta+1)}{\Gamma(h-k+1)\Gamma(\zeta(m-h+k)+1)\Gamma(k+1)}\\ \times\frac{(-1)^{h}\sigma^{h}}{\Gamma(\xi(n-k)+1)\Gamma(\alpha k+\beta+1)\Gamma(\gamma(h-k)+\delta+1)}\end{array}\right]$$
$$\times\frac{h!y^{s}\sigma^{-h}}{\left(2\sigma\right)^{s+1}}\exp\left(-\frac{y^{2}}{4\sigma}\right)L_{h}^{s}\left(\frac{y^{2}}{4\sigma}\right).$$

This can also be written as

(2.7)

$$I = \frac{y^s}{(2\sigma)^{s+1}} \exp\left(-\frac{y^2}{4\sigma}\right) \sum_{h=0}^{m+n} \sum_{k=0}^{h} L_h^s\left(\frac{y^2}{4\sigma}\right) \binom{h}{k}$$
$$\times \frac{\Gamma(\alpha n+\beta+1)\,\Gamma(\gamma m+\delta+1)\,(-1)^h}{\Gamma(\zeta(m-h+k)+1)\,\Gamma(\xi(n-k)+1)\,\Gamma(\alpha k+\beta+1)\,\Gamma(\gamma(h-k)+\delta+1)}.$$

Thus, we have

$$I = \frac{y^s}{(2\sigma)^{s+1}} \exp\left(-\frac{y^2}{4\sigma}\right) \sum_{h=0}^{m+n} \Delta_{\alpha,\beta;\gamma,\delta}^{n,m,\xi.\zeta}(h)\, L_h^s\left(\frac{y^2}{4\sigma}\right),$$

where $\Delta_{\alpha,\beta;\gamma,\delta}^{n,m,\xi.\zeta}(h)$ is defined by (2.2).

(b) Let $\xi, \zeta \in \mathbb{N}$ and using (1.18), (2.7) becomes

$$I = \frac{y^s}{(2\sigma)^{s+1}} \exp\left(-\frac{y^2}{4\sigma}\right) \frac{\Gamma(\alpha n+\beta+1)\,\Gamma(\gamma m+\delta+1)}{\Gamma(\zeta m+1)\,\Gamma(\xi n+1)}$$
$$\times \sum_{h=0}^{m+n} L_h^s\left(\frac{y^2}{4\sigma}\right) \sum_{k=0}^{h} \left[\binom{h}{k} \frac{(-1)^{h-\zeta(h-k)-\xi k}\,(-\zeta m)_{\zeta(h-k)}\,(-\xi n)_{\xi k}}{\Gamma(\alpha k+\beta+1)\,\Gamma(\gamma(h-k)+\delta+1)}\right].$$

Thus, we have

$$I = \frac{y^s}{(2\sigma)^{s+1}} \exp\left(-\frac{y^2}{4\sigma}\right) \sum_{h=0}^{m+n} \nabla_{\alpha,\beta;\gamma,\delta}^{n,m,\xi.\zeta}(h)\, L_h^s\left(\frac{y^2}{4\sigma}\right),$$

where $\nabla_{\alpha,\beta;\gamma,\delta}^{n,m,\xi.\zeta}(h)$ is defined by (2.4).

This completes the proof. □

3. Integral formula

Theorem 3.1. *Some integral formulae* (*as a special case of* (2.1)) *have been obtained as,*

(a)

(3.1)
$$\int_0^\infty x^{s+1} e^{-\sigma x^2} L_m^{(\gamma,\delta)}(\sigma x^2)\, L_n^{(\alpha,\beta)}(\sigma x^2)\, J_s(xy)\, dx$$
$$= \frac{y^s}{(2\sigma)^{s+1}} \exp\left(-\frac{y^2}{4\sigma}\right) \frac{\Gamma(\alpha n+\beta+1)\,\Gamma(\gamma m+\delta+1)}{m!n!}$$
$$\times \sum_{h=0}^{m+n} L_h^s\left(\frac{y^2}{4\sigma}\right) \sum_{k=0}^{h} \binom{h}{k} \frac{(-m)_{h-k}\,(-n)_k}{\Gamma(\alpha k+\beta+1)\,\Gamma(\gamma(h-k)+\delta+1)}.$$

(b)

$$\int_0^\infty x^{s+1}e^{-\sigma x^2}L_m^\delta\left(\sigma x^2\right)L_n^\beta\left(\sigma x^2\right)J_s\left(xy\right)dx \tag{3.2}$$

$$= \frac{y^s}{(2\sigma)^{s+1}}\exp\left(-\frac{y^2}{4\sigma}\right)\frac{\Gamma\left(n+\beta+1\right)\Gamma\left(m+\delta+1\right)}{m!n!}$$

$$\times\sum_{h=0}^{m+n}L_h^s\left(\frac{y^2}{4\sigma}\right)\sum_{k=0}^{h}\left(\begin{array}{c}h\\k\end{array}\right)\frac{(-m)_{h-k}\,(-n)_k}{\Gamma\left(k+\beta+1\right)\Gamma\left(h-k+\delta+1\right)}.$$

(c)

$$\int_0^\infty xe^{-\sigma x}L_m^{(\gamma,\delta)}\left(\sigma x\right)L_n^{(\alpha,\beta)}\left(\sigma x\right)dx$$

$$=\frac{\Gamma\left(\alpha n+\beta+1\right)\Gamma\left(m\gamma+\delta+1\right)}{2\sigma\Gamma\left(n+1\right)\Gamma\left(m+1\right)}$$

$$\times\sum_{h=0}^{m+n}\sum_{k=0}^{h}\left(\begin{array}{c}h\\k\end{array}\right)\frac{(-n)_k\,(-m)_{h-k}}{\Gamma\left(\alpha k+\beta+1\right)\Gamma\left(\gamma\left(h-k\right)+\delta+1\right)}.$$

(d)

$$\int_0^\infty x^{s+1}e^{-\sigma x^2}L_m^{s-\beta}\left(\sigma x^2\right)L_n^\beta\left(\sigma x^2\right)J_s\left(xy\right)dx \tag{3.3}$$

$$= \frac{1}{2\sigma}\left(\frac{y}{2\sigma}\right)^s\exp\left(-\frac{y^2}{4\sigma}\right)(-1)^{m+n}\,L_m^{\beta-m+n}\left(\frac{y^2}{4\sigma}\right)L_n^{s-\beta+m-n}\left(\frac{y^2}{4\sigma}\right).$$

(e)

$$\int_0^\infty x^{s+1}e^{-\sigma x^2}Z_m^{(\zeta,\delta)}\left(x;2\right)Z_n^{(\xi,\beta)}\left(x;2\right)J_s\left(xy\right)dx$$

$$= \frac{y^s}{(2)^{s+1}}\exp\left(-\frac{y^2}{4}\right)\frac{\Gamma\left(2n+\beta+1\right)\Gamma\left(2m+\delta+1\right)}{\Gamma\left(\zeta m+1\right)\Gamma\left(\xi n+1\right)}\sum_{h=0}^{m+n}L_h^s\left(\frac{y^2}{4}\right)$$

$$\times\sum_{k=0}^{h}\left[\left(\begin{array}{c}h\\k\end{array}\right)\frac{(-1)^h}{\Gamma\left(2k+\beta+1\right)\Gamma\left(2\left(h-k\right)+\delta+1\right)\Gamma\left(\zeta\left(m-h+k\right)+1\right)\Gamma\left(\xi\left(n-k\right)+1\right)}\right].$$

(f)

$$\int_0^\infty x^{s+1}e^{-\sigma x^2}Z_m^{\delta}\left(x;2\right)Z_n^{\beta}\left(x;2\right)J_s\left(xy\right)dx$$

$$= \frac{y^s}{(2)^{s+1}}\exp\left(-\frac{y^2}{4}\right)\frac{\Gamma\left(2n+\beta+1\right)\Gamma\left(2m+\delta+1\right)}{m!n!}\sum_{h=0}^{m+n}L_h^s\left(\frac{y^2}{4}\right)$$

$$\times\sum_{k=0}^{h}\left[\left(\begin{array}{c}h\\k\end{array}\right)\frac{(-1)^h}{\Gamma\left(2k+\beta+1\right)\Gamma\left(2\left(h-k\right)+\delta+1\right)\left(m-h+k\right)!\left(n-k\right)!}\right].$$

Proof. (a) On setting $\xi = \zeta = 1$ in (2.3) and (2.4), we get

$$\int_0^\infty x^{s+1} e^{-\sigma x^2} L_m^{(\gamma,\delta)}\left(1;\sigma x^2\right) L_n^{(\alpha,\beta)}\left(1;\sigma x^2\right) J_s(xy)\,dx$$
$$= \frac{y^s}{(2\sigma)^{s+1}} \exp\left(-\frac{y^2}{4\sigma}\right) \frac{\Gamma(\alpha n+\beta+1)\,\Gamma(\gamma m+\delta+1)}{\Gamma(m+1)\,\Gamma(n+1)}$$
$$\times \sum_{h=0}^{m+n}\sum_{k=0}^{h}\left[\binom{h}{k}\frac{(-m)_{h-k}(-n)_k}{\Gamma(\alpha k+\beta+1)\,\Gamma(\gamma(h-k)+\delta+1)}\right] L_h^s\left(\frac{y^2}{4\sigma}\right).$$

Further using (1.10), we arrive at (3.1).

(b) On setting $\gamma = \alpha = 1$ in (3.1), we get

$$\int_0^\infty x^{s+1} e^{-\sigma x^2} L_m^{(1,\delta)}\left(\sigma x^2\right) L_n^{(1,\beta)}\left(\sigma x^2\right) J_s(xy)\,dx$$
$$= \frac{y^s}{(2\sigma)^{s+1}} \exp\left(-\frac{y^2}{4\sigma}\right) \frac{\Gamma(n+\beta+1)\,\Gamma(m+\delta+1)}{\Gamma(m+1)\,\Gamma(n+1)}$$
$$\times \sum_{h=0}^{m+n}\sum_{k=0}^{h}\left[\binom{h}{k}\frac{(-m)_{h-k}(-n)_k}{\Gamma(k+\beta+1)\,\Gamma((h-k)+\delta+1)}\right] L_h^s\left(\frac{y^2}{4\sigma}\right).$$

Further using (1.2), we arrive at (3.2).

(c) Now using (2.4), we have

$$\nabla_{0,\beta;0,\delta}^{n,m,1,1}(h) = \frac{1}{\Gamma(n+1)\,\Gamma(m+1)}\sum_{k=0}^{h}\binom{h}{k}(-n)_k(-m)_{h-k}.$$

Afterwards (1.19) gives,

$$\nabla_{0,\beta;0,\delta}^{n,m,1,1}(h) = \frac{1}{\Gamma(n+1)\,\Gamma(m+1)}(-m-n)_h\,.$$

Let $\alpha = \gamma = 0, \xi = \zeta = 1$ in (2.3), we have

$$\text{(3.4)}\qquad \int_0^\infty x^{s+1} e^{-\sigma x^2} L_m^{(0,\delta)}\left(1;\sigma x^2\right) L_n^{(0,\beta)}\left(1;\sigma x^2\right) J_s(xy)\,dx$$
$$= \frac{y^s}{(2\sigma)^{s+1}\,\Gamma(n+1)\,\Gamma(m+1)} \exp\left(-\frac{y^2}{4\sigma}\right)\sum_{h=0}^{m+n}(-m-n)_h\,L_h^s\left(\frac{y^2}{4\sigma}\right).$$

Also, applying (1.2) and (1.18)-(1.21), we have

$$\text{(3.5)}\qquad L_n^{(0,\beta)}(x) = \frac{1}{\Gamma(n+1)}\sum_{k=0}^{n}\binom{n}{k}(-x)^k = \frac{1}{\Gamma(n+1)}(1-x)^n.$$

From (3.4) and (3.5), we have

$$\int_0^\infty x^{s+1}e^{-\sigma x^2}\left(1-\sigma x^2\right)^{m+n}J_s(xy)\,dx$$
$$=\frac{y^s}{(2\sigma)^{s+1}}\exp\left(-\frac{y^2}{4\sigma}\right)\sum_{h=0}^{m+n}(-m-n)_h\,L_h^s\left(\frac{y^2}{4\sigma}\right).$$

On replacing $m+n$ by n, we have

$$\int_0^\infty x^{s+1}e^{-\sigma x^2}\left(1-\sigma x^2\right)^{n}J_s(xy)\,dx \tag{3.6}$$
$$=\frac{y^s}{(2\sigma)^{s+1}}\exp\left(-\frac{y^2}{4\sigma}\right)\sum_{h=0}^{n}(-n)_h\,L_h^s\left(\frac{y^2}{4\sigma}\right).$$

Let $s=0$, $y=0$, $\xi=\zeta=1$ in (2.4), we get

$$\int_0^\infty xe^{-\sigma x^2}L_m^{(\gamma,\delta)}\left(\sigma x^2\right)L_n^{(\alpha,\beta)}\left(\sigma x^2\right)dx=\frac{1}{2\sigma}\sum_{h=0}^{m+n}\nabla_{\alpha,\beta;\gamma,\delta}^{n,m,1,1}(h).$$

This can be written in a simplified form as,

$$\int_0^\infty xe^{-\sigma x}L_m^{(\gamma,\delta)}\left(\sigma x^2\right)L_n^{(\alpha,\beta)}\left(\sigma x^2\right)dx \tag{3.7}$$
$$=\frac{\Gamma(\alpha n+\beta+1)\,\Gamma(m\gamma+\delta+1)}{2\sigma\Gamma(n+1)\,\Gamma(m+1)}$$
$$\times\sum_{h=0}^{m+n}\sum_{k=0}^{h}\begin{pmatrix}h\\k\end{pmatrix}\frac{(-n)_k\,(-m)_{h-k}}{\Gamma(\alpha k+\beta+1)\,\Gamma(\gamma(h-k)+\delta+1)}.$$

This completes the proof of (c).

Now, we deduce the Hankel transform integral containing an exponential function and two Laguerre polynomials from (3.1).

(d) On setting $\alpha=1$, $\gamma=1$ and $\delta=s-\beta$ in (3.1) and using (1.2), we have

$$\int_0^\infty x^{s+1}e^{-\sigma x^2}L_m^{s-\beta}\left(\sigma x^2\right)L_n^{\beta}\left(\sigma x^2\right)J_s(xy)\,dx \tag{3.8}$$
$$=\frac{y^s}{(2\sigma)^{s+1}}\exp\left(-\frac{y^2}{4\sigma}\right)\frac{\Gamma(n+\beta+1)\,\Gamma(m+s-\beta+1)}{\Gamma(n+1)\,\Gamma(m+1)}$$
$$\times\sum_{h=0}^{m+n}L_h^s\left(\frac{y^2}{4\sigma}\right)\sum_{k=0}^{h}\begin{pmatrix}h\\k\end{pmatrix}\frac{(-n)_k\,(-m)_{h-k}}{\Gamma(k+\beta+1)\,\Gamma((h-k)+s-\beta+1)}.$$

From (1.21) and (3.8), we have

(3.9)

$$=\frac{y^s}{(2\sigma)^{s+1}}\exp\left(-\frac{y^2}{4\sigma}\right)\sum_{h=0}^{m+n}L_h^s\left(\frac{y^2}{4\sigma}\right)\sum_{k=0}^{h}(-1)^h\begin{pmatrix}h\\k\end{pmatrix}\begin{pmatrix}m+s-\beta\\h-k+s-\beta\end{pmatrix}\begin{pmatrix}n+\beta\\k+\beta\end{pmatrix}.$$

On applying $\binom{n}{k} = \binom{n}{n-k}$ to (3.9), we obtain

(3.10)

$$= \frac{y^s}{(2\sigma)^{s+1}} \exp\left(-\frac{y^2}{4\sigma}\right) \sum_{h=0}^{m+n} L_h^s\left(\frac{y^2}{4\sigma}\right) \sum_{k=0}^{h} (-1)^h \binom{h}{h-k} \binom{n+\beta}{n-k} \binom{m+s-\beta}{m-h+k}.$$

In second summation, starting the terms from the end

$$= \frac{y^s}{(2\sigma)^{s+1}} \exp\left(-\frac{y^2}{4\sigma}\right) \sum_{h=0}^{m+n} L_h^s\left(\frac{y^2}{4\sigma}\right) \sum_{k=0}^{h} (-1)^h \binom{h}{k} \binom{n+\beta}{n-h+k} \binom{m+s-\beta}{m-k}.$$

And by using (1.17), we get

$$\int_0^\infty x^{s+1} e^{-\sigma x^2} L_m^{s-\beta}\left(\sigma x^2\right) L_n^{\beta}\left(\sigma x^2\right) J_s\left(xy\right) dx$$
$$= \frac{1}{2\sigma}\left(\frac{y}{2\sigma}\right)^s \exp\left(-\frac{y^2}{4\sigma}\right) (-1)^{m+n} L_m^{\beta-m+n}\left(\frac{y^2}{4\sigma}\right) L_n^{s-\beta+m-n}\left(\frac{y^2}{4\sigma}\right).$$

This completes the proof of (d).

(e) On setting $\sigma = 1$ and $\alpha = \gamma = 2$ in (2.1)

$$\int_0^\infty x^{s+1} e^{-\sigma x^2} L_m^{(2,\delta)}\left(\zeta; x^2\right) L_n^{(2,\beta)}\left(\xi; x^2\right) J_s\left(xy\right) dx$$
$$= \frac{y^s}{(2)^{s+1}} \exp\left(-\frac{y^2}{4}\right) \sum_{h=0}^{m+n} \Delta_{2,\beta;2,\delta}^{n,m,\xi,\zeta}\left(h\right) L_h^s\left(\frac{y^2}{4}\right),$$

where $\Delta_{\alpha,\beta;\gamma,\delta}^{n,m,\xi.\zeta}\left(h\right)$ is given by (2.2). Further using (1.11), we get result (e).

(f) On setting $\xi = \zeta = 1$ in result (e) and using (1.7) gives result (f).

This completes the proof. □

4. Maple implementation

In this section, we examine the implementation of the scientific-technical computing system Maple to obtain

$$I = \int_0^\infty x^{s+1} e^{-\sigma x^2} L_m^{(\chi,\delta)}\left(\eta; \sigma x^2\right) L_n^{(\alpha,\beta)}\left(\xi; \sigma x^2\right) J_s\left(xy\right) dx \tag{4.1}$$
$$= \frac{y^s}{(2\sigma)^{s+1}} \exp\left(-\frac{y^2}{4\sigma}\right) \sum_{h=0}^{m+n} \Delta_{\alpha,\beta;\chi,\delta}^{n,m,\xi,\eta}\left(h\right) L_h^s\left(\frac{y^2}{4\sigma}\right),$$

where $y \in \mathbb{R}^+; \sigma, \xi, \zeta \in \mathbb{C}^+;\ s, \alpha, \beta, \chi, \delta \in \mathbb{C}_{-1}^+$ and $\Delta_{\alpha,\beta;\chi,\delta}^{n,m,\xi,\eta}\left(h\right)$ is defined by (2.2).

To find (4.1), start new Maple windows in '**Worksheet Mode**' with default '**Typesetting Rules**' and type the following maple code:

```
> restart;
> assume(y > 0);
> assume(n > 0);
> assume(m > 0);
```

```
> additionally(n::integer);
> additionally(m::integer);
> assume(Re(alpha) > -1);
> assume(Re(beta) > -1);
> assume(Re(chi) > -1);
> assume(Re(delta)> -1);
> assume(Re(sigma) > 0);
> assume(Re(xi) > 0);
> assume(Re(eta) > 0);
> assume(Re(s) > -1);
> f := proc (alpha, beta, chi, delta, m, n, xi, eta, sigma, s, y)
options operator, arrow;
y^s*exp((1/4)*y^2/sigma)*(sum(sum(GAMMA(alpha*n+beta+1)
*GAMMA(chi*m+delta+1)*factorial(h)*(-1)^h
*LaguerreL(h, s, (1/4)*y^2/sigma)(factorial(h-k)*factorial(k)
*GAMMA(alpha*k+beta+1)*GAMMA(eta*(m-h+k)+1)
*GAMMA(xi*(n-k)+1)*GAMMA(chi*(h-k)+delta+1)),k = 0 .. h),
h = 0 .. m+n))/(2*sigma)^(s+1) end proc;
```

On putting particular value for different parameters in following expression,

```
> f(alpha, beta, chi, delta, m, n, xi, eta, sigma, s, y);
```

yields (4.1) for that particular values.

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