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ALGORITHMIC PROOF OF MaxMult(T) = $p(T)$

In-Jae Kim

Abstract. For a given graph G we consider a set $S(G)$ of all symmetric matrices $A = [a_{ij}]$ whose nonzero entries are placed according to the location of the edges of the graph, i.e., for $i \neq j$, $a_{ij} \neq 0$ if and only if vertex i is adjacent to vertex j. The minimum rank $\mathrm{mr}(G)$ of the graph G is defined to be the smallest rank of a matrix in $S(G)$. In general the computation of $\mathrm{mr}(G)$ is complicated, and so is that of the maximum multiplicity $\mathrm{MaxMult}(G)$ of an eigenvalue of a matrix in $S(G)$ which is equal to $n - \mathrm{mr}(G)$ where n is the number of vertices in G. However, for trees T, there is a recursive formula to compute $\mathrm{MaxMult}(T)$. In this note we show that this recursive formula for $\mathrm{MaxMult}(T)$ also computes the path cover number $p(T)$ of the tree T. This gives an alternative proof of the interesting result, $\mathrm{MaxMult}(T) = p(T)$.

1. Introduction

Let $G = (V, E)$ be a graph on n vertices. We define a set of symmetric matrices associated with G as follows:

$$S(G) = \{A \in \mathbb{R}^{n\times n} \mid A \text{ is symmetric, and } a_{ij} \neq 0\ (i \neq j) \text{ if and only if } i \sim j\},$$

where $i \sim j$ means that vertex i is adjacent to vertex j. The *minimum rank* $\mathrm{mr}(G)$ *of* G is the smallest rank of a matrix in $S(G)$, i.e.,

$$\mathrm{mr}(G) = \min_{A\in S(G)} \mathrm{rank}(A).$$

Since the order of $A \in S(G)$ is n, the maximum corank of G is equal to

$$n - \mathrm{mr}(G).$$

Note that the main diagonal entries of $A \in S(G)$ is not related to the topology of the graph G, and hence $A - \lambda I$ is also in $S(G)$. For an eigenvalue λ of A, the corank of $A - \lambda I$ is equal to the multiplicity of λ as an eigenvalue of A. This implies that the maximum corank of the graph G on n vertices is equal to the

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maximum of the multiplicities of eigenvalues of matrices in $S(G)$, $\mathrm{MaxMult}(G)$ (called the *maximum multiplicity of* G). Hence, we have

$$\mathrm{mr}(G) = n - \mathrm{MaxMult}(G).$$

For example, it can be shown by considering the all ones matrix that the complete graph K_n on n vertices has $\mathrm{mr}(K_n) = 1$, and hence $\mathrm{MaxMult}(K_n) = n - 1$. For path P_n on n vertices, the rank of a matrix A in $S(P_n)$ is either $n-1$ or n since the $(n-1) \times (n-1)$ submatrix of A, obtained by deleting the last row and the first column of A, is nonsingular. By choosing main diagonal entries properly, we can construct a singular matrix in $S(P_n)$ with each row sum equal to zero. Hence, $\mathrm{mr}(P_n) = n - 1$ and $\mathrm{MaxMult}(P_n) = 1$.

In general the computation of the minimum rank of a graph is complicated (For recent development in the computation of minimum ranks of graphs, see [1]). For trees T, however, there is a recursive way to compute $\mathrm{MaxMult}(T)$ and hence $\mathrm{mr}(T)$, using the path cover number of T. The *path cover number* $p(T)$ of a tree T is the minimum number of vertex disjoint paths, occurring as induced subgraphs of T, that cover all the vertices of T. It was shown in [2] that

$$\mathrm{MaxMult}(T) = \Delta(T) = p(T),$$

where $\Delta(T) = \max[p - q]$ for p and q such that there exist q vertices of T whose deletion leaves p paths. In this note we give an alternative proof for $\mathrm{MaxMult}(T) = p(T)$, by showing that the recursive algorithm for $\mathrm{MaxMult}(T)$ also computes $p(T)$.

2. Main result

Let T be a tree on n vertices and $V(T)$ be the vertex set of T. For a subset U of $V(T)$, the graph $T \setminus U$ is the subgraph of T obtained by deleting vertices in U and all edges incident to the vertices in U. In particular, for $p \in V(T)$, we use T_p to denote the acyclic subgraph $T \setminus \{p\}$. If the degree of p is k, then we call the k connected components $T_p^1, \ldots, T_p^k$ of T_p as the *branches* of T at p. If at least two of branches at p are paths (on one or more vertices) which are connected to p in T through an endpoint, then we call p an *appropriate vertex* of T.

Proposition 2.1 ([4, Lemma 3.1])**.** *Every tree T with at least three vertices has an appropriate vertex.*

We now give a recursive formula for $\mathrm{mr}(T)$ in [4].

Theorem 2.2 ([4, Corollary 3.3])**.** *Let T be a tree on $n \geq 3$ vertices and p an appropriate vertex of T, and let $T_p^1, \ldots, T_p^k$ be the branches of T at p. Then*

$$\mathrm{mr}(T) = \mathrm{mr}(T_p^1) + \cdots + \mathrm{mr}(T_p^k) + 2. \tag{1}$$

To write the result in Theorem 2.2 in terms of maximum multiplicities, we use $\mathrm{mr}(G) = n - \mathrm{MaxMult}(G)$. For a vertex p of degree k in T, let n_i be the number of vertices in T_p^i for $i = 1, \ldots, k$. Note that $\sum_{i=1}^k n_i = n - 1$ where n is the number of vertices in T. From (1) we get

$$\begin{aligned} n - \mathrm{MaxMult}(T) &= (n_1 - \mathrm{MaxMult}(T_p^1)) + \cdots + (n_k - \mathrm{MaxMult}(T_p^k)) + 2 \\ &= \left(\left[\sum_{i=1}^k n_i \right] + 2 \right) - (\mathrm{MaxMult}(T_p^1) + \cdots + \mathrm{MaxMult}(T_p^k)), \end{aligned}$$

and hence

$$\mathrm{MaxMult}(T) = \mathrm{MaxMult}(T_p^1) + \cdots + \mathrm{MaxMult}(T_p^k) - 1.$$

Since at least two of the branches at p are paths which are connected to p in T through an endpoint, without loss of generality, we may assume that T_p^1 and T_p^2 are such paths. Since the maximum multiplicity of a path is 1, we have

$$\begin{aligned} \mathrm{MaxMult}(T) &= \mathrm{MaxMult}(T_p^3) + \cdots + \mathrm{MaxMult}(T_p^k) + 1 \\ &= \mathrm{MaxMult}(T \setminus (V(T_p^1) \cup V(T_p^2) \cup \{p\})) + 1. \end{aligned}$$

Let P be the induced subgraph (that is a path) of T with the vertex set $V(T_p^1) \cup V(T_p^2) \cup \{p\}$. Then

$$\mathrm{MaxMult}(T) = \mathrm{MaxMult}(T \setminus V(P)) + 1. \tag{2}$$

Note that P is a path in T such that the its end vertices are pendant vertices of T, and at most one vertex (if any, that is p) of the path P has degree 3 or more in T. The existence of such a path P in T is guaranteed by the existence of an appropriate vertex (see Proposition 2.1).

The following result shows that the computation of $p(T)$ can be done by the same recursive formula as in (2).

Lemma 2.3 ([3, Proposition 13]). *Let T be a tree that is not a path. Suppose that P is a path in T such that P's end vertices are pendant vertices of T and P has exactly one vertex p of degree 3 or more in T. Then*

$$p(T) = p(T \setminus V(P)) + 1.$$

Therefore, we have proved the following result.

Theorem 2.4. *Let T be a tree. Then*

$$\mathrm{MaxMult}(T) = p(T).$$

Example 2.5. Consider the tree T in Figure 1. To compute MaxMult(T) we compute its path cover number $p(T)$ recursively by deleting the following paths sequentially:

(i) $1 - 2 - 3$
(ii) $4 - 5 - 6$
(iii) $7 - 8 - 9$

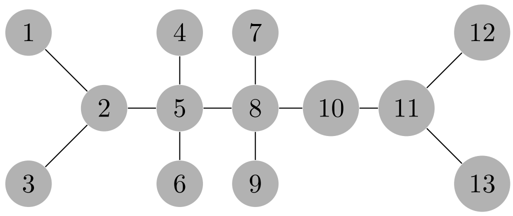

Figure 1. Tree T

(iv) $10 - 11 - 12$
(v) 13

After deleting the five paths, there is no vertex left. Hence $p(T) = \text{MaxMult}(T) = 5$.

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Department of Mathematics and Statistics
MN Modeling and Simulation Center
Minnesota State University
Mankato, MN 56001, USA
E-mail address: in-jae.kim@mnsu.edu