Commun. Korean Math. Soc. **27** (2012), No. 3, pp. 629–644
http://dx.doi.org/10.4134/CKMS.2012.27.3.629

A PARABOLIC SYSTEM WITH NONLOCAL BOUNDARY CONDITIONS AND NONLOCAL SOURCES

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Abstract. In this work, the authors study the blow-up properties of solutions to a parabolic system with nonlocal boundary conditions and nonlocal sources. Conditions for the existence of global or blow-up solutions are given. Global blow-up property and precise blow-up rate estimates are also obtained.

1. Introduction

In this article, we consider the positive classical solutions to the following porous medium system with nonlocal boundary conditions and nonlocal sources

$$\begin{cases} u_t = \Delta u^m + a\int_\Omega v^p \mathrm{d}x, & x\in\Omega,\ t>0,\\ v_t = \Delta v^n + b\int_\Omega u^q \mathrm{d}x, & x\in\Omega,\ t>0,\\ u(x,t) = \int_\Omega k_1(x,y)u(y,t)\mathrm{d}y, & x\in\partial\Omega,\ t>0,\\ v(x,t) = \int_\Omega k_2(x,y)v(y,t)\mathrm{d}y, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $m,\ n>1,\ a,\ b,\ p,\ q>0$ are constants and Ω is a bounded domain in $\mathbb{R}^N$ $(N\ge 1)$, with smooth boundary $\partial\Omega$. $k_1(x,y)$, $k_2(x,y)\not\equiv 0$ are nonnegative continuous functions defined for $x\in\partial\Omega$ and $y\in\overline{\Omega}$, while $u_0(x)$, $v_0(x)$ are positive continuous functions and satisfy the compatibility conditions $u_0(x)=\int_\Omega k_1(x,y)u_0(y)\mathrm{d}y$ and $v_0(x)=\int_\Omega k_2(x,y)v_0(y)\mathrm{d}y$ for $x\in\partial\Omega$.

A vector valued function $(u(x,t),v(x,t))$ is called a classical solution to Problem (1.1) if $(u,v)\in[C^{1,2}(\Omega\times(0,T))\cap C(\overline{\Omega}\times[0,T))]^2$ for some T, $0<T\le+\infty$ and satisfies (1.1). If $T=+\infty$, (u,v) is called a global solution.

In the past few decades, many physical phenomena have been formulated into nonlocal parabolic equations, see [14, 21]. It has also been suggested that

Received January 11, 2011; Revised April 6, 2011.

2010 *Mathematics Subject Classification.* 35K40, 35K51.

Key words and phrases. global existence, blow-up, nonlocal sources, nonlocal boundary conditions, global blow-up, blow-up rate.

The project is supported by Key Lab of Symbolic Computation and Knowledge Engineering of Ministry of Education, by the 985 program of Jilin University and by Graduate Innovation Fund of Jilin University (20111034).

nonlocal growth terms present more realistic model in physics for compressible reactive gases; see [2].

There have been many articles which investigate the properties of solutions to partial differential equations with local boundary conditions. However, there are also some important phenomena formulated into parabolic equations coupled with nonlocal boundary conditions in mathematical modelling such as thermoelasticity theory (see [4, 7, 8]). In this case, the solution $u(x,t)$ describes entropy per volume of material.

The parabolic problem with nonlocal boundary condition of the following type

$$\begin{cases} u_t = \Delta u + g(x,u), & x \in \Omega,\ t > 0, \\ u(x,t) = \int_\Omega k(x,y)u(y,t)\mathrm{d}y, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

was studied by Friedman [12]. He established the global existence of solution, and showed that the unique solution tends to 0 monotonically and exponentially as $t \to +\infty$ in the case of $g(x,u) = c(x)u$ with $c(x) \le 0$ and $\int_\Omega |k(x,y)|dy < 1$ for all $x \in \partial\Omega$. In 1992, Deng [9] gave the comparison principle and local existence of classical solution to (1.2) with general $g(x,u)$. For the case $g(x,u) = c(x)u$, he showed that the solution exists globally and may increase at most exponentially with t under some weaker assumptions than those in [13]. Blow-up results of Problem (1.2) are due to Seo [19]. He investigated Problem (1.2) with $g(x,u) = g(u)$ and gave the blow-up condition of the positive solutions by using supersolution and subsolution method. The blow-up rate estimates for the special case $g(u) = u^p$ and $g(u) = e^u$ were also derived.

As for more general discussions on dynamics of parabolic problem with nonlocal boundary conditions, we refer the reader to [16, 17] by Pao, where the following problem

$$\begin{cases} u_t = Lu + g(x,u), & x \in \Omega,\ t > 0, \\ Bu = \int_\Omega k(x,y)u(y,t)\mathrm{d}y, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{1.3}$$

with uniformly elliptic operator

$$L = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i x_j} + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

and

$$Bu = \alpha_0 \frac{\partial u}{\partial n} + u$$

was studied. Later Pao gave the numerical solution to this problem in [18].

Semilinear parabolic equations and systems with both nonlocal reaction terms and nonlocal boundary conditions have also been studied. For example, the scalar problem

$$
\begin{cases} u_t - \Delta u = \int_\Omega g(u)\mathrm{d}x, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \int_\Omega \varphi(x,y)u(y,t)\mathrm{d}y, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{1.4}
$$

was studied by Lin and Liu [15], and parabolic system of the following type (1.5)

$$
\begin{cases} u_t - \Delta u = u^m(x,t)\int_\Omega v^n(y,t)\mathrm{d}y, & (x,t) \in \Omega \times (0,T), \\ v_t - \Delta v = v^q(x,t)\int_\Omega u^p(y,t)\mathrm{d}y, & (x,t) \in \Omega \times (0,T), \\ u = \int_\Omega \varphi(x,y)u(y,t)\mathrm{d}y, \quad v = \int_\Omega \psi(x,y)v(y,t)\mathrm{d}y, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega \end{cases}
$$

was investigated by Zheng and Kong [26]. Local existence, global existence and nonexistence of classical solutions were established and blow-up properties were discussed in their work, respectively.

Recently, Cui and Yang [6] studied the following porous medium problem with nonlocal boundary condition and nonlocal reaction term

$$
\begin{cases} u_t = \Delta u^m + au^q \int_\Omega u^p(y,t)\mathrm{d}y, & x \in \Omega,\ t > 0, \\ u(x,t) = \int_\Omega k(x,y)u(y,t)\mathrm{d}y, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.6}
$$

where $m > 1$, $a,\ p > 0$, $q \geq 0$ are constants and $u_0(x)$ and $k(x,y)$ satisfy the same assumptions as given in Problem (1.1). They proved that if $p+q > 1$ and $\int_\Omega k(x,y)\mathrm{d}y \geq 1$ for $x \in \partial\Omega$, the solution to Problem (1.6) blows up in finite time, while if $\int_\Omega k(x,y)\mathrm{d}y < 1$, there exist both global and blow-up solutions to Problem (1.6) depending on the initial datum and the constants $m,\ p$ and q. Moreover, they obtained the blow-up rate estimates under some conditions. For more related works, we refer the readers to [3, 20, 22, 24, 25] and references therein.

The above studies show that the growth and decay properties of solution to Problem (1.2)-(1.6) depend on the growth of $g(x,u)$, which is similar to general semilinear equation with homogeneous boundary condition. On the other hand, due to the appearance of the nonlocal boundary, the properties of solution heavily depend on the weight function $k(x,y)$ as well.

The present work is partially motivated by [6, 15, 26]. It is known that the problem

$$
u_t = \Delta u^m + au^p \tag{1.7}
$$

and the nonlocal one

$$
u_t = \Delta u^m + a\int_\Omega u^p(y,t)\mathrm{d}y \tag{1.8}
$$

with homogeneous Dirichlet boundary condition share the same blow-up criteria and blow-up rate. But there do exist some essential differences from the two problems. For example, the blow-up set of Problem (1.7) consists of a single point under some conditions for initial datum (such as symmetry and monotonicity of $u_0(x)$), while Problem (1.8) has global blow-up property (see [21]). The main purpose of this paper is to study the blow-up properties of Problem (1.1). Moreover, since the reaction terms are nonlocal, we obtain that the blow-up set is the whole domain whenever blow-up occurs.

This paper is organized as follows. Section 2 establishes the comparison principle for Problem (1.1) and Section 3 is devoted to the global existence and blow-up results of classical solutions. In Section 4, we give the blow-up rate estimates.

2. Comparison principle

In this section, we establish the comparison principle for Problem (1.1). Let $Q_T = \Omega \times (0,T)$ and $\Gamma_T = \partial\Omega \times (0,T) \cup \overline{\Omega} \times \{t = 0\}$. We begin with the definition of subsolutions and supersolutions to Problem (1.1).

Definition 2.1. A vector valued function $(\underline{u}(x,t), \underline{v}(x,t))$ is called a subsolution to Problem (1.1) in Q_T, if $(\underline{u}, \underline{v}) \in [C^{2,1}(Q_T) \cap C(Q_T \cup \Gamma_T)]^2$ satisfies

$$\begin{cases} \underline{u}_t \le \Delta \underline{u}^m + a\int_\Omega \underline{v}^p \mathrm{d}x, & (x,t) \in Q_T, \\ \underline{v}_t \le \Delta \underline{v}^n + b\int_\Omega \underline{u}^q \mathrm{d}x, & (x,t) \in Q_T, \\ \underline{u}(x,t) \le \int_\Omega k_1(x,y)\underline{u}(y,t)\mathrm{d}y, & (x,t) \in \partial\Omega \times (0,T), \\ \underline{v}(x,t) \le \int_\Omega k_2(x,y)\underline{v}(y,t)\mathrm{d}y, & (x,t) \in \partial\Omega \times (0,T), \\ \underline{u}(x,0) \le u_0(x),\ \underline{v}(x,0) \le v_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

A supersolution is defined in a similar way with each inequality reversed.

Theorem 2.1. *Let $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ be a nonnegative subsolution and a supersolution to (1.1), respectively, with $(\underline{u}(x,0), \underline{v}(x,0)) \le (\overline{u}(x,0), \overline{v}(x,0))$ for $x \in \overline{\Omega}$. Then $(\underline{u}, \underline{v}) \le (\overline{u}, \overline{v})$ in Q_T if either $(\underline{u}, \underline{v}) \ge (\epsilon, \epsilon) > (0,0)$ or $(\overline{u}, \overline{v}) \ge (\epsilon, \epsilon) > (0,0)$ holds.*

Proof. The technique for proving the comparison principle is quite standard. For example, see [1]. We shall sketch the argument for the convenience of the readers.

Let $\varphi_i(x,t) \in C^{2,1}(\overline{Q}_T)$ $(i = 1,2)$ be nonnegative functions with $\varphi_i \mid_{\partial\Omega \times (0,T)} = 0$. Multiplying the first inequality in (2.1) by $\varphi_1(x,t)$ and then integrating on Q_t for $0 < t < T$, we get

$$\begin{aligned} &\int_\Omega \underline{u}\varphi_1 \mathrm{d}x \\ \le &\int_\Omega \underline{u}(x,0)\varphi_1(x,0)\mathrm{d}x + \iint_{Q_t} (\underline{u}\varphi_{1\tau} + \underline{u}^m \Delta\varphi_1 + a\varphi_1 \int_\Omega \underline{v}^p \mathrm{d}x)\mathrm{d}x\mathrm{d}\tau \end{aligned}$$

$$- \int_0^t \int_{\partial\Omega} \frac{\partial \varphi_1}{\partial n} (\int_\Omega k_1(x,y)\underline{u}(y,\tau)\mathrm{d}y)^m \mathrm{d}S\mathrm{d}\tau,$$

where n is the unit outward normal vector on $\partial\Omega$. On the other hand, the supersolution $\overline{u}$ satisfies the reversed inequality

$$\begin{aligned}
&\int_\Omega \overline{u}\varphi_1 \mathrm{d}x \\
&\geq \int_\Omega \overline{u}(x,0)\varphi_1(x,0)\mathrm{d}x + \iint_{Q_t} (\overline{u}\varphi_{1\tau} + \overline{u}^m \Delta\varphi_1 + a\varphi_1 \int_\Omega \overline{v}^p \mathrm{d}x)\mathrm{d}x\mathrm{d}\tau \\
&\quad - \int_0^t \int_{\partial\Omega} \frac{\partial \varphi_1}{\partial n} (\int_\Omega k_1(x,y)\overline{u}(y,\tau)\mathrm{d}y)^m \mathrm{d}S\mathrm{d}\tau.
\end{aligned}$$

Set $w(x,t) = \underline{u}(x,t) - \overline{u}(x,t)$, $z(x,t) = \underline{v}(x,t) - \overline{v}(x,t)$, we have

$$\begin{aligned}
&\int_\Omega w(x,t)\varphi_1(x,t)\mathrm{d}x - \int_\Omega w(x,0)\varphi_1(x,0)\mathrm{d}x \\
&\leq \iint_{Q_t} (\varphi_{1\tau} + \Phi_1(x,\tau)\Delta\varphi_1) w \mathrm{d}x\mathrm{d}\tau + \iint_{Q_t} \varphi_1 (\int_\Omega \Phi_2(x,\tau) z(x,\tau)\mathrm{d}x)\mathrm{d}x\mathrm{d}\tau \\
&\quad - \int_0^t \int_{\partial\Omega} \frac{\partial \varphi_1}{\partial n} m\xi^{m-1} (\int_\Omega k_1(x,y) w(y,\tau)\mathrm{d}y)\mathrm{d}S\mathrm{d}\tau,
\end{aligned}$$

where

$$\Phi_1(x,\tau) = \int_0^1 m(\theta\underline{u}(x,\tau) + (1-\theta)\overline{u}(x,\tau))^{m-1}\mathrm{d}\theta,$$

$$\Phi_2(x,\tau) = a\int_0^1 p(\theta\underline{v}(x,\tau) + (1-\theta)\overline{v}(x,\tau))^{p-1}\mathrm{d}\theta,$$

and ξ is a function between $\int_\Omega k_1(x,y)\underline{u}(y,\tau)\mathrm{d}y$ and $\int_\Omega k_1(x,y)\overline{u}(y,\tau)\mathrm{d}y$. Noticing that $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ are bounded functions, it follows from $m > 1,\ p \geq 1$ that $\Phi_1,\ \Phi_2$ are bounded nonnegative functions. Now if $0 < p < 1$, we have $\Phi_2 \leq a\epsilon^{p-1}$ by using the assumption that $\underline{v} \geq \epsilon$ or $\overline{v} \geq \epsilon$. Thus an appropriate $\varphi_1(x,t)$ may be chosen exactly as in [1, pp. 118–123] to obtain

$$\begin{aligned}
&\int_\Omega w_+ \mathrm{d}x \\
&\leq C_1 \int_\Omega w(x,0)_+ \mathrm{d}x + C_2 \iint_{Q_t} w(x,\tau)_+ \mathrm{d}x\mathrm{d}\tau + C_3 \iint_{Q_t} z(x,\tau)_+ \mathrm{d}x\mathrm{d}\tau \\
&\leq C_2 \iint_{Q_t} w(x,\tau)_+ \mathrm{d}x\mathrm{d}\tau + C_3 \iint_{Q_t} z(x,\tau)_+ \mathrm{d}x\mathrm{d}\tau,
\end{aligned}$$

where $w_+ = \max\{w,\ 0\}$ and $C_i > 0$. Similarly, we can prove

$$\int_\Omega z_+ \mathrm{d}x \leq C_4 \iint_{Q_t} w(x,\tau)_+ \mathrm{d}x\mathrm{d}\tau + C_5 \iint_{Q_t} z(x,\tau)_+ \mathrm{d}x\mathrm{d}\tau.$$

Now, the above two inequalities combined with Gronwall's inequality show that $(w,z) \leq (0,0)$. This completes the proof. □

Remark 2.1. From the above proof, it is easy to see that the comparison principle still holds without the assumption $(\underline{u}, \underline{v})$ or $(\overline{u}, \overline{v}) \geq (\epsilon, \epsilon)$ in the case of $p,\ q \geq 1$.

To obtain the blow-up rate estimates, we need the following positivity lemma, whose proof is much the same as that of [9].

Lemma 2.1. *Suppose that* $w(x,t), z(x,t) \in C(Q_T \cup \Gamma_T) \cap C^{2,1}(Q_T)$ *satisfy*

$$\begin{cases} w_t - d_1(x,t)\Delta w \geq c_1(x,t)w(x,t) + c_2(x,t)\int_\Omega c_7(x,t)z(x,t)\mathrm{d}x, & (x,t) \in Q_T, \\ z_t - d_2(x,t)\Delta z \geq c_3(x,t)z(x,t) + c_4(x,t)\int_\Omega c_8(x,t)w(x,t)\mathrm{d}x, & (x,t) \in Q_T, \\ w(x,t) \geq \int_\Omega c_5(x,y)w(y,t)\mathrm{d}y, & (x,t) \in \partial\Omega \times (0,T), \\ z(x,t) \geq \int_\Omega c_6(x,y)z(y,t)\mathrm{d}y, & (x,t) \in \partial\Omega \times (0,T), \end{cases}$$

where $d_i(x,t) > 0$ $(i = 1,2)$ *in* Q_T, c_i $(i = 1, \ldots, 8)$ *are bounded functions in their respective domains,* $c_2, c_4, c_7, c_8 \geq 0$ *in* Q_T, $c_5(x,y), c_6(x,y) \geq 0$ *for* $x \in \partial\Omega$, $y \in \Omega$ *and are not identically zero. Then* $(w(x,0), z(x,0)) > (0,0)$ *for* $x \in \overline{\Omega}$ *implies* $(w(x,t), z(x,t)) > (0,0)$ *in* Q_T. *Moreover, if* $c_5(x,y),\ c_6(x,y) \equiv 0$ *or if* $c_5(x,y),\ c_6(x,y) \geq 0$ *and* $\int_\Omega c_5(x,y)\mathrm{d}y \leq 1,\ \int_\Omega c_6(x,y)\mathrm{d}y \leq 1$ $x \in \partial\Omega$, *then* $(w(x,0), z(x,0)) \geq (0,0)$ *implies* $(w(x,t), z(z,t)) \geq (0,0)$ *in* Q_T.

Local (in time) existence of positive classical solutions to Problem (1.1) can be obtained by using fixed point theorem (see [9, 24]). By the above comparison principle and Remark 2.1, we can get the uniqueness of classical solution to Problem (1.1) in the case of $p,\ q \geq 1$.

3. Global existence and blow-up

Compared with the homogeneous Dirichlet boundary conditions, the weight functions $k_i(x,y)$ $(i = 1,2)$ play important roles in the global existence or blow-up results for Problem (1.1).

Theorem 3.1. *Assume that* $\int_\Omega k_1(x,y)\mathrm{d}y,\ \int_\Omega k_2(x,y)\mathrm{d}y \geq 1$ *for all* $x \in \partial\Omega$. *Then every solution to* (1.1) *blows up in finite time if* $pq > 1$.

Proof. Let (f,g) be the unique solution to the following ODE

$$\begin{cases} f'(t) = a|\Omega|g^p,\ g'(t) = b|\Omega|f^q, \\ f(0) = f_0 > 0,\ g(0) = g_0 > 0, \end{cases} \tag{3.1}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. It can be seen from [23] that for any positive initial datum (f_0, g_0), (f,g) exists globally if and only if $pq \leq 1$. If we choose $(f_0, g_0) \leq (u_0, v_0)$, then it is easy to see that (f,g) is a subsolution to (1.1) since $\int_\Omega k_1(x,y)\mathrm{d}y,\ \int_\Omega k_2(x,y)\mathrm{d}y \geq 1$ for all $x \in \partial\Omega$. Noticing that $(f,g) \geq (f_0,g_0) > (0,0)$, by using the comparison principle, we know that $(f,g) \leq (u,v)$. Since (f,g) blows up in finite time when $pq > 1$, so does (u,v). The proof is complete. □

Theorem 3.2. *Assume that* $\int_\Omega k_1(x,y)\mathrm{d}y = \int_\Omega k_2(x,y)\mathrm{d}y = 1$ *for all* $x \in \partial\Omega$. *Then* (u,v) *exists globally provided that* $pq \leq 1$.

Proof. It is easy to verify that the solution to (3.1) is a supersolution to (1.1) if (f_0, g_0) is chosen to satisfy $(f_0, g_0) \geq (u_0, v_0)$. Since $(f, g) \geq (f_0, g_0) > (0, 0)$, the comparison principle implies that $(f, g) \geq (u, v)$. Thus (u, v) exists globally if $pq \leq 1$. The proof is complete. $\square$

Theorem 3.3. *Assume that* $\int_\Omega k_1(x, y)\mathrm{d}y,\ \int_\Omega k_2(x, y)\mathrm{d}y < 1$ *for all* $x \in \partial\Omega$.

(i) *If* $pq < mn$, *then every solution to* (1.1) *exists globally.*

(ii) *If* $pq = mn$, *then every solution to* (1.1) *exists globally if* (a, b) *is sufficiently small.*

(iii) *If* $p, q > 1$ *with* $pq > mn$, *then the unique solution to* (1.1) *exists globally provided that* (u_0, v_0) *or* (a, b) *is small, while it blows up in finite time if* (u_0, v_0) *is large enough.*

Proof. Let $\psi(x)$ be the unique positive classical solution to the linear elliptic problem

$$\begin{cases} -\Delta\psi = \epsilon_0, & x \in \Omega, \\ \psi(x) = k(x), & x \in \partial\Omega, \end{cases} \tag{3.2}$$

where $k(x)$ satisfying $\max\{\int_\Omega k_1(x, y)\mathrm{d}y,\ \int_\Omega k_2(x, y)\mathrm{d}y\} \leq k(x) < 1$ is a smooth function. Choose a positive constant ϵ_0 such that $0 < \psi(x) < 1$ for all $x \in \overline{\Omega}$ (such ϵ_0 exists since $0 < k(x) < 1$). Let $\overline{K} = \max_{x\in\overline{\Omega}} \psi(x)$, $\underline{K} = \min_{x\in\overline{\Omega}} \psi(x)$.

Define a vector valued function $(w(x), z(x))$ as follows:

$$w(x) = M^{l_1}\psi^{\frac{1}{m}}(x), \quad z(x) = M^{l_2}\psi^{\frac{1}{n}}(x), \tag{3.3}$$

where M, l_1, l_2 are positive constants to be determined later. Then, we have

$$\begin{aligned} w\,|_{\partial\Omega} &\geq M^{l_1}\Big(\int_\Omega k_1(x, y)\mathrm{d}y\Big)^{\frac{1}{m}} \geq M^{l_1}\int_\Omega k_1(x, y)\mathrm{d}y \\ &\geq M^{l_1}\int_\Omega k_1(x, y)\psi^{\frac{1}{m}}(y)\mathrm{d}y = \int_\Omega k_1(x, y)w(y)\mathrm{d}y, \\ z\,|_{\partial\Omega} &\geq M^{l_2}\Big(\int_\Omega k_2(x, y)\mathrm{d}y\Big)^{\frac{1}{n}} \geq M^{l_2}\int_\Omega k_2(x, y)\mathrm{d}y \\ &\geq M^{l_2}\int_\Omega k_2(x, y)\psi^{\frac{1}{n}}(y)\mathrm{d}y = \int_\Omega k_2(x, y)z(y)\mathrm{d}y. \end{aligned} \tag{3.4}$$

Here we use the assumptions $\int_\Omega k_i(x, y)\mathrm{d}y < 1$ for all $x \in \partial\Omega$ and $0 < \psi(x) < 1$.

On the other hand, we have

$$\begin{aligned} w_t - \Delta w^m - a\int_\Omega z^p\mathrm{d}x &= M^{ml_1}\epsilon_0 - aM^{pl_2}\int_\Omega \psi^{\frac{p}{n}}\mathrm{d}x \\ &\geq M^{ml_1}\epsilon_0 - a|\Omega|M^{pl_2}\overline{K}^{\frac{p}{n}}, \\ z_t - \Delta z^n - b\int_\Omega w^q\mathrm{d}x &= M^{nl_2}\epsilon_0 - bM^{ql_1}\int_\Omega \psi^{\frac{q}{m}}\mathrm{d}x \\ &\geq M^{nl_2}\epsilon_0 - b|\Omega|M^{ql_1}\overline{K}^{\frac{q}{m}}. \end{aligned} \tag{3.5}$$

(i) In the case of $pq < mn$, we can choose $l_1,\ l_2 > 0$ such that $\frac{m}{p} > \frac{l_2}{l_1} > \frac{q}{n}$. Combining (3.4) with (3.5), we know that if we take

$$M = \max\{(a|\Omega|\overline{K}^{\frac{p}{n}}\epsilon_0^{-1})^{\frac{1}{ml_1-pl_2}}, (b|\Omega|\overline{K}^{\frac{q}{m}}\epsilon_0^{-1})^{\frac{1}{nl_2-ql_1}}, (\underline{K}^{-\frac{1}{m}}\max_{\overline{\Omega}} u_0(x))^{1/l_1}, (\underline{K}^{-\frac{1}{n}}\max_{\overline{\Omega}} v_0(x))^{1/l_2}\},$$

then (w,z) defined as in (3.3) is a supersolution to Problem (1.1) and $(w,z) \geq (M^{l_1}\underline{K}^{\frac{1}{m}}, M^{l_2}\underline{K}^{\frac{1}{n}})$. The comparison principle guarantees that $(u,v) \leq (w,z)$, and hence (u,v) exists globally.

(ii) In the case of $pq = mn$, we can choose $l_1,\ l_2 > 0$ such that $\frac{m}{p} = \frac{l_2}{l_1} = \frac{q}{n}$. Then for any given (u_0,v_0), we first choose $M > 0$ suitable large such that $(u_0,v_0) \leq (M^{l_1}\psi^{\frac{1}{m}}(x), M^{l_2}\psi^{\frac{1}{n}}(x))$. Set $a_0 = \epsilon_0|\Omega|^{-1}\overline{K}^{-\frac{p}{n}},\ b_0 = \epsilon_0|\Omega|^{-1}\overline{K}^{-\frac{q}{m}}$, then it is easy to verify that (w,z) is a supersolution to Problem (1.1) provided $a \leq a_0,\ b \leq b_0$. Thus we know that (u,v) exists globally by using the comparison principle again.

(iii) In the case of $pq > mn$, there exist both global and blow-up solutions depending on the initial data (u_0,v_0) and the coefficients (a,b). For the global existence part, the proof is similar to that of (i) and (ii). First, choose $l_1,\ l_2 > 0$ such that $\frac{m}{p} < \frac{l_2}{l_1} < \frac{q}{n}$. For any given $a,\ b > 0$, if we take

$$M = \min\{(\epsilon_0 a^{-1}|\Omega|^{-1}\overline{K}^{-\frac{p}{n}})^{\frac{1}{pl_2-ml_1}}, (\epsilon_0 b^{-1}|\Omega|^{-1}\overline{K}^{-\frac{q}{m}})^{\frac{1}{ql_1-nl_2}}\},$$

then (w,z) is a supersolution to (1.1) provided that

$$(u_0,v_0) \leq (M^{l_1}\psi^{\frac{1}{m}}(x),\ M^{l_2}\psi^{\frac{1}{n}}(x)). \tag{3.6}$$

By the comparison principle, we know that (u,v) exists globally provided that (u_0,v_0) satisfies (3.6).

On the other hand, for any given initial datum (u_0,v_0), there exists a suitable large constant $M > 0$ such that $(u_0,v_0) \leq (M^{l_1}\psi^{\frac{1}{m}}(x), M^{l_2}\psi^{\frac{1}{n}}(x))$. For such a fixed M, set $a_0 = \epsilon_0|\Omega|^{-1}\overline{K}^{-\frac{p}{n}}M^{ml_1-pl_2},\ b_0 = \epsilon_0|\Omega|^{-1}\overline{K}^{-\frac{q}{m}}M^{nl_2-ql_1}$. Then we know that (w,z) is a supersolution to (1.1) if $(a,b) \leq (a_0,b_0)$. Again by using the comparison principle, we obtain the global existence of (u,v).

To prove the blow-up result, we consider the following porous medium problem

$$\begin{cases} \underline{u}_t = \Delta\underline{u}^m + a\int_\Omega \underline{v}^p \mathrm{d}x, & x \in \Omega,\ t > 0, \\ \underline{v}_t = \Delta\underline{v}^n + b\int_\Omega \underline{u}^q \mathrm{d}x, & x \in \Omega,\ t > 0, \\ \underline{u}(x,t) = \underline{v}(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ \underline{u}(x,0) = \underline{u}_0(x) \leq u_0(x), \underline{v}(x,0) = \underline{v}_0(x) \leq v_0(x), & x \in \Omega. \end{cases} \tag{3.7}$$

Let $(\underline{u},\underline{v})$ be the unique solution to Problem (3.7). It is obvious that $(\underline{u},\underline{v})$ is a subsolution to Problem (1.1). It can be seen from [11] that $(\underline{u},\underline{v})$ blows up in finite time if $(\underline{u}_0(x),\underline{v}_0(x))$ is large enough. Thus, the unique solution to

(1.1) blows up in finite time if (u_0, v_0) is large enough by using the comparison principle. The proof is complete. □

At the end of this section, we shall point out that the blow-up is global whenever blow-up occurs.

Definition 3.1. Suppose that (u, v) blows up in a finite time T. We say that x^* is a blow-up point of (u, v) if $\limsup_{t\to T^-}(u(x^*, t) + v(x^*, t)) = +\infty$.

We say that the blow-up is global if every point in Ω is a blow-up point. To prove the global blow-up property, we need the following lemma first.

Lemma 3.1. *Assume that the solution* (u, v) *to Problem* (1.1) *blows up in a finite time* T*. Let*

$$h_1(t) = a\int_\Omega v^p \mathrm{d}x, \quad H_1(t) = \int_0^t h_1(s)\mathrm{d}s,$$
$$h_2(t) = b\int_\Omega u^q \mathrm{d}x, \quad H_2(t) = \int_0^t h_2(s)\mathrm{d}s.$$

Then we have

$$\limsup_{t\to T^-} h_i(t) = +\infty, \qquad \lim_{t\to T^-} H_i(t) = +\infty, \quad i = 1, 2.$$

Proof. Let $U(t) = \max_{x\in\overline{\Omega}} u(x, t)$, $V(t) = \max_{x\in\overline{\Omega}} v(x, t)$. Then $U(t)$ and $V(t)$ are Lipschitz continuous and satisfy (see [13], Theorem 4.5)

$$U'(t) \le h_1(t), \quad V'(t) \le h_2(t) \quad \text{a.e. } t \in (0, T). \tag{3.8}$$

Integrating the inequalities in (3.8) over $(0, t)$, we get

$$U(t) \le U(0) + H_1(t), \quad V(t) \le V(0) + H_2(t), \quad t \in (0, T). \tag{3.9}$$

Since (u, v) blows up in a finite time T, it can be deduced that

$$U(t) \to +\infty \quad \text{or} \quad V(t) \to +\infty \quad \text{as} \quad t \to T^-.$$

Without loss of generality, we may assume that $U(t) \to +\infty$ as $t \to T^-$. Then we see from the first inequality in (3.9) that $\lim_{t\to T^-} H_1(t) = +\infty$, which in turn implies $\limsup_{t\to T^-} h_1(t) = +\infty$. By the definition of $h_1(t)$, we have $\limsup_{t\to T^-} V(t) = +\infty$. Applying the second inequality in (3.9), we obtain $\lim_{t\to T^-} H_2(t) = +\infty$ and $\limsup_{t\to T^-} h_2(t) = +\infty$. The proof is complete. □

Theorem 3.4. *If the solution* (u, v) *to* (1.1) *blows up in a finite time* T*, then* (u, v) *blows up globally.*

Proof. The method used in our paper to prove the global blow-up property is similar to that of [5]. For any given $x_1 \in \Omega$, set $R_1 = \mathrm{dist}(x_1, \partial\Omega)$, $\Omega_1 = \{x; |x - x_1| < R_1\}$ and $r = |x - x_1|$. Choose two functions $w_0(r)$ and $z_0(r)$ such that

$$w_0(r), z_0(r) > 0 \quad \text{for} \quad 0 \le r < R_1, \quad w_0(R_1) = z_0(R_1) = 0,$$

$$w_0(r) \le u_0(x), \quad z_0(r) \le v_0(x), \quad w_0'(r), z_0'(r) \le 0 \quad \text{for} \quad 0 \le r \le R_1.$$

Consider the following problem

$$\begin{cases} w_t = \Delta w^m + h_1(t), & x \in \Omega_1, t > 0, \\ z_t = \Delta z^n + h_2(t), & x \in \Omega_1, t > 0, \\ w(x,t) = z(x,t) = 0, & x \in \partial\Omega_1, t > 0, \\ w(x,0) = w_0(r), \quad z(x,0) = z_0(r), & x \in \Omega_1. \end{cases} \tag{3.10}$$

Then by a similar method used in [13] one sees that $w(x,t) = w(r,t)$, $z(x,t) = z(r,t)$ and $w_r(r,t)$, $z_r(r,t) \le 0$ for $0 \le r \le R_1$ and $t \ge 0$. By the classical comparison principle for porous medium system, we know that

$$w(x,t) \le u(x,t), \quad z(x,t) \le v(x,t), \quad x \in \Omega_1, \quad 0 \le t < T. \tag{3.11}$$

Denote by $\lambda_1 > 0$ and $\phi(x) > 0$ ($x \in \Omega_1$) the first eigenvalue and the corresponding eigenfunction of the eigenvalue problem

$$-\Delta\phi = \lambda\phi, \quad x \in \Omega_1, \quad \phi(x) = 0, \quad x \in \partial\Omega_1.$$

Normalizing: $\int_{\Omega_1} \phi(x)dx = 1$.

Multiplying both sides of the first equation in (3.10) by $\phi(x)$ and integrating the resulting equality over $Q_{1t} \triangleq \Omega_1 \times (0,t)$, we get

$$\begin{aligned} \int_{\Omega_1} w(x,t)\phi(x)\mathrm{d}x &= \int_{\Omega_1} w_0(x)\phi(x)\mathrm{d}x - \lambda_1 \iint_{Q_{1t}} w^m\phi\mathrm{d}x\mathrm{d}s + H_1(t) \\ &\quad - \int_0^t \int_{\partial\Omega_1} w^m \frac{\partial\phi}{\partial\nu}\mathrm{d}\sigma\mathrm{d}s \\ &= \int_{\Omega_1} w_0(x)\phi(x)\mathrm{d}x - \lambda_1 \iint_{Q_{1t}} w^m\phi\mathrm{d}x\mathrm{d}s + H_1(t). \end{aligned} \tag{3.12}$$

From (3.12) and the fact that $w_r \le 0$, one obtains

$$w(x_1,t) \ge \int_{\Omega_1} w_0(x)\phi(x)\mathrm{d}x - \lambda_1 \iint_{Q_{1t}} w^m\phi\mathrm{d}x\mathrm{d}s + H_1(t). \tag{3.13}$$

Letting $t \to T^-$ and applying Lemma 3.1 we know that if $\int_0^T \int_{\Omega_1} w^m\phi\mathrm{d}x\mathrm{d}s < +\infty$, then

$$\limsup_{t\to T^-} w(x_1,t) = +\infty.$$

It is obvious that if $\int_0^T \int_{\Omega_1} w^m\phi\mathrm{d}x\mathrm{d}s = +\infty$, then $\limsup_{t\to T^-} w(x_1,t) = +\infty$. Using (3.11) and the arbitrariness of x_1, we see that $u(x,t)$ blows up globally. Applying similar arguments to the second equation in (3.10), we can deduce that $v(x,t)$ also blows up globally. Thus we complete the proof. □

Remark 3.1. From Theorem 3.4, we know that the blow-up set of porous medium system with nonlocal boundary conditions is the same as that of the homogeneous Dirichlet boundary conditions when the sources are nonlocal.

4. Blow-up rate estimates

In this section, we will show the blow-up rate of the solution to Problem (1.1) in the case of $p,\ q > 1$ and $\int_\Omega k_i(x,y)\mathrm{d}y \le 1$ $(i = 1,2)$ for $x \in \partial\Omega$. To achieve this, we need an additional assumption on the initial datum (u_0, v_0):

(H) $u_0,\ v_0 \in C^{2+\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ and there exists a constant $\delta > \delta_0 > 0$ such that

$$\Delta u_0^m + a\int_\Omega v_0^p \mathrm{d}x \ge \delta u_0^{mk_1+1}, \quad \Delta v_0^n + b\int_\Omega u_0^q \mathrm{d}x \ge \delta v_0^{nk_2+1},$$

where $\delta_0,\ k_1,\ k_2$ will be given later. The main result of this section is the following theorem on the blow-up rate.

Theorem 4.1. *Suppose that* (u_0, v_0) *satisfies the assumption* (H), $p,\ q > 1$ *and* $\int_\Omega k_i(x,y)\mathrm{d}y \le 1$ $(i = 1,2)$. *If* (u,v) *is the classical solution to* (1.1) *and blows up in a finite time* T, *then there exist four positive constants* C_1, C_2, C_3, C_4 *such that*

$$C_1 \le \max_{x\in\overline{\Omega}} u(x,t)(T-t)^{\frac{p+1}{pq-1}} \le C_2,$$
$$C_3 \le \max_{x\in\overline{\Omega}} v(x,t)(T-t)^{\frac{q+1}{pq-1}} \le C_4.$$

In order to get the blow-up rate estimates, we firstly introduce some transformations. Let $u^m = U(x,t),\ v^n = V(x,t)$. Then (1.1) becomes

$$(4.1)\quad \begin{cases} U_t = mU^{r_1}(\Delta U + a\int_\Omega V^{p_1}\mathrm{d}x), & x\in\Omega,\ t>0, \\ V_t = nV^{r_2}(\Delta V + b\int_\Omega U^{q_1}\mathrm{d}x), & x\in\Omega,\ t>0, \\ U(x,t) = (\int_\Omega k_1(x,y)U^{\frac{1}{m}}(y,t)\mathrm{d}y)^m, & x\in\partial\Omega,\ t>0, \\ V(x,t) = (\int_\Omega k_2(x,y)V^{\frac{1}{n}}(y,t)\mathrm{d}y)^n, & x\in\partial\Omega,\ t>0, \\ U(x,0) = U_0(x) = u_0^m(x),\ V(x,0) = V_0(x) = v_0^n(x), & x\in\Omega, \end{cases}$$

where $0 < r_1 = (m-1)/m < 1$, $0 < r_2 = (n-1)/n < 1$, $p_1 = p/n$ and $q_1 = q/m$.

Under these transformations, the assumption (H) becomes

(H′) $U_0,\ V_0 \in C^{2+\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ and there exists a constant $\delta > \delta_0 > 0$ such that

$$(4.2)\quad \Delta U_0 + a\int_\Omega V_0^{p_1}\mathrm{d}x \ge \delta U_0^{k_1+1-r_1}, \quad \Delta V_0 + b\int_\Omega U_0^{q_1}\mathrm{d}x \ge \delta V_0^{k_2+1-r_2},$$

where $\delta_0,\ k_1,\ k_2$ will be determined later.

Suppose that the solution to (4.1) blows up in a finite time T, and set $M_1(t) = \max_{x\in\overline{\Omega}} U(x,t)$, $M_2(t) = \max_{x\in\overline{\Omega}} V(x,t)$, then we can get the blow-up rate from the following lemmas.

Lemma 4.1. *Suppose that* $U_0,\ V_0$ *satisfy* (H′). *Then there exists a positive constant* K_1 *such that*

$$M_1^{q_1-r_1+1}(t)+M_2^{p_1-r_2+1}(t)\geq K_1(T-t)^{-\frac{(q_1-r_1+1)(p_1-r_2+1)}{p_1q_1-(1-r_1)(1-r_2)}}. \tag{4.3}$$

Proof. We can easily see that $M_1(t),\ M_2(t)$ are Lipschitz continuous (see [13], Theorem 4.5) and thus they are differentiable almost everywhere.

By the equations in (4.1) and $\Delta M_1(t)\leq 0,\ \Delta M_2(t)\leq 0$, we have

$$M_1'(t)\leq ma|\Omega|M_1^{r_1}(t)M_2^{p_1}(t),\ M_2'(t)\leq nb|\Omega|M_1^{q_1}(t)M_2^{r_2}(t) \text{ a.e. } t\in(0,T). \tag{4.4}$$

Noticing that $q_1-r_1+1>0$ and $p_1-r_2+1>0$, we get the following inequalities by virtue of Young's inequality

$$\begin{aligned}&(M_1^{q_1-r_1+1}(t)+M_2^{p_1-r_2+1}(t))'\\ &\leq (ma|\Omega|(q_1-r_1+1)+nb|\Omega|(p_1-r_2+1))M_1^{q_1}M_2^{p_1}\\ &\leq C(M_1^{q_1-r_1+1}(t)+M_2^{p_1-r_2+1}(t))^{\frac{(q_1-r_1+1)p_1+(p_1-r_2+1)q_1}{(q_1-r_1+1)(p_1-r_2+1)}}.\end{aligned} \tag{4.5}$$

Here $\frac{(q_1-r_1+1)p_1+(p_1-r_2+1)q_1}{(q_1-r_1+1)(p_1-r_2+1)}>1$, since $p,\ q>1$. Integrating (4.5) over (t,T), we obtain (4.3). □

Lemma 4.2. *Suppose that* $U_0,\ V_0$ *satisfy* (H′). *Then we have*

$$U_t\geq\delta_0U^{k_1+1},\ V_t\geq\delta_0V^{k_2+1},\ (x,t)\in Q_T, \tag{4.6}$$

where

$$\begin{aligned}&k_1=\frac{p_1q_1-(1-r_1)(1-r_2)}{p_1+1-r_2},\ k_2=\frac{p_1q_1-(1-r_1)(1-r_2)}{q_1+1-r_1},\\ &\delta>\delta_0=\max\{\delta_1,\delta_2\}>0,\\ &\delta_1=\frac{mak_1(k_1+1)|\Omega|}{r_1(2k_1+1-r_1)}(\frac{k_1+1}{p_1+k_2})^{\frac{p_1}{k_2}},\\ &\delta_2=\frac{nbk_2(k_2+1)|\Omega|}{r_2(2k_2+1-r_2)}(\frac{k_2+1}{q_1+k_1})^{\frac{q_1}{k_1}}.\end{aligned}$$

Proof. Set $J_1(x,t)=U_t-\delta_0U^{k_1+1},\ J_2(x,t)=V_t-\delta_0V^{k_2+1}$ for $(x,t)\in Q_T$, then by assumption (H′), we have

$$J_1(x,0)>0,\quad J_2(x,0)>0,\quad x\in\Omega. \tag{4.7}$$

A straightforward computation yields

$$\begin{aligned}&J_{1t}-mU^{r_1}\Delta J_1-2r_1\delta_0U^{k_1}J_1-map_1U^{r_1}\int_\Omega V^{p_1-1}J_2\mathrm{d}x\\ &=r_1U^{-1}J^2+m\delta_0k_1(k_1+1)U^{r_1+k_1-1}|\nabla U|^2+r_1\delta_0^2U^{2k_1+1}\\ &\quad+map_1\delta_0U^{r_1}\int_\Omega V^{p_1+k_2}\mathrm{d}x-ma\delta_0(1+k_1)U^{k_1+r_1}\int_\Omega V^{p_1}\mathrm{d}x\end{aligned} \tag{4.8}$$

$$\geq r_1\delta_0^2 U^{2k_1+1} + map_1\delta_0 U^{r_1}\int_\Omega V^{p_1+k_2}\mathrm{d}x - ma\delta_0(1+k_1)U^{k_1+r_1}\int_\Omega V^{p_1}\mathrm{d}x$$
$$\geq r_1\delta_0^2 U^{2k_1+1} + map_1\delta_0 U^{r_1}\int_\Omega V^{p_1+k_2}\mathrm{d}x$$
$$- ma\delta_0(1+k_1)|\Omega|^{k_2/(p_1+k_2)}U^{k_1+r_1}\left(\int_\Omega V^{p_1+k_2}\mathrm{d}x\right)^{p_1/(p_1+k_2)}.$$

Since $k_1/(2k_1+1-r_1)+p_1/(p_1+k_2)=1$, by virtue of Young's inequality, we have

$$U^{k_1}\left(\int_\Omega V^{p_1+k_2}\mathrm{d}x\right)^{p_1/(p_1+k_2)}$$
$$\leq \frac{k_1\theta^{\frac{2k_1+1-r_1}{k_1}}}{2k_1+1-r_1}U^{2k_1+1-r_1} + \frac{p_1\theta^{-\frac{p_1+k_2}{p_1}}}{p_1+k_2}\int_\Omega V^{p_1+k_2}\mathrm{d}x, \tag{4.9}$$

where

$$\theta = ((k_1+1)/(p_1+k_2))^{p_1/(p_1+k_2)}|\Omega|^{p_1k_2/(p_1+k_2)^2}. \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.8) deduces

$$J_{1t} - mU^{r_1}\Delta J_1 - 2r_1\delta_0 U^{k_1}J_1 - map_1U^{r_1}\int_\Omega V^{p_1-1}J_2\mathrm{d}x$$
$$\geq r_1\delta_0^2U^{2k_1+1} + map_1\delta_0U^{r_1}\int_\Omega V^{p_1+k_2}\mathrm{d}x \tag{4.11}$$
$$- \frac{ma\delta_0k_1(k_1+1)}{2k_1+1-r_1}|\Omega|^{k_2/(p_1+k_2)}\theta^{\frac{2k_1+1-r_1}{k_1}}U^{2k_1+1}$$
$$- \frac{ma\delta_0p_1(k_1+1)}{p_1+k_2}|\Omega|^{k_2/(p_1+k_2)}\theta^{-\frac{p_1+k_2}{p_1}}U^{r_1}\int_\Omega V^{p_1+k_2}\mathrm{d}x$$
$$= r_1\delta_0(\delta_0-\delta_1)U^{2k_1+1}$$
$$\geq 0.$$

Similarly, we have

$$J_{2t} - nV^{r_2}\Delta J_2 - 2r_2\delta_0V^{k_2}J_2 - nbq_1V^{r_2}\int_\Omega U^{q_1-1}J_1\mathrm{d}x \geq 0. \tag{4.12}$$

Fix $(x,t)\in\partial\Omega\times(0,T)$, we have

$$J_1(x,t) = \left(\int_\Omega k_1(x,y)u(y,t)\mathrm{d}y\right)^{m-1}$$
$$\times\left(\int_\Omega mk_1(x,y)u_t(y,t)\mathrm{d}y - \delta_0\left(\int_\Omega k_1(x,y)u(y,t)\mathrm{d}y\right)^{mk_1+1}\right).$$

Since $U_t(y,t) = J_1(y,t)+\delta_0U^{k_1+1}(y,t)$, we have

$$\int_\Omega mk_1(x,y)u_t(y,t)\mathrm{d}y - \delta_0\left(\int_\Omega k_1(x,y)u(y,t)\mathrm{d}y\right)^{mk_1+1}$$

$$= \delta_0 \left(\int_\Omega k_1(x,y) U^{\frac{1+mk_1}{m}}(y,t)\mathrm{d}y - \left(\int_\Omega k_1(x,y) U^{\frac{1}{m}}(y,t)\mathrm{d}y \right)^{mk_1+1} \right) + \int_\Omega k_1(x,y) U^{\frac{1-m}{m}}(y,t) J_1(y,t)\mathrm{d}y.$$

Noticing that $0 < F_1(x) = \int_\Omega k_1(x,y)\mathrm{d}y \le 1,\ x \in \partial\Omega$, we can apply Jensen's inequality to get

$$\begin{aligned} &\int_\Omega k_1(x,y) U^{\frac{1+mk_1}{m}}(y,t)\mathrm{d}y - \left(\int_\Omega k_1(x,y) U^{\frac{1}{m}}(y,t)\mathrm{d}y \right)^{mk_1+1} \\ &\ge F_1(x) \left(\int_\Omega k_1(x,y) U^{\frac{1}{m}}(y,t) \frac{\mathrm{d}y}{F_1(x)} \right)^{mk_1+1} - \left(\int_\Omega k_1(x,y) U^{\frac{1}{m}}(y,t)\mathrm{d}y \right)^{mk_1+1} \\ &\ge 0. \end{aligned}$$

Hence, for $(x,t) \in \partial\Omega \times (0,T)$, we have

$$J_1(x,t) \ge \left(\int_\Omega k_1(x,y) U^{\frac{1}{m}}(y,t)\mathrm{d}y \right)^{m-1} \left(\int_\Omega k_1(x,y) U^{\frac{1-m}{m}}(y,t) J_1(y,t)\mathrm{d}y \right). \tag{4.13}$$

Similarly, we have

$$J_2(x,t) \ge \left(\int_\Omega k_2(x,y) V^{\frac{1}{n}}(y,t)\mathrm{d}y \right)^{n-1} \left(\int_\Omega k_2(x,y) V^{\frac{1-n}{n}}(y,t) J_2(y,t)\mathrm{d}y \right). \tag{4.14}$$

Since $U(x,t),\ V(x,t)$ are positive continuous functions for $(x,t) \in \overline{\Omega} \times [0,T)$, it follows from (4.7), (4.11)-(4.14) and Lemma 2.1 that $J_1(x,t),\ J_2(x,t) \ge 0$ for $(x,t) \in \overline{\Omega} \times [0,T)$, i.e., $U_t \ge \delta_0 U^{k_1+1},\ V_t \ge \delta_0 V^{k_2+1}$. The proof is complete. □

Integrating (4.6) over (t,T), we conclude that

$$\begin{aligned} U(x,t) &\le K_3 (T-t)^{-(p_1+1-r_2)/(p_1q_1-(1-r_1)(1-r_2))}, \\ V(x,t) &\le K_4 (T-t)^{-(q_1+1-r_1)/(p_1q_1-(1-r_1)(1-r_2))}, \end{aligned} \tag{4.15}$$

where $K_3,\ K_4$ are positive constants independent of t. Combining (4.3) with (4.15), we obtain the following result.

Lemma 4.3. *Suppose that $U_0,\ V_0$ satisfy* (H$'$). *If (U,V) is the solution to system* (4.1) *and blows up in a finite time T, then there exist four positive constants $K_3,\ K_4,\ K_5,\ K_6$ such that*

$$K_5 \le \max_{x\in\overline{\Omega}} U(x,t)(T-t)^{\frac{p_1+1-r_2}{p_1q_1-(1-r_1)(1-r_2)}} \le K_3,$$

$$K_6 \le \max_{x\in\overline{\Omega}} V(x,t)(T-t)^{\frac{q_1+1-r_1}{p_1q_1-(1-r_1)(1-r_2)}} \le K_4.$$

According the transformations and Lemma 4.3, we obtain Theorem 4.1.

Remark 4.1. From Theorem 4.1, we know that in the case of $\int_\Omega k_i(x,y)\mathrm{d}y \leq 1$, $x \in \partial\Omega$, $p > 1$ and $q > 1$, the blow-up rate of porous medium system with nonlocal boundary conditions is the same as that of porous medium system with homogeneous Dirichlet boundary conditions when the reaction terms are nonlocal.

Acknowledgment. The authors would like to thank the referees for their valuable comments and suggestions regarding the original manuscript.

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