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IMPLICATIVE FILTERS OF R_0-ALGEBRAS BASED ON FUZZY POINTS

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Abstract. As a generalization of the concept of a fuzzy implicative filter which is introduced by Liu and Li [3], the notion of $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy implicative filters is introduced, and related properties are investigated. The relationship between $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy filters and $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy implicative filters is established. Conditions for an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy filter to be an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy implicative filter are considered. Characterizations of an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy implicative filter are provided, and the implication-based fuzzy implicative filters of an R_0-algebra is discussed.

1. Introduction

One important task of artificial intelligence is to make the computers simulate beings in dealing with certainty and uncertainty in information. Logic appears in a "sacred" (respectively, a "profane") form which is dominant in proof theory (respectively, model theory). The role of logic in mathematics and computer science is twofold – as a tool for applications in both areas, and a technique for laying the foundations. Non-classical logic including many-valued logic, fuzzy logic, etc., takes the advantage of classical logic to handle information with various facets of uncertainty (see [11] for generalized theory of uncertainty), such as fuzziness, randomness etc. Non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. Among all kinds of uncertainties, incomparability is an important one which can be encountered in our life. The concept of R_0-algebras was first introduced by Wang in [7] by providing an algebraic proof of the completeness theorem of a formal deductive system [8]. Obviously, R_0-algebras are different from the BL-algebras. Jun and Liu [1] studied filters of R_0-algebras. Liu and Li [3] discussed the fuzzy set theory of implicative filters in R_0-algebras, and introduced the notion of a fuzzy implicative filter. As a generalization of the notion of fuzzy filters, Ma et al. [4] dealt with the

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notion of ($\in$, $\in\vee$q)-fuzzy filters in R_0-algebras. In [2], Jun et al. discussed more general form of the notion of ($\in$, $\in\vee$q)-fuzzy filters.

In this article, as a generalization of the concept of fuzzy implicative filters, we introduce the notion of ($\in$, $\in\vee\mathrm{q}_k$)-fuzzy implicative filters, and deal with related properties. We investigate the relationship between ($\in$, $\in\vee\mathrm{q}_k$)-fuzzy filters and ($\in$, $\in\vee\mathrm{q}_k$)-fuzzy implicative filters. We consider conditions for an ($\in$, $\in\vee\mathrm{q}_k$)-fuzzy filter to be an ($\in$, $\in\vee\mathrm{q}_k$)-fuzzy implicative filter. We establish characterizations of an ($\in$, $\in\vee\mathrm{q}_k$)-fuzzy implicative filter, and finally discuss the implication-based fuzzy implicative filters of an R_0-algebra.

2. Preliminaries

Let L be a bounded distributive lattice with order-reversing involution $\neg$ and a binary operation $\to$. Then $(L, \wedge, \vee, \neg, \to)$ is called an *R_0-algebra* (see [7]) if it satisfies the following axioms:

(R1) $x \to y = \neg y \to \neg x$,
(R2) $1 \to x = x$,
(R3) $(y \to z) \wedge ((x \to y) \to (x \to z)) = y \to z$,
(R4) $x \to (y \to z) = y \to (x \to z)$,
(R5) $x \to (y \vee z) = (x \to y) \vee (x \to z)$,
(R6) $(x \to y) \vee ((x \to y) \to (\neg x \vee y)) = 1$.

Let L be an R_0-algebra. For any $x, y \in L$, we define $x \odot y = \neg(x \to \neg y)$ and $x \oplus y = \neg x \to y$. It is proved that $\odot$ and $\oplus$ are commutative, associative and $x \oplus y = \neg(\neg x \odot \neg y)$, and $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice.

For any elements x, y and z of a R_0-algebra L, we have the following properties (see [5]).

(a1) $x \le y$ if and only if $x \to y = 1$,
(a2) $x \le y \to x$,
(a3) $\neg x = x \to 0$,
(a4) $(x \to y) \vee (y \to x) = 1$,
(a5) $x \le y$ implies $y \to z \le x \to z$,
(a6) $x \le y$ implies $z \to x \le z \to y$,
(a7) $((x \to y) \to y) \to y = x \to y$,
(a8) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,
(a9) $x \odot \neg x = 0$ and $x \oplus \neg x = 1$,
(a10) $x \odot y \le x \wedge y$ and $x \odot (x \to y) \le x \wedge y$,
(a11) $(x \odot y) \to z = x \to (y \to z)$,
(a12) $x \le y \to (x \odot y)$,
(a13) $x \odot y \le z$ if and only if $x \le y \to z$,
(a14) $x \le y$ implies $x \odot z \le y \odot z$,
(a15) $x \to y \le (y \to z) \to (x \to z)$,
(a16) $(x \to y) \odot (y \to z) \le x \to z$.

A non-empty subset A of an R_0-algebra L is called an *implicative filter* of L if it satisfies:

(b1) $1 \in A$,

(b2) $(\forall x, y, z \in A)$ $(x \to (y \to z) \in A \;\&\; x \to y \in A \Longrightarrow x \to z \in A)$.

A fuzzy set μ in an R_0-algebra L is called a *fuzzy implicative filter* of L if it satisfies:

(b3) $(\forall x \in L)$ $(\mu(1) \geq \mu(x))$,

(b4) $(\forall x, y, z \in L)$ $(\mu(x \to z) \geq \min\{\mu(x \to (y \to z)), \mu(x \to y)\})$.

For any fuzzy set μ in L and $t \in (0, 1]$, the set

$$U(\mu; t) = \{x \in L \mid \mu(x) \geq t\}$$

is called a *level subset* of L. A fuzzy set μ in a set L of the form

$$\mu(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases} \tag{1}$$

is said to be a *fuzzy point* with support x and value t and is denoted by (x, t).

For a fuzzy point (x, t) and a fuzzy set μ in a set L, Pu and Liu [6] introduced the symbol $(x, t)\alpha\mu$, where $\alpha \in \{\in, \mathrm{q}, \in\vee\mathrm{q}, \in\wedge\mathrm{q}\}$. To say that $(x, t) \in \mu$ (resp. $(x, t)\mathrm{q}\mu$), we mean $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, (x, t) is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set μ. To say that $(x, t) \in\vee\mathrm{q}\,\mu$ (resp. $(x, t) \in\wedge\mathrm{q}\,\mu$), we mean $(x, t) \in \mu$ or $(x, t)\mathrm{q}\mu$ (resp. $(x, t) \in \mu$ and $(x, t)\mathrm{q}\mu$).

3. Implicative filters based on fuzzy points

In what follows, L is an R_0-algebra and let k denote an arbitrary element of $[0, 1)$ unless otherwise specified. To say that $(x, t)\ \mathrm{q}_k\ \mu$, we mean $\mu(x) + t + k > 1$. To say that $(x, t) \in\vee\mathrm{q}_k\ \mu$, we mean $(x, t) \in \mu$ or $(x, t)\ \mathrm{q}_k\ \mu$. For $\alpha \in \{\in, \in\vee\mathrm{q}_k\}$, to say that $(x, t)\overline{\alpha}\mu$, we mean $(x, t)\alpha\mu$ does not hold.

Definition 3.1. A fuzzy set μ in L is called an $(\in, \in\vee\mathrm{q}_k)$*-fuzzy implicative filter* of L if it satisfies:

(c1) $(x, t) \in \mu \implies (1, t) \in\vee\mathrm{q}_k\,\mu$,

(c2) $(x \to y, t) \in \mu \;\&\; (x \to (y \to z), r) \in \mu \;\Rightarrow\; (x \to z, \min\{t, r\}) \in\vee\mathrm{q}_k\,\mu$

for all $x, y, z \in L$ and $t, r \in (0, 1]$.

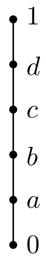
An $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter of L with $k = 0$ is called an $(\in, \in\vee\mathrm{q})$*-fuzzy implicative filter* of L.

Example 3.2. Let $L = \{0, a, b, c, d, 1\}$ be a set with Hasse diagram and Cayley tables which are given in Table 1. Then $(L, \wedge, \vee, \neg, \to, 0, 1)$ is an R_0-algebra (see [3]), where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Define a fuzzy set μ in L by

$$\mu = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.3 & 0.2 & 0.3 & 0.7 & 0.7 & 0.45 \end{pmatrix}.$$

It is routine to verify that μ is an $(\in, \in\vee\mathrm{q}_{0.16})$-fuzzy implicative filter of L. But it is neither a fuzzy implicative filter nor an $(\in, \in\vee\mathrm{q})$-fuzzy implicative filter of L.

Table 1. Hasse diagram and Cayley tables



x	$\neg x$
0	1
a	d
b	c
c	b
d	a
1	0

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	c	c	1	1	1	1
c	b	b	b	1	1	1
d	a	a	b	c	1	1
1	0	a	b	c	d	1

We establish characterizations of an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter.

Theorem 3.3. *A fuzzy set μ in L is an $(\in,\in\vee\mathrm{q}_k)$-fuzzy implicative filter of L if and only if it satisfies two conditions*:

(c3) $(\forall x \in L)\ (\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\})$,

(c4) $(\forall x, y, z \in L)\ (\mu(x \to z) \geq \min\{\mu(x \to y), \mu(x \to (y \to z)), \frac{1-k}{2}\})$.

Proof. Assume that μ is an $(\in,\in\vee\mathrm{q}_k)$-fuzzy implicative filter of L. If (c3) is not valid, then $\mu(1) < t_a \leq \min\{\mu(a), \frac{1-k}{2}\}$ for some $a \in L$ and $t_a \in (0, \frac{1-k}{2}]$. Thus $(a, t_a) \in \mu$ but $(1, t_a)\,\overline{\in}\,\mu$. Also, $\mu(1) + t_a < 2t_a \leq 1 - k$, i.e., $(1, t_a)\,\overline{\mathrm{q}_k}\,\mu$. Therefore $(1, t_a)\,\overline{\in\vee\mathrm{q}_k}\,\mu$, a contradiction. Consequently, $\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\}$ for all $x \in L$. Assume that (c4) is not valid. Then there exist $a, b, c \in L$ and $t \in (0, \frac{1-k}{2}]$ such that

$$\mu(a \to c) < t \leq \min\left\{\mu(a \to b), \mu(a \to (b \to c)), \tfrac{1-k}{2}\right\}.$$

If $\min\{\mu(a \to b), \mu(a \to (b \to c))\} < \frac{1-k}{2}$, then

$$\mu(a \to c) < t \leq \min\{\mu(a \to b), \mu(a \to (b \to c))\}.$$

Hence $(a \to b, t) \in \mu$ and $(a \to (b \to c), t) \in \mu$ but $(a \to c, t)\,\overline{\in}\,\mu$. Moreover,

$$\mu(a \to c) + t < 2t \leq 1 - k,$$

and so $(a \to c, t)\,\overline{\mathrm{q}_k}\,\mu$. Therefore $(a \to c, t)\,\overline{\in\vee\mathrm{q}_k}\,\mu$, a contradiction. If

$$\min\{\mu(a \to b), \mu(a \to (b \to c))\} \geq \tfrac{1-k}{2},$$

then $(a \to b, \frac{1-k}{2}) \in \mu$ and $(a \to (b \to c), \frac{1-k}{2}) \in \mu$ but $(a \to c, \frac{1-k}{2})\,\overline{\in}\,\mu$. Also, $\mu(a \to c) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$, i.e., $(a \to c, \frac{1-k}{2})\,\overline{\mathrm{q}_k}\,\mu$. Hence $(a \to c, \frac{1-k}{2})\,\overline{\in\vee\mathrm{q}_k}\,\mu$ which is a contradiction. Consequently,

$$\mu(x \to z) \geq \min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\}$$

for all $x, y, z \in L$. Conversely, let μ be a fuzzy set in L satisfying (c3) and (c4). Let $x \in L$ and $t \in (0, 1]$ be such that $(x, t) \in \mu$. Then $\mu(x) \geq t$, and so $\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\}$. If $t \leq \frac{1-k}{2}$, then $\mu(1) \geq t$, i.e., $(1, t) \in \mu$. If $t > \frac{1-k}{2}$, then $\mu(1) \geq \frac{1-k}{2}$. Thus $\mu(1) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$, i.e., $(1, t)\,\mathrm{q}_k\,\mu$. Hence $(1, t) \in\vee\mathrm{q}_k\,\mu$, which proves (c1). Let $x, y, z \in L$ and $t, r \in (0, 1]$ be such that $(x \to y, t) \in \mu$ and $(x \to (y \to z), r) \in \mu$. Then $\mu(x \to y) \geq t$ and $\mu(x \to (y \to z)) \geq r$. It follows from (c4) that

$$\begin{aligned} \mu(x \to z) &\geq \min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\} \\ &\geq \min\left\{t, r, \tfrac{1-k}{2}\right\} \\ &= \begin{cases} \min\{t, r\} & \text{if } t \leq \frac{1-k}{2} \text{ or } r \leq \frac{1-k}{2}, \\ \frac{1-k}{2} & \text{if } t > \frac{1-k}{2} \text{ and } r > \frac{1-k}{2}. \end{cases} \end{aligned}$$

The case $\mu(x \to z) \geq \min\{t, r\}$ implies that $(x \to z, \min\{t, r\}) \in \mu$. From the case $\mu(x \to z) \geq \frac{1-k}{2}$, we have

$$\mu(x \to z) + \min\{t, r\} > \tfrac{1-k}{2} + \tfrac{1-k}{2} = 1 - k,$$

i.e., $(x \to z, \min\{t, r\})\, \mathrm{q}_k\, \mu$. Hence $(x \to z, \min\{t, r\}) \in\vee\, \mathrm{q}_k\, \mu$. Therefore the condition (c2) is valid. Consequently, μ is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy implicative filter of L. □

Proposition 3.4. *Every $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy implicative filter μ of L satisfies the following inequality*:

$$\mu(x \to z) \geq \min\left\{\mu(y \to z), \mu(x \to (\neg z \to y)), \tfrac{1-k}{2}\right\} \tag{2}$$

for all $x, y, z \in L$.

Proof. Note that

$$\begin{aligned} \neg z \to (\neg y \to \neg x) &= \neg(\neg y \to \neg x) \to \neg\neg z = \neg(x \to y) \to z \\ &= \neg z \to \neg\neg(x \to y) = \neg z \to (x \to y) \\ &= x \to (\neg z \to y) \end{aligned}$$

for all $x, y, z \in L$. It follows from (c4)and (R1) that

$$\begin{aligned} \mu(x \to z) &= \mu(\neg z \to \neg x) \\ &\geq \min\left\{\mu(\neg z \to \neg y), \mu(\neg z \to (\neg y \to \neg x)), \tfrac{1-k}{2}\right\} \\ &= \min\left\{\mu(y \to z), \mu(x \to (\neg z \to y)), \tfrac{1-k}{2}\right\} \end{aligned}$$

for all $x, y, z \in L$. □

Corollary 3.5. *Every $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy implicative filter μ of L satisfies the following inequality*:

$$\mu(x \to y) \geq \min\left\{\mu(x \to (\neg y \to y)), \tfrac{1-k}{2}\right\} \tag{3}$$

for all $x, y \in L$.

Proof. Taking $z = y$ in (2) and using (c3) imply that

$$\begin{aligned} \mu(x \to y) &\geq \min\left\{\mu(y \to y), \mu(x \to (\neg y \to y)), \tfrac{1-k}{2}\right\} \\ &= \min\left\{\mu(1), \mu(x \to (\neg y \to y)), \tfrac{1-k}{2}\right\} \\ &\geq \min\left\{\min\left\{\mu(x \to (\neg y \to y)), \tfrac{1-k}{2}\right\}, \mu(x \to (\neg y \to y)), \tfrac{1-k}{2}\right\} \\ &= \min\left\{\mu(x \to (\neg y \to y)), \tfrac{1-k}{2}\right\} \end{aligned}$$

for all $x, y \in L$. $\square$

Proposition 3.6. *Let μ be a fuzzy set in L satisfying two conditions* (c3) *and* (2). *Then μ is an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter of L.*

Proof. It is sufficient to show that μ satisfies the condition (c4). From (R1), (R4) and (2), we have

$$\begin{aligned}\mu(x \to z) &= \mu(\neg z \to \neg x)\\ &\geq \min\left\{\mu(\neg y \to \neg x), \mu(\neg z \to (\neg\neg x \to \neg y)), \tfrac{1-k}{2}\right\}\\ &= \min\left\{\mu(x \to y), \mu(\neg z \to (x \to \neg y)), \tfrac{1-k}{2}\right\}\\ &= \min\left\{\mu(x \to y), \mu(x \to (\neg z \to \neg y)), \tfrac{1-k}{2}\right\}\\ &= \min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\}\end{aligned}$$

for all $x, y, z \in L$. By means of Theorem 3.3, μ is an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter of L. $\square$

We investigate the relationship between an $(\in, \in\vee\mathrm{q}_k)$-fuzzy filter and $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter. We recall the notion of $(\in, \in\vee\mathrm{q}_k)$-fuzzy filters.

Definition 3.7 ([2])**.** A fuzzy set μ in L is said to be an $(\in, \in\vee\mathrm{q}_k)$*-fuzzy filter* of L if it satisfies:

(1) $(x, t) \in \mu \;\&\; (y, r) \in \mu \implies (x \odot y, \min\{t, r\}) \in\vee\mathrm{q}_k\, \mu$,
(2) $(x, t) \in \mu \;\&\; x \leq y \implies (y, t) \in\vee\mathrm{q}_k\, \mu$

for all $x, y \in L$ and $t, r \in (0, 1]$.

An $(\in, \in\vee\mathrm{q}_k)$-fuzzy filter of L with $k = 0$ is called an $(\in, \in\vee\mathrm{q})$*-fuzzy filter* of L.

Lemma 3.8 ([2])**.** *A fuzzy set μ in L is an $(\in, \in\vee\mathrm{q}_k)$-fuzzy filter of L if and only if it satisfies*:

(1) $(\forall x \in L)$ $(\mu(1) \geq \min\left\{\mu(x), \tfrac{1-k}{2}\right\})$,
(2) $(\forall x, y \in L)$ $(\mu(y) \geq \min\left\{\mu(x), \mu(x \to y), \tfrac{1-k}{2}\right\})$.

Lemma 3.9 ([2])**.** *A fuzzy set μ in L is an $(\in, \in\vee\mathrm{q}_k)$-fuzzy filter of L if and only if it satisfies*:

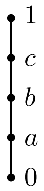
(1) $(\forall x, y \in L)$ $(\mu(x \odot y) \geq \min\left\{\mu(x), \mu(y), \tfrac{1-k}{2}\right\})$,
(2) $(\forall x, y \in L)$ $(x \leq y \implies \mu(y) \geq \min\left\{\mu(x), \tfrac{1-k}{2}\right\})$.

Theorem 3.10. *Every $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter is an $(\in, \in\vee\mathrm{q}_k)$-fuzzy filter.*

Proof. Let μ be an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter of L. Taking $x = 1$ in (c4) and using (R2) and Lemma 3.8, we have the desired result. $\square$

The following example shows that the converse of Theorem 3.10 is not true.

Table 2. Hasse diagram and Cayley tables



x	$\neg x$
0	1
a	c
b	b
c	a
1	0

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	b	1	1	1
c	a	a	b	1	1
1	0	a	b	c	1

Example 3.11. Let $L = \{0, a, b, c, 1\}$ be a set with Hasse diagram and Cayley tables which are given in Table 2. Then $(L, \wedge, \vee, \neg, \to, 0, 1)$ is an R_0-algebra (see [3]), where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Define a fuzzy set μ in L by

$$\mu = \begin{pmatrix} 0 & a & b & c & 1 \\ 0.3 & 0.3 & 0.3 & 0.8 & 0.45 \end{pmatrix}.$$

Then μ is an $(\in, \in\vee q_{0.2})$-fuzzy filter of L (see [2]). But it is not an $(\in, \in\vee q_{0.2})$-fuzzy implicative filter of L since $(b \to (b \to a), 0.35) \in \mu$ and $(b \to b, 0.4) \in \mu$, but $(b \to a, \min\{0.4, 0.35\}) \overline{\in\vee q_{0.2}}\, \mu$.

Proposition 3.12. *Every $(\in, \in\vee q_k)$-fuzzy implicative filter μ of L satisfies the following inequality*:

$$\mu(x \to z) \geq \min\left\{\mu(x \to (y \to (\neg z \to z)), \mu(y), \tfrac{1-k}{2}\right\} \tag{4}$$

for all $x, y, z \in L$.

Proof. Let μ be an $(\in, \in\vee q_k)$-fuzzy implicative filter of L. Then μ is an $(\in, \in\vee q_k)$-fuzzy filter of L by Theorem 3.10. Using Lemma 3.8(2), we have

$$\mu(x \to (\neg z \to z)) \geq \min\left\{\mu(y), \mu(y \to (x \to (\neg z \to z))), \tfrac{1-k}{2}\right\}$$

for all $x, y, z \in L$. It follows from (3) and (R4) that

$$\begin{aligned} \mu(x \to z) &\geq \min\left\{\mu(x \to (\neg z \to z)), \tfrac{1-k}{2}\right\} \\ &\geq \min\left\{\mu(x \to (y \to (\neg z \to z))), \mu(y), \tfrac{1-k}{2}\right\} \end{aligned}$$

for all $x, y, z \in L$. □

Theorem 3.13. *Every $(\in, \in\vee q_k)$-fuzzy filter satisfying* (4) *is an $(\in, \in\vee q_k)$-fuzzy implicative filter.*

Proof. Let μ be an $(\in, \in\vee q_k)$-fuzzy filter of L that satisfies the condition (4). Since $(x \odot \neg z) \to y \leq (y \to z) \to ((x \odot \neg z) \to z)$ for all $x, y, z \in L$, it follows from Lemma 3.9(2) that

$$\mu((y \to z) \to ((x \odot \neg z) \to z)) \geq \min\left\{\mu((x \odot \neg z) \to y), \tfrac{1-k}{2}\right\}$$

so from Lemma 3.8(2) that

$$\begin{aligned}\mu((x\odot\neg z)\to z) &\geq \min\left\{\mu(y\to z),\mu((y\to z)\to((x\odot\neg z)\to z)),\tfrac{1-k}{2}\right\}\\ &\geq \min\left\{\mu(y\to z),\min\{\mu((x\odot\neg z)\to y),\tfrac{1-k}{2}\},\tfrac{1-k}{2}\right\}\\ &= \min\left\{\mu(y\to z),\mu((x\odot\neg z)\to y),\tfrac{1-k}{2}\right\},\end{aligned}$$

that is,

$$(5)\qquad \mu(x\to(\neg z\to z))\geq \min\left\{\mu(y\to z),\mu(x\to(\neg z\to y)),\tfrac{1-k}{2}\right\}$$

for all $x,y,z\in L$. Taking $y=1$ in (4) and using (5), (R2) and (c3), we have

$$\begin{aligned}\mu(x\to z) &\geq \min\left\{\mu(x\to(1\to(\neg z\to z))),\mu(1),\tfrac{1-k}{2}\right\}\\ &= \min\left\{\mu(x\to(\neg z\to z)),\tfrac{1-k}{2}\right\}\\ &\geq \min\left\{\mu(y\to z),\mu(x\to(\neg z\to y)),\tfrac{1-k}{2}\right\}\end{aligned}$$

for all $x,y,z\in L$. By means of Proposition 3.6, we conclude that μ is an $(\in,\in\vee\,\mathrm{q}_k)$-fuzzy implicative filter of L. □

Proposition 3.14. *Every* $(\in,\in\vee\,\mathrm{q}_k)$*-fuzzy implicative filter* μ *of* L *satisfies the following inequalities*:

(1) $(\forall x\in L)\ (\mu(x)\geq \min\left\{\mu(\neg x\to x),\tfrac{1-k}{2}\right\})$.
(2) $(\forall x,y\in L)\ (\mu(x)\geq \min\left\{\mu((x\to y)\to x),\tfrac{1-k}{2}\right\})$.
(3) $(\forall x,y,z\in L)\ (\mu(x)\geq \min\left\{\mu(z\to((x\to y)\to x)),\mu(z),\tfrac{1-k}{2}\right\})$.

Proof. (1) From (3) and (R2), we get

$$\mu(x)=\mu(1\to x)\geq \min\left\{\mu(1\to(\neg x\to x)),\tfrac{1-k}{2}\right\}=\min\{\mu(\neg x\to x),\tfrac{1-k}{2}\}$$

for all $x\in L$.

(2) Note that $(x\to y)\to x\leq \neg x\to x$ for all $x,y\in L$. Since μ is an $(\in,\in\vee\,\mathrm{q}_k)$-fuzzy filter of L by Theorem 3.10, it follows from (1) and Lemma 3.9(2) that

$$\begin{aligned}\mu(x) &\geq \min\left\{\mu(\neg x\to x),\tfrac{1-k}{2}\right\}\\ &\geq \min\left\{\min\left\{\mu((x\to y)\to x),\tfrac{1-k}{2}\right\},\tfrac{1-k}{2}\right\}\\ &= \min\left\{\mu((x\to y)\to x),\tfrac{1-k}{2}\right\}\end{aligned}$$

for all $x,y\in L$.

(3) Note that $\mu((x\to y)\to x)\geq \min\{\mu(z),\mu(z\to((x\to y)\to x)),\tfrac{1-k}{2}\}$ for all $x,y,z\in L$. Since μ satisfies the condition (2), it follows that

$$\begin{aligned}\mu(x) &\geq \min\left\{\mu((x\to y)\to x),\tfrac{1-k}{2}\right\}\\ &\geq \min\left\{\min\left\{\mu(z),\mu(z\to((x\to y)\to x)),\tfrac{1-k}{2}\right\},\tfrac{1-k}{2}\right\}\\ &= \min\left\{\mu(z),\mu(z\to((x\to y)\to x)),\tfrac{1-k}{2}\right\}\end{aligned}$$

for all $x,y,z\in L$. □

Theorem 3.15. *Every* $(\in, \in\vee\mathrm{q}_k)$*-fuzzy filter* μ *of* L *that satisfies the condition* (3) *is an* $(\in, \in\vee\mathrm{q}_k)$*-fuzzy implicative filter of* L.

Proof. By the proof of Proposition 3.12, we know that μ satisfies the condition (4). Using Theorem 3.13, μ is an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter of L. □

Theorem 3.16. *Let* μ *be an* $(\in, \in\vee\mathrm{q}_k)$*-fuzzy filter of* L *in which the condition* (3) *of Proposition 3.14 is valid. Then* μ *is an* $(\in, \in\vee\mathrm{q}_k)$*-fuzzy implicative filter of* L.

Proof. Since $y \leq x \to y$ for all $x, y \in L$, we have $\neg(x \to y) \leq \neg y$ and $\neg y \to (x \to y) \leq \neg(x \to y) \to (x \to y)$ by (a5). Using Lemma 3.9(2), we obtain

$$\mu(\neg(x \to y) \to (x \to y)) \geq \min\left\{\mu(\neg y \to (x \to y)), \tfrac{1-k}{2}\right\}.$$

Taking $x = x \to y$, $z = 1$ and $y = 0$ in Proposition 3.14(3) and using (R2), (a3), (c3) and (R4), it follows that

$$\begin{aligned}\mu(x \to y) &\geq \min\left\{\mu(1 \to (((x \to y) \to 0) \to (x \to y))), \mu(1), \tfrac{1-k}{2}\right\}\\ &= \min\left\{\mu(\neg(x \to y) \to (x \to y)), \tfrac{1-k}{2}\right\}\\ &\geq \min\left\{\mu(\neg y \to (x \to y)), \tfrac{1-k}{2}\right\}\\ &= \min\left\{\mu(x \to (\neg y \to y)), \tfrac{1-k}{2}\right\}\end{aligned}$$

for all $x, y \in L$. Using Theorem 3.15, μ is an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter of L. □

Theorem 3.17. *A fuzzy set* μ *in* L *is an* $(\in,\in\vee\mathrm{q}_k)$*-fuzzy implicative filter of* L *if and only if it satisfies*:

(6) $\left(\forall t \in (0, \tfrac{1-k}{2}]\right)$ $(U(\mu; t) \neq \emptyset \Rightarrow U(\mu; t)$ *is an implicative filter of* $L)$.

Proof. Let μ be an $(\in,\in\vee\mathrm{q}_k)$-fuzzy implicative filter of L. Let $t \in (0, \frac{1-k}{2}]$ be such that $U(\mu; t) \neq \emptyset$. Obviously, $1 \in U(\mu; t)$ for all $t \in (0, \frac{1-k}{2}]$. Let $x, y, z \in L$ be such that $x \to y \in U(\mu; t)$ and $x \to (y \to z) \in U(\mu; t)$. Then $\mu(x \to y) \geq t$ and $\mu(x \to (y \to z)) \geq t$. It follows from (c4) that

$$\mu(x \to z) \geq \min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\} \geq \min\left\{t, \tfrac{1-k}{2}\right\} = t$$

so that $x \to z \in U(\mu; t)$. Hence $U(\mu; t)$ is an implicative filter of L.

Conversely, let μ be a fuzzy set in L in which (6) is valid. If there exists $a \in L$ such that $\mu(1) < \min\{\mu(a), \frac{1-k}{2}\}$, then $\mu(1) < t_a \leq \min\{\mu(a), \frac{1-k}{2}\}$ for some $t_a \in (0, \frac{1-k}{2}]$. Thus $(a, t_a) \in \mu$ but $(1, t_a)\,\overline{\in}\,\mu$. Also, $\mu(1) + t_a < 2t_a \leq 1-k$, i.e., $(1, t_a)\,\overline{\mathrm{q}_k}\,\mu$. Hence $(1, t_a)\,\overline{\in\vee\mathrm{q}_k}\,\mu$, which is a contradiction. Therefore $\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\}$ for all $x \in L$. Assume that there exist $a, b, c \in L$ such that

$$\mu(a \to c) < \min\left\{\mu(a \to b), \mu(a \to (b \to c)), \tfrac{1-k}{2}\right\}.$$

Then $\mu(a \to c) < t \leq \min\{\mu(a \to b), \mu(a \to (b \to c)), \frac{1-k}{2}\}$ for some $t \in (0, \frac{1-k}{2}]$, and so $a \to b \in U(\mu; t)$ and $a \to (b \to c) \in U(\mu; t)$, but $a \to c \notin U(\mu; t)$. Since $U(\mu; t)$ is an implicative filter of L, it is a contradiction. Therefore

$$\mu(x \to z) \geq \min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\}$$

for all $x, y, z \in L$. Consequently, we conclude that μ is an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter of L by Theorem 3.3. □

If we take $k = 0$ in Theorem 3.17, then we have the following corollary.

Corollary 3.18. *A fuzzy set μ in L is an $(\in, \in\vee\mathrm{q})$-fuzzy implicative filter of L if and only if it satisfies*:

(7) $(\forall t \in (0, 0.5])\ (U(\mu; t) \neq \emptyset \ \Rightarrow\ U(\mu; t)$ *is an implicative filter of* $L)$.

Theorem 3.19. *If A is an implicative filter of L, then a fuzzy set μ in L defined by*

$$\mu : L \to [0, 1],\ x \mapsto \begin{cases} t_1 & \textit{if } x \in A, \\ t_2 & \textit{if otherwise} \end{cases}$$

where $t_1 \in [\frac{1-k}{2}, 1]$ and $t_2 \in (0, \frac{1-k}{2})$, is an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter of L.

Proof. Note that

$$U(\mu; r) = \begin{cases} A & \text{if } r \in (t_2, \frac{1-k}{2}], \\ L & \text{if } r \in (0, t_2] \end{cases}$$

which is an implicative filter of L. It follows from Theorem 3.17 that μ is an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter of L. □

Corollary 3.20. *If A is an implicative filter of L, then a fuzzy set μ in L defined by*

$$\mu : L \to [0, 1],\ x \mapsto \begin{cases} t_1 & \textit{if } x \in A, \\ t_2 & \textit{if otherwise} \end{cases}$$

where $t_1 \in [0.5, 1]$ and $t_2 \in (0, 0.5)$, is an $(\in, \in\vee\mathrm{q})$-fuzzy implicative filter of L.

Theorem 3.21. *Every fuzzy implicative filter is an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter.*

Proof. Straightforward. □

Example 3.2 shows that the converse of Theorem 3.21 may not be true. We provide a condition for an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter to be a fuzzy implicative filter.

Theorem 3.22. *If μ is an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter satisfying $\mu(1) < \frac{1-k}{2}$, then μ is a fuzzy implicative filter.*

Proof. Let μ be an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter of L such that $\mu(1) < \frac{1-k}{2}$. Using (c3), we have $\min\{\mu(x), \frac{1-k}{2}\} \leq \mu(1) < \frac{1-k}{2}$ and so $\mu(x) \leq \frac{1-k}{2}$ for all $x \in L$. It follows from (c3) and (c4) that $\mu(1) \geq \mu(x)$ and

$$\mu(x \to z) \geq \min\{\mu(x \to y), \mu(x \to (y \to z))\}$$

for all $x, y, z \in L$. Hence μ is a fuzzy implicative filter of L. □

Corollary 3.23. *If μ is an $(\in, \in\vee\mathrm{q})$-fuzzy implicative filter satisfying $\mu(1) < 0.5$, then μ is a fuzzy implicative filter.*

Proposition 3.24. *For any $k_1, k_2 \in (0, 1]$ with $k_1 < k_2$, every $(\in, \in\vee\mathrm{q}_{k_1})$-fuzzy implicative filter is an $(\in, \in\vee\mathrm{q}_{k_2})$-fuzzy implicative filter.*

Proof. Straightforward. □

The following example shows that the converse of Proposition 3.24 is not true.

Example 3.25. Consider the R_0-algebra L which is given in Example 3.2. Define a fuzzy set μ in L by

$$\mu = \begin{pmatrix} 0 & a & b & c & d & 1 \\ 0.2 & 0.1 & 0.2 & 0.6 & 0.6 & 0.4 \end{pmatrix}.$$

Then μ is an $(\in, \in\vee\mathrm{q}_{0.2})$-fuzzy implicative filter of L. But it is not an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter of L for $k < 0.2$.

For any fuzzy set μ in L and any $t \in (0, 1]$, we consider four subsets:

$$Q(\mu; t) := \{x \in L \mid (x, t)\,\mathrm{q}\,\mu\},\ [\mu]_t := \{x \in L \mid (x, t) \in\vee\mathrm{q}\,\mu\},$$
$$Q^k(\mu; t) := \{x \in L \mid (x, t)\,\mathrm{q}_k\,\mu\},\ [\mu]_t^k := \{x \in L \mid (x, t) \in\vee\mathrm{q}_k\,\mu\}.$$

It is clear that $[\mu]_t = U(\mu; t) \cup Q(\mu; t)$ and $[\mu]_t^k = U(\mu; t) \cup Q^k(\mu; t)$.

Theorem 3.26. *If μ is an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter of L, then $Q^k(\mu; t)$ is an implicative filter of L whenever $Q^k(\mu; t) \neq \emptyset$ for all $t \in (\frac{1-k}{2}, 1]$.*

Proof. Assume that μ is an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter of L and let $t \in (\frac{1-k}{2}, 1]$ be such that $Q^k(\mu; t) \neq \emptyset$. Then there exists $x \in Q^k(\mu; t)$, and so $\mu(x) + t + k > 1$. It follows from (c3) that

$$\mu(1) \geq \min\left\{\mu(x), \tfrac{1-k}{2}\right\} \geq \min\left\{1 - t - k, \tfrac{1-k}{2}\right\} = 1 - t - k$$

so that $1 \in Q^k(\mu; t)$. Let $x, y, z \in L$ be such that $x \to y \in Q^k(\mu; t)$ and $x \to (y \to z) \in Q^k(\mu; t)$. Then $(x \to y, t)\,\mathrm{q}_k\,\mu$ and $(x \to (y \to z), t)\,\mathrm{q}_k\,\mu$, i.e., $\mu(x \to y) + t + k > 1$ and $\mu(x \to (y \to z)) + t + k > 1$. Using (c4), we have

$$\begin{aligned} \mu(x \to z) &\geq \min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\} \\ &\geq \min\left\{1 - t - k, \tfrac{1-k}{2}\right\} = 1 - t - k \end{aligned}$$

and so $(x \to z, t)\,\mathrm{q}_k\,\mu$, that is, $x \to z \in Q^k(\mu; t)$. Therefore $Q^k(\mu; t)$ is an implicative filter of L. □

Corollary 3.27. *If μ is an $(\in, \in\vee \mathrm{q})$-fuzzy implicative filter of L, then*

(8) $$\Big(\forall t \in (0.5, 1]\Big)\Big(Q(\mu; t) \neq \emptyset \ \Rightarrow\ Q(\mu; t) \text{ is an implicative filter of } L\Big).$$

Proof. It is clear by taking $k = 0$ in Theorem 3.26. □

Corollary 3.28. *Let $k, r \in (0, 1]$ with $k < r$. If μ is an $(\in, \in\vee \mathrm{q}_k)$-fuzzy implicative filter of L, then $Q^r(\mu; t)$ is an implicative filter of L whenever $Q^r(\mu; t) \neq \emptyset$ for all $t \in (\frac{1-r}{2}, 1]$.*

Proof. It is straightforward by Proposition 3.24 and Theorem 3.26. □

Theorem 3.29. *For any fuzzy set μ in L, the following are equivalent:*

(1) *μ is an $(\in, \in\vee \mathrm{q}_k)$-fuzzy implicative filter of L.*

(2) $(\forall t \in (0, 1])$ $\Big([\mu]_t^k \neq \emptyset \implies [\mu]_t^k$ is an implicative filter of $L\Big)$.

Proof. Assume that μ is an $(\in, \in\vee \mathrm{q}_k)$-fuzzy implicative filter of L and let $t \in (0, 1]$ be such that $[\mu]_t^k \neq \emptyset$. Then there exists $x \in [\mu]_t^k = U(\mu; t) \cup Q^k(\mu; t)$, and so $x \in U(\mu; t)$ or $x \in Q^k(\mu; t)$. If $x \in U(\mu; t)$, then (c3) implies that

$$\begin{aligned}\mu(1) &\geq \min\left\{\mu(x), \tfrac{1-k}{2}\right\} \geq \min\left\{t, \tfrac{1-k}{2}\right\}\\ &= \begin{cases} t & \text{if } t \leq \frac{1-k}{2},\\ \frac{1-k}{2} > 1-t-k & \text{if } t > \frac{1-k}{2}.\end{cases}\end{aligned}$$

Thus $1 \in U(\mu; t) \cup Q^k(\mu; t) = [\mu]_t^k$. Assume that $x \in Q^k(\mu; t)$. Then $(x, t)\, \mathrm{q}_k\, \mu$, i.e., $\mu(x) + t + k > 1$. Thus if $t > \frac{1-k}{2}$, then

$$\begin{aligned}\mu(1) &\geq \min\left\{\mu(x), \tfrac{1-k}{2}\right\}\\ &= \begin{cases} \mu(x) > 1-t-k & \text{if } \mu(x) < \frac{1-k}{2},\\ \frac{1-k}{2} > 1-t-k & \text{if } \mu(x) \geq \frac{1-k}{2}\end{cases}\end{aligned}$$

and so $1 \in Q^k(\mu; t) \subseteq [\mu]_t^k$. If $t \leq \frac{1-k}{2}$, then

$$\begin{aligned}\mu(1) &\geq \min\left\{\mu(x), \tfrac{1-k}{2}\right\}\\ &= \begin{cases} \mu(x) > 1-t-k & \text{if } \mu(x) < \frac{1-k}{2},\\ \frac{1-k}{2} \geq t & \text{if } \mu(x) \geq \frac{1-k}{2}\end{cases}\end{aligned}$$

which implies that $1 \in U(\mu; t) \cup Q^k(\mu; t) = [\mu]_t^k$. Let $x, y, z \in L$ be such that $x \to y \in [\mu]_t^k$ and $x \to (y \to z) \in [\mu]_t^k$. Then

$$\mu(x \to) \geq t \text{ or } \mu(x \to y) + t + k > 1,$$

and

$$\mu(x \to (y \to z)) \geq t \text{ or } \mu(x \to (y \to z)) + t + k > 1.$$

We can consider four cases:

(9) $$\mu(x \to y) \geq t \text{ and } \mu(x \to (y \to z)) \geq t,$$

(10) $$\mu(x \to y) \geq t \text{ and } \mu(x \to (y \to z)) + t + k > 1,$$

(11) $$\mu(x \to y) + t + k > 1 \text{ and } \mu(x \to (y \to z)) \geq t,$$

(12) $\mu(x \to y) + t + k > 1$ and $\mu(x \to (y \to z)) + t + k > 1.$

For the first case, (c4) implies that

$$\begin{aligned}\mu(x \to z) &\geq \min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\} \\ &\geq \min\left\{t, \tfrac{1-k}{2}\right\} = \begin{cases} \frac{1-k}{2} & \text{if } t > \frac{1-k}{2}, \\ t & \text{if } t \leq \frac{1-k}{2}, \end{cases}\end{aligned}$$

and so $\mu(x \to z) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, i.e., $(x \to z, t)\, \mathrm{q}_k\, \mu$, or $x \to z \in U(\mu; t)$. Therefore $x \to z \in U(\mu; t) \cup Q^k(\mu; t) = [\mu]_t^k$. For the case (10), assume that $t > \frac{1-k}{2}$. Then $1 - t - k \leq 1 - t < \frac{1-k}{2}$, and so

$$\begin{aligned}\mu(x \to z) &\geq \min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\} \\ &= \min\left\{\mu(x \to (y \to z)), \tfrac{1-k}{2}\right\} > 1 - t - k\end{aligned}$$

whenever $\min\{\mu(x \to (y \to z)), \frac{1-k}{2}\} \leq \mu(x \to y)$; and

$$\mu(x \to z) \geq \min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\} = \mu(x \to y) \geq t$$

whenever $\min\{\mu(x \to (y \to z)), \frac{1-k}{2}\} > \mu(x \to y)$. Thus $x \to z \in U(\mu; t) \cup Q^k(\mu; t) = [\mu]_t^k$. Suppose that $t \leq \frac{1-k}{2}$. Then $1 - t \geq \frac{1-k}{2}$, which implies that

$$\begin{aligned}\mu(x \to z) &\geq \min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\} \\ &= \min\left\{\mu(x \to y), \tfrac{1-k}{2}\right\} \geq t\end{aligned}$$

whenever $\min\{\mu(x \to y), \frac{1-k}{2}\} \leq \mu(x \to (y \to z))$; and

$$\begin{aligned}\mu(x \to z) &\geq \min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\} \\ &= \mu(x \to (y \to z)) > 1 - t - k\end{aligned}$$

whenever $\min\{\mu(x \to y), \frac{1-k}{2}\} > \mu(x \to (y \to z))$, and thus $x \to z \in U(\mu; t) \cup Q^k(\mu; t) = [\mu]_t^k$. We have similar result for the case (11). For the final case, if $t > \frac{1-k}{2}$, then $1 - t - k \leq 1 - t < \frac{1-k}{2}$. Hence

$$\mu(x \to z) \geq \min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\} = \tfrac{1-k}{2} > 1 - t - k$$

whenever $\min\{\mu(x \to y), \mu(x \to (y \to z))\} \geq \frac{1-k}{2}$; and

$$\begin{aligned}\mu(x \to z) &\geq \min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\} \\ &= \min\left\{\mu(x \to y), \mu(x \to (y \to z))\right\} > 1 - t - k\end{aligned}$$

whenever $\min\{\mu(x \to y), \mu(x \to (y \to z))\} < \frac{1-k}{2}$. Hence $x \to z \in Q^k(\mu; t) \subseteq [\mu]_t^k$. If $t \leq \frac{1-k}{2}$, then

$$\mu(x \to z) \geq \min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\} = \tfrac{1-k}{2} \geq t$$

whenever $\min\{\mu(x \to y), \mu(x \to (y \to z))\} \geq \frac{1-k}{2}$; and

$$\begin{aligned}\mu(x \to z) &\geq \min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\} \\ &= \min\{\mu(x \to y), \mu(x \to (y \to z))\} > 1 - t - k\end{aligned}$$

whenever $\min\{\mu(x \to y), \mu(x \to (y \to z))\} < \frac{1-k}{2}$. Thus $x \to z \in U(\mu; t) \cup Q^k(\mu; t) = [\mu]_t^k$. Therefore $[\mu]_t^k$ is an implicative filter of L.

Conversely, let μ be a fuzzy set in L such that $[\mu]_t^k$ is an implicative filter of L whenever it is non-empty for all $t \in (0, 1]$. If there exists $a \in L$ such that $\mu(1) < \min\{\mu(a), \frac{1-k}{2}\}$, then $\mu(1) < t_a \leq \min\{\mu(a), \frac{1-k}{2}\}$ for some $t_a \in (0, \frac{1-k}{2}]$. It follows that $a \in U(\mu; t_a) \subseteq [\mu]_{t_a}^k$ but $1 \notin U(\mu; t_a)$. Also, $\mu(1) + t_a < 2t_a \leq 1 - k$, and so $(1, t_a)\, \overline{\mathrm{q}_k}\, \mu$, i.e., $1 \notin Q^k(\mu; t_a)$. Therefore $1 \notin [\mu]_{t_a}^k$, a contradiction. Hence $\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\}$ for all $x \in L$. Suppose that there exist $a, b, c \in L$ such that

$$\mu(a \to c) < \min\left\{\mu(a \to b), \mu(a \to (b \to c)), \tfrac{1-k}{2}\right\}.$$

Then

$$\mu(a \to c) < t_b \leq \min\left\{\mu(a \to b), \mu(a \to (b \to c)), \tfrac{1-k}{2}\right\} \tag{13}$$

for some $t_b \in (0, \frac{1-k}{2}]$, which implies that $a \to b, a \to (b \to c) \in U(\mu; t_b) \subseteq [\mu]_{t_b}^k$ so from (b2) that $a \to c \in [\mu]_{t_b}^k = U(\mu; t_b) \cup Q^k(\mu; t_b)$ since $[\mu]_{t_b}^k$ is an implicative filter of L. But, (13) implies that $a \to c \notin U(\mu; t_b)$ and $\mu(a \to c) + t_b < 2t_b \leq 1 - k$, i.e., $a \to c \notin Q^k(\mu; t_b)$. This is a contradiction, and therefore $\mu(x \to z) \geq \min\{\mu(x \to y), \mu(x \to (y \to z)), \frac{1-k}{2}\}$ for all $x, y, z \in L$. Using Theorem 3.3, we conclude that μ is an $(\in, \in \vee\, \mathrm{q}_k)$-fuzzy implicative filter of L. □

If we take $k = 0$ in Theorem 3.29, then we have the following corollary.

Corollary 3.30. *For any fuzzy set μ in L, the following are equivalent*:

(1) *μ is an $(\in, \in \vee\, \mathrm{q})$-fuzzy implicative filter of L.*

(2) $(\forall t \in (0, 1])\ \Big([\mu]_t \neq \emptyset \implies [\mu]_t$ is an implicative filter of $L\Big)$.

4. Implication-based fuzzy implicative filters

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example $\wedge$, $\vee$, $\neg$, $\to$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition Φ is denoted by $[\Phi]$. For a universe U of discourse, we display the fuzzy logical and corresponding set-theoretical notations used in this paper

$$[x \in \mu] = \mu(x), \tag{14}$$

$$[\Phi \wedge \Psi] = \min\{[\Phi], [\Psi]\}, \tag{15}$$

$$[\Phi \to \Psi] = \min\{1, 1 - [\Phi] + [\Psi]\}, \tag{16}$$

$$[\forall x \Phi(x)] = \inf_{x \in U} [\Phi(x)], \tag{17}$$

$$\models \Phi \text{ if and only if } [\Phi] = 1 \text{ for all valuations.} \tag{18}$$

The truth valuation rules given in (16) are those in the Łukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a selection of them in the following.

(a) Gaines-Rescher implication operator (I_{GR}):

$$I_{\text{GR}}(a,b) = \begin{cases} 1 & \text{if} \quad a \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Gödel implication operator (I_{G}):

$$I_{\text{G}}(a,b) = \begin{cases} 1 & \text{if} \quad a \leq b, \\ b & \text{otherwise.} \end{cases}$$

(c) The contraposition of Gödel implication operator (I_{cG}):

$$I_{\text{cG}}(a,b) = \begin{cases} 1 & \text{if} \quad a \leq b, \\ 1-a & \text{otherwise.} \end{cases}$$

Ying [9] introduced the concept of fuzzifying topology. We can expand his/her idea to R_0-algebras, and we define a fuzzifying implicative filter as follows.

Definition 4.1. A fuzzy subset μ of L is called a *fuzzifying implicative filter* of L if it satisfies the following conditions:

(1) for all $x \in L$, we have

$$\models \ [x \in \mu] \to [1 \in \mu]. \tag{19}$$

(2) for all $x, y \in R$, we get

$$\models \ [x \to y \in \mu] \wedge [x \to (y \to z) \in \mu] \to [x \to z \in \mu]. \tag{20}$$

Obviously, conditions (19) and (20) are equivalent to (b3) and (b4), respectively. Therefore a fuzzifying implicative filter is an ordinary fuzzy implicative filter.

In [10], the concept of t-tautology is introduced, i.e.,

$$\models_t \Phi \text{ if and only if } [\Phi] \geq t \text{ for all valuations.} \tag{21}$$

Definition 4.2. Let μ be a fuzzy set in L and $t \in (0,1]$. Then μ is called a *t-implication-based fuzzy implicative filter* of L if it satisfies the following conditions:

(1) for all $x \in L$, we have

$$\models_t [x \in \mu] \to [1 \in \mu]. \tag{22}$$

(2) for all $x, y \in R$, we get

$$\models_t [x \to y \in \mu] \wedge [x \to (y \to z) \in \mu] \to [x \to z \in \mu]. \tag{23}$$

Let I be an implication operator. Clearly, μ is a t-implication-based fuzzy implicative filter of L if and only if it satisfies:

(1) $(\forall x \in L)\ (I(\mu(x), \mu(1)) \geq t)$,

(2) $(\forall x, y \in L)$ $(I(\min\{\mu(x \to y), \mu(x \to (y \to z))\}, \mu(x \to z)) \geq t)$.

Theorem 4.3. *For any fuzzy set μ in L, we have*

(1) *If $I = I_{\mathrm{GR}}$, then μ is a 0.5-implication-based fuzzy implicative filter of L if and only if μ is a fuzzy implicative filter of L.*
(2) *If $I = I_{\mathrm{G}}$, then μ is a $\frac{1-k}{2}$-implication-based fuzzy implicative filter of L if and only if μ is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy implicative filter of L.*
(3) *If $I = I_{\mathrm{cG}}$, then μ is a $\frac{1-k}{2}$-implication-based fuzzy implicative filter of L if and only if μ satisfies the following conditions*:
  (3.1) $\max\{\mu(1), \frac{1-k}{2}\} \geq \min\{\mu(x), 1\}$,
  (3.2) $\max\{\mu(x \to z), \frac{1-k}{2}\} \geq \min\{\mu(x \to y), \mu(x \to (y \to z)), 1\}$ *for all $x, y, z \in L$.*

Proof. (1) Straightforward.

(2) Assume that μ is a $\frac{1-k}{2}$-implication-based fuzzy implicative filter of L. Then

(i) $(\forall x \in L)$ $\left(I_{\mathrm{G}}(\mu(x), \mu(1)) \geq \frac{1-k}{2}\right)$,
(ii) $(\forall x, y \in L)$ $\left(I_{\mathrm{G}}(\min\{\mu(x \to y), \mu(x \to (y \to z))\}, \mu(x \to z)) \geq \frac{1-k}{2}\right)$.

From (i), we have $\mu(1) \geq \mu(x)$ or $\mu(x) \geq \mu(1) \geq \frac{1-k}{2}$, and so $\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\}$ for all $x \in L$. The second case implies that

$$\mu(x \to z) \geq \min\left\{\mu(x \to y), \mu(x \to (y \to z))\right\}$$

or $\min\{\mu(x \to y), \mu(x \to (y \to z))\} > \mu(x \to z) \geq \frac{1-k}{2}$. It follows that

$$\mu(x \to z) \geq \min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\}$$

for all $x, y, z \in L$. Using Theorem 3.3, we know that μ is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy implicative filter of L.

Conversely, suppose that μ is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy implicative filter of L. From (c3), if $\min\{\mu(x), \frac{1-k}{2}\} = \mu(x)$, then $I_{\mathrm{G}}(\mu(x), \mu(1)) = 1 \geq \frac{1-k}{2}$. Otherwise, $I_{\mathrm{G}}(\mu(x), \mu(1)) \geq \frac{1-k}{2}$. From (c4), if

$$\min\left\{\mu(x \to y), \mu(x \to (y \to z)), \tfrac{1-k}{2}\right\} = \min\{\mu(x \to y), \mu(x \to (y \to z))\},$$

then $\mu(x \to z) \geq \min\{\mu(x \to y), \mu(x \to (y \to z))\}$ and so

$$I_{\mathrm{G}}(\min\{\mu(x \to y), \mu(x \to (y \to z))\}, \mu(x \to z)) = 1 \geq \tfrac{1-k}{2}.$$

If $\min\{\mu(x \to y), \mu(x \to (y \to z)), \frac{1-k}{2}\} = \frac{1-k}{2}$, then $\mu(x \to z) \geq \frac{1-k}{2}$ and thus

$$I_{\mathrm{G}}(\min\{\mu(x \to y), \mu(x \to (y \to z))\}, \mu(x \to z)) \geq \tfrac{1-k}{2}.$$

Consequently, μ is a $\frac{1-k}{2}$-implication-based fuzzy implicative filter of L.

(3) Suppose that μ satisfies (3.1) and (3.2). In (3.1), if $\mu(x) = 1$, then $\max\{\mu(1), \frac{1-k}{2}\} = 1$ and hence $I_{\mathrm{cG}}(\mu(x), \mu(1)) = 1 \geq \frac{1-k}{2}$. If $\mu(x) < 1$, then

$$\max\left\{\mu(1), \tfrac{1-k}{2}\right\} \geq \mu(x). \tag{24}$$

If $\max\{\mu(1), \frac{1-k}{2}\} = \mu(1)$ in (24), then $\mu(1) \geq \mu(x)$. Hence

$$I_{\mathrm{cG}}(\mu(x), \mu(1)) = 1 \geq \tfrac{1-k}{2}.$$

If $\max\{\mu(1), \frac{1-k}{2}\} = \frac{1-k}{2}$ in (24), then $\mu(x) \leq \frac{1-k}{2}$ which implies that

$$I_{\mathrm{cG}}(\mu(x), \mu(1)) = \begin{cases} 1 \geq \frac{1-k}{2} & \text{if } \mu(1) \geq \mu(x), \\ 1-\mu(x) \geq \frac{1-k}{2} & \text{otherwise.} \end{cases}$$

In (3.2), if $\min\{\mu(x \to y), \mu(x \to (y \to z)), 1\} = 1$, then

$$\max\left\{\mu(x \to z), \tfrac{1-k}{2}\right\} = 1$$

and so $\mu(x \to z) = 1 \geq \min\{\mu(x \to y), \mu(x \to (y \to z))\}$. Therefore

$$I_{\mathrm{cG}}(\min\{\mu(x \to y), \mu(x \to (y \to z))\}, \mu(x \to z)) = 1 \geq \tfrac{1-k}{2}.$$

If $\min\{\mu(x \to y), \mu(x \to (y \to z)), 1\} = \min\{\mu(x \to y), \mu(x \to (y \to z))\}$, then

$$\max\left\{\mu(x \to z), \tfrac{1-k}{2}\right\} \geq \min\{\mu(x \to y), \mu(x \to (y \to z))\}. \tag{25}$$

Thus, if $\max\{\mu(x \to z), \frac{1-k}{2}\} = \frac{1-k}{2}$ in (25), then $\mu(x \to z) \leq \frac{1-k}{2}$ and

$$\min\{\mu(x \to y), \mu(x \to (y \to z))\} \leq \tfrac{1-k}{2}.$$

Therefore

$$I_{\mathrm{cG}}(\min\{\mu(x \to y), \mu(x \to (y \to z))\}, \mu(x \to z)) = 1 \geq \tfrac{1-k}{2}$$

whenever $\mu(x \to z) \geq \min\{\mu(x \to y), \mu(x \to (y \to z))\}$, and

$$\begin{aligned} &I_{\mathrm{cG}}(\min\{\mu(x \to y), \mu(x \to (y \to z))\}, \mu(x \to z)) \\ &= 1 - \min\{\mu(x \to y), \mu(x \to (y \to z))\} \geq \tfrac{1-k}{2} \end{aligned}$$

whenever $\mu(x \to z) < \min\{\mu(x \to y), \mu(x \to (y \to z))\}$. Now, if

$$\max\left\{\mu(x \to z), \tfrac{1-k}{2}\right\} = \mu(x \to z)$$

in (25), then $\mu(x \to z) \geq \min\{\mu(x \to y), \mu(x \to (y \to z))\}$ and so

$$I_{\mathrm{cG}}(\min\{\mu(x \to y), \mu(x \to (y \to z))\}, \mu(x \to z)) = 1 \geq \tfrac{1-k}{2}.$$

Consequently, μ is a $\frac{1-k}{2}$-implication-based fuzzy implicative filter of L.

Conversely assume that μ is a $\frac{1-k}{2}$-implication-based fuzzy implicative filter of L. Then

(iii) $I_{\mathrm{cG}}(\mu(x), \mu(1)) \geq \frac{1-k}{2}$,

(iv) $I_{\mathrm{cG}}(\min\{\mu(x \to y), \mu(x \to (y \to z))\}, \mu(x \to z)) \geq \frac{1-k}{2}$

for all $x, y, z \in L$. The case (iii) implies that $I_{\mathrm{cG}}(\mu(x), \mu(1)) = 1$, i.e., $\mu(x) \leq \mu(1)$, or $1 - \mu(x) \geq \frac{1-k}{2}$ and so $\mu(x) \leq \frac{1-k}{2}$. It follows that

$$\max\left\{\mu(1), \tfrac{1-k}{2}\right\} \geq \mu(x) = \min\{\mu(x), 1\}.$$

From (iv), we have

$$I_{\mathrm{cG}}(\min\{\mu(x \to y), \mu(x \to (y \to z))\}, \mu(x \to z)) = 1,$$

i.e., $\min\{\mu(x \to y), \mu(x \to (y \to z))\} \le \mu(x \to z)$, or

$$1 - \min\{\mu(x \to y), \mu(x \to (y \to z))\} \ge \frac{1-k}{2}.$$

Hence

$$\begin{aligned} \max\left\{\mu(x \to z), \tfrac{1-k}{2}\right\} &\ge \min\{\mu(x \to y), \mu(x \to (y \to z))\} \\ &= \min\{\mu(x \to y), \mu(x \to y), 1\} \end{aligned}$$

for all $x, y, z \in L$. This completes the proof. □

Corollary 4.4. *For any fuzzy set μ in L, we have*

(1) *If $I = I_{\mathrm{G}}$, then μ is a 0.5-implication-based fuzzy implicative filter of L if and only if μ is an $(\in, \in\vee\mathrm{q})$-fuzzy implicative filter of L.*

(2) *If $I = I_{\mathrm{cG}}$, then μ is a 0.5-implication-based fuzzy implicative filter of L if and only if μ satisfies the following conditions:*

(2.1) $\max\{\mu(1), 0.5\} \ge \min\{\mu(x), 1\}$,

(2.2) $\max\{\mu(x \to z), 0.5\} \ge \min\{\mu(x \to y), \mu(x \to (y \to z)), 1\}$ *for all $x, y, z \in L$.*

5. Conclusion

In this paper we have introduced a natural generalization of the concept of a fuzzy implicative filter in an R_0-algebra. We have introduced the notion of an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter in an R_0-algebra, and investigated related properties. We have shown that every $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative is an $(\in, \in\vee\mathrm{q}_k)$-fuzzy filter, but the converse is not true by providing an example. We have found conditions for an $(\in, \in\vee\mathrm{q}_k)$-fuzzy filter to be an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter. We have dealt with characterizations of an $(\in, \in\vee\mathrm{q}_k)$-fuzzy implicative filter in R_0-algebras, and have discussed the implication-based fuzzy implicative filters of an R_0-algebra. Hopefully, the rich supply of characterizations at hand suffices in making persuasive the relative argument that these structures are definitely worth investigating.

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