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$\mathcal{N}$-SUBALGEBRAS AND $\mathcal{N}$-IDEALS BASED ON A SUB-BCK-ALGEBRA OF A BCI-ALGEBRA

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Abstract. Based on a sub-BCK-algebra K of a BCI-algebra X, the notions of $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals of X are introduced, and their relations/properties are investigated.

1. Introduction

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [2] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. They discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI-algebras.

In this paper, by using a sub-BCK-algebra K of a BCI-algebra X and a number $\varrho \in [-1, 0]$, we introduce the notions of $\mathcal{N}(K, \varrho)$-subalgebras and $\mathcal{N}(K, \varrho)$-ideals in BCI-algebras. We investigate their properties, and show that these two notions are independent each other by providing examples.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras with type $\tau = (2, 0)$. By a *BCI-algebra* we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,
(a2) $(x * (x * y)) * y = 0$,

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(a3) $x * x = 0$,
(a4) $x * y = y * x = 0 \Longrightarrow x = y$,

where x, y and z are elements of X. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*. We can define a partial ordering $\preceq$ by

$$(\forall x, y \in X)\,(x \preceq y \iff x * y = 0).$$

In a BCK/BCI-algebra X, the following hold:

(b1) $x * 0 = x$,
(b2) $(x * y) * z = (x * z) * y$,

where x, y and z are elements of X.

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in A, \tag{2.1}$$

$$x * y \in A,\ y \in A \Rightarrow x \in A, \tag{2.2}$$

where x and y are elements of X.

We refer the reader to the books [1] and [3] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\vee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise,} \end{cases}$$

$$\wedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, *$\mathcal{N}$-function* on X). By an *$\mathcal{N}$-structure* we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X.

Definition 2.1 ([2])**.** By a *subalgebra* of a BCK/BCI-algebra X based on $\mathcal{N}$-function f (briefly, *$\mathcal{N}$-subalgebra* of X), we mean an $\mathcal{N}$-structure (X, f) in which f satisfies the following condition: for any $x, y \in X$,

$$f(x * y) \leq \vee\{f(x), f(y)\}. \tag{2.3}$$

Definition 2.2 ([2])**.** By an *ideal* of a BCK/BCI-algebra X based on $\mathcal{N}$-function f (briefly, *$\mathcal{N}$-ideal* of X), we mean an $\mathcal{N}$-structure (X, f) in which f satisfies the following condition: for any $x, y \in X$,

$$f(0) \leq f(x) \leq \vee\left\{f(x * y), f(y)\right\}. \tag{2.4}$$

For any $\mathcal{N}$-structure (X, f) and $\alpha \in [-1, 0]$, the set

$$C(f; \alpha) := \{x \in X \mid f(x) \leq \alpha\}$$

TABLE 1. $*$-operation

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	0	a	a
2	2	2	0	b	a
a	a	a	a	0	0
b	b	b	a	2	0

is called the *closed support* of (X, f) related to α, and the set

$$O(f;\alpha) := \{x \in X \mid f(x) < \alpha\}$$

is called the *open support* of (X, f) related to α.

Proposition 2.3 ([2]). *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subalgebra (resp. ideal) of a BCK/BCI-algebra X if and only if every closed support of (X, f) related to α is a subalgebra (resp. ideal) of X for all $\alpha \in [-1, 0]$.*

For our convenience, the empty set $\emptyset$ is regarded as a subalgebra (resp. ideal) of X.

3. $\mathcal{N}$-subalgebras based on a sub-BCK-algebra

Definition 3.1. Let $(X; *, 0)$ be a BCI-algebra. By a *sub-BCK-algebra* of X we mean a subset K of X such that $0 \in K$ and $(K; *, 0)$ is a BCK-algebra.

Example 3.2. Let $X = \{0, 1, 2, a, b\}$ be a set with the $*$-operation given by Table 1. Then $(X; *, 0)$ is a BCI-algebra and $(K = \{0, 1, 2\}; *, 0)$ is a sub-BCK-algebra of X.

Definition 3.3. Let K be a sub-BCK-algebra of a BCI-algebra X and let $\varrho \in [-1, 0]$. An $\mathcal{N}$-structure (X, f) is called an *$\mathcal{N}$-subalgebra* of X based on K and ϱ (briefly, *$\mathcal{N}(K, \varrho)$-subalgebra* of X) if it is an $\mathcal{N}$-subalgebra of X that satisfies the following condition:

$$(\forall x \in K)\ (\forall y \in X \setminus K)\ \ (f(x) \leq \varrho \leq f(y))\,. \tag{3.1}$$

Example 3.4. Let X and K be as in Example 3.2.

(1) An $\mathcal{N}$-structure (X, f) in which f is given by

$$f = \begin{pmatrix} 0 & 1 & 2 & a & b \\ -0.7 & -0.6 & -0.5 & -0.3 & -0.3 \end{pmatrix}$$

is an $\mathcal{N}(K, \varrho)$-subalgebra of X for $\varrho \in [-0.5, -0.3]$.

(2) Let (X, g) be an $\mathcal{N}$-structure in which g is given by

$$g = \begin{pmatrix} 0 & 1 & 2 & a & b \\ -0.7 & -0.5 & -0.2 & -0.4 & -0.2 \end{pmatrix}.$$

TABLE 2. $*$-operation

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Then (X, g) is an $\mathcal{N}$-subalgebra of X, but it does not satisfy (3.1) since $g(2) = -0.2 > -0.4 = g(a)$.

The following example shows that there exists an $\mathcal{N}$-structure (X, f) in a BCI-algebra X such that it satisfies the condition (3.1), but it is not an $\mathcal{N}$-subalgebra of X.

Example 3.5. Let $X = \{0, 1, a, b, c\}$ be a set with the $*$-operation given by Table 2. Then $(X; *, 0)$ is a BCI-algebra and $(K = \{0, 1\}; *, 0)$ is only a sub-BCK-algebra of X. Let (X, f) be an $\mathcal{N}$-structure in which f is given by

$$f = \begin{pmatrix} 0 & 1 & a & b & c \\ -0.5 & -0.6 & -0.2 & -0.4 & -0.3 \end{pmatrix}.$$

Then (X, f) satisfies the condition (3.1) for $\varrho \in [-0.5, -0.4]$, but it is not an $\mathcal{N}$-subalgebra of X since $f(b * c) = f(a) = -0.2 > -0.3 = \vee\{f(b), f(c)\}$.

Theorem 3.6. *Let K be a sub-BCK-algebra of a BCI-algebra X. If an $\mathcal{N}$-structure (X, f) satisfies the following condition*:

$$(\forall x \in K)\ (\forall y \in X \setminus K)\ \ (f(x) \leq f(y)), \tag{3.2}$$

then (X, f) is an $\mathcal{N}(K, \varrho)$-subalgebra of X for every $\varrho \in \left[\bigwedge_{y \in X\setminus K} f(y), \bigvee_{x \in K} f(x)\right]$.

Proof. Straightforward. □

Obviously, a restriction of an $\mathcal{N}(K, \varrho)$-subalgebra of a BCI-algebra X to a sub-BCK-algebra K of X is a $\mathcal{N}$-subalgebra of $(K; *, 0)$.

Theorem 3.7. *Let $\varrho \in [-1, 0]$ and let K be a sub-BCK-algebra of a BCI-algebra X. Then every $\mathcal{N}(K, \varrho)$-subalgebra (X, f) of X satisfies the following assertions*:

(1) $K \subseteq C(f; \varrho)$.
(2) $(\forall \beta \in [-1, 0])\ (\beta < \varrho \ \Rightarrow\ C(f; \beta)$ *is a subalgebra of* $K)$.

Proof. Assume that (X, f) is an $\mathcal{N}(K, \varrho)$-subalgebra of X. Obviously, $K \subseteq C(f; \varrho)$. Let $\beta \in [-1, 0]$ be such that $\beta < \varrho$. Then $C(f; \beta) \subseteq K$. Let $x, y \in C(f; \beta)$. Then $f(x) \leq \beta$ and $f(y) \leq \beta$. Thus $f(x * y) \leq \vee\{f(x), f(y)\} \leq \beta$, and so $x * y \in C(f; \beta)$. Therefore $C(f; \beta)$ is a subalgebra of K. □

TABLE 3. $*$-operation

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

We give conditions for an $\mathcal{N}$-subalgebra to be an $\mathcal{N}(K, \varrho)$-subalgebra.

Theorem 3.8. *Let $\varrho \in [-1, 0]$ and let K be a sub-BCK-algebra of a BCI-algebra X. If (X, f) is an $\mathcal{N}$-subalgebra of X satisfying two conditions* (1) *and* (2) *in Theorem 3.7, then (X, f) is an $\mathcal{N}(K, \varrho)$-subalgebra of X.*

Proof. Let $x \in K$ and $y \in X \setminus K$. Then $x \in C(f; \varrho)$ by (1) in Theorem 3.7, and so $f(x) \leq \varrho$. Let $f(y) = \beta$. If $\beta < \varrho$, then $y \in C(f; \beta) \subseteq K$ by (2) in Theorem 3.7. This is a contradiction, and thus $f(x) \leq \varrho \leq \beta = f(y)$. Consequently, (X, f) is an $\mathcal{N}(K, \varrho)$-subalgebra of X. □

4. $\mathcal{N}$-ideals based on a sub-BCK-algebra

Definition 4.1. Let $\varrho \in [-1, 0]$ and let K be a sub-BCK-algebra of a BCI-algebra X. An $\mathcal{N}$-structure (X, f) is called an *$\mathcal{N}$-ideal* of X based on K and ϱ (briefly, *$\mathcal{N}(K, \varrho)$-ideal* of X) if it satisfies:

$$(\forall x \in K)(\forall y \in X \setminus K)\,(f(0) \leq f(x) \leq \varrho \leq f(y))\,. \tag{4.1}$$

$$(\forall x, y \in K)\,(f(x) \leq \vee\{f(x * y), f(y)\})\,. \tag{4.2}$$

Example 4.2. Let $X = \{0, 1, 2, a, b\}$ be a set with the $*$-operation given by Table 3. Then $(X; *, 0)$ is a BCI-algebra and $(K = \{0, 1, 2\}; *, 0)$ is a sub-BCK-algebra of X. Let (X, f) be an $\mathcal{N}$-structure in which f is given by

$$f = \begin{pmatrix} 0 & 1 & 2 & a & b \\ -0.8 & -0.5 & -0.7 & -0.1 & -0.2 \end{pmatrix}.$$

Then (X, f) is an $\mathcal{N}(K, \varrho)$-ideal of X. But it is not an $\mathcal{N}$-ideal of X since $f(a) = -0.1 \nleq -0.2 = \vee\{f(a * b), f(b)\}$.

Theorem 4.3. *Let $\varrho \in [0, 1]$ and let K be a sub-BCK-algebra of a BCI-algebra X. If (X, f) is an $\mathcal{N}(K, \varrho)$-ideal of X, then*

(1) $K \subseteq C(f; \varrho)$.
(2) $(\forall \beta \in [-1, 0])(\beta < \varrho \Rightarrow C(f; \varrho)$ *is an ideal of* $K)$.

Proof. Let $x \in K$. Then $f(x) \leq \varrho$ by (4.1), and so $x \in C(f; \varrho)$. Hence $K \subseteq C(f; \varrho)$. Let $\beta \in [-1, 0]$ be such that $\beta < \varrho$. If $x \in C(f; \beta)$, then $f(x) \leq \beta < \varrho$ and thus $x \in K$. Hence $C(f; \beta) \subseteq K$. From (4.1), we know that $f(0) \leq f(x)$

for all $x \in X$. Hence $f(0) \le f(x) \le \beta$ for $x \in C(f;\beta)$, and so $0 \in C(f;\beta)$. Let $x, y \in K$ be such that $x * y \in C(f;\beta)$ and $y \in C(f;\beta)$. Then $f(x*y) \le \beta$ and $f(y) \le \beta$. It follows from (4.2) that $f(x) \le \vee\{f(x*y), f(y)\} \le \beta$ so that $x \in C(f;\beta)$. Hence $C(f;\beta)$ is an ideal of K. □

For a sub-BCK-algebra K of a BCI-algebra X and $\varrho \in [-1, 0]$, the following example shows that an $\mathcal{N}$-ideal (X, f) of X may not be an $\mathcal{N}(K,\varrho)$-ideal of X.

Example 4.4. Let X and K be as in Example 4.2. Consider an $\mathcal{N}$-structure (X, f) in which f is given by

$$f = \begin{pmatrix} 0 & 1 & 2 & a & b \\ -0.8 & -0.3 & -0.7 & -0.5 & -0.3 \end{pmatrix}.$$

Then

$$C(f;\beta) = \begin{cases} X & \text{if } \beta \in [-0.3, 0], \\ \{0, 2, a\} & \text{if } \beta \in [-0.5, -0.3), \\ \{0, 2\} & \text{if } \beta \in [-0.7, -0.5), \\ \{0\} & \text{if } \beta \in [-0.8, -0.7), \\ \emptyset & \text{if } \beta \in [-1, -0.8), \end{cases}$$

and so $C(f;\beta)$ is an ideal of X for all $\beta \in [-1, 0]$. Hence (X, f) is an $\mathcal{N}$-ideal of X by Proposition 2.3. But (X, f) is not an $\mathcal{N}(K,\varrho)$-ideal of X for $\varrho \in [-0.5, -0.3)$ because $f(1) = -0.3 > \varrho \ge -0.5 = f(a)$.

We provide conditions for an $\mathcal{N}$-ideal to be an $\mathcal{N}(K,\varrho)$-ideal.

Theorem 4.5. *Let $\varrho \in [0, 1]$ and let K be a sub-BCK-algebra of a BCI-algebra X. If an $\mathcal{N}$-ideal (X, f) of X satisfies conditions* (1) *and* (2) *in Theorem 4.3, then (X, f) is an $\mathcal{N}(K,\varrho)$-ideal of X.*

Proof. Let $x \in K$ and $y \in X \setminus K$. Then $x \in C(f;\varrho)$ by (1) of Theorem 4.3, which implies $f(x) \le \varrho$. If $f(y) < \varrho$, then $y \in C(f; f(y)) \subseteq K$ by (2) of Theorem 4.3. This is a contradiction, and so $f(y) \ge \varrho$. Since $f(0) \le f(x)$ for all $x \in X$, it follows that $f(0) \le f(x) \le \varrho \le f(y)$ so that (4.1) is valid. Since f is an $\mathcal{N}$-ideal of X, the condition (4.2) is obvious. Therefore (X, f) is an $\mathcal{N}(K,\varrho)$-ideal of X. □

The following example shows that an $\mathcal{N}(K,\varrho)$-subalgebra may not be an $\mathcal{N}(K,\varrho)$-ideal, and vice versa.

Example 4.6. (1) Let X and K be as in Example 3.2. Let (X, g) be an $\mathcal{N}$-structure in which g is given by

$$g = \begin{pmatrix} 0 & 1 & 2 & a & b \\ -0.8 & -0.5 & -0.7 & -0.2 & -0.2 \end{pmatrix}.$$

Then (X, g) is an $\mathcal{N}(K,\varrho)$-subalgebra of X for $\varrho \in [-0.5, -0.2]$. But (X, g) is not an $\mathcal{N}(K,\varrho)$-ideal of X for $\varrho \in [-0.5, -0.2]$ since $g(1) = -0.5 \not\le -0.7 = \vee\{f(1*2), f(2)\}$.

(2) Let X and K be as in Example 3.2. Consider an $\mathcal{N}$-structure (X, f) in which f is given by

$$f = \begin{pmatrix} 0 & 1 & 2 & a & b \\ -0.7 & -0.6 & -0.5 & -0.4 & -0.2 \end{pmatrix}.$$

Then (X, f) is an $\mathcal{N}(K, \varrho)$-ideal of X for $\varrho \in [-0.5, -0.4]$. Since $f(2 * a) = f(b) = -0.2 \not\leq -0.4 = \vee\{f(2), f(a)\}$, (X, f) is not an $\mathcal{N}$-subalgebra of X. Therefore (X, f) is not an $\mathcal{N}(K, \varrho)$-subalgebra of X for $\varrho \in [-0.5, -0.4]$.

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