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STAR OPERATORS ON *sn*-NETWORKS

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Abstract. Star operations are defined by R. E. Hodel in 1994. In this paper some relations among star operators, sequential closure operators and closure operators are discussed. Moreover, we introduce an induced topology by a family of subsets of a space, and some interesting results about star operators are established by the induced topology.

1. Introduction

Weak bases are an important concept in generalized metric spaces. F. Siwiec [9] proved that a Hausdorff space is first-countable if and only if it is a Fréchet, weakly first-countable space. R. E. Hodel [5] introduced the star operator on a weakly first-countable space and also showed the result of F. Siwiec by using the star operator. Recently, Woo Chorl Hong [6] proved that the star operator and the sequential closure operator on a weakly first-countable space are the same and get the result of F. Siwiec and R. E. Hodel by a different method from R. E. Hodel.

It is well known that the closure operator and sequential closure operator are defined in an arbitrary topological space. Therefore, we shall consider a star operator in an arbitrary topological space, and also discuss some relations among closure operators, sequential closure operators and star operators. In particular, the sequential closure operators are equivalent to the star operators under some conditions.

Let X be a topological space, and $\mathcal{P}$ a family of subsets of X. The family $\mathcal{P}$ is called a *standard network* for X if it satisfies the following conditions:

(1) $\mathcal{P} = \cup_{x\in X}\mathcal{P}_x$;

(2) $\mathcal{P}_x$ is a network at x in X for each $x \in X$, i.e., if $x \in U$ and U is open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}_x$;

(3) if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

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Definition 1.1. Let $A \subset X$. Put

$$[A] = \{x \in X : \text{ there is a sequence in } A \text{ converging to } x\}.$$

The operation $[\bullet] : \mathcal{P}(X) \to \mathcal{P}(X)$ is called a *sequential closure operator* [3] on the space X.

If $\mathcal{P} = \cup_{x \in X} \mathcal{P}_x$ is a standard network for X, for each $A \subset X$ put

$$(A)^*_{\mathcal{P}} = \{x \in X : P \cap A \neq \emptyset \text{ for each } P \in \mathcal{P}_x\}.$$

The operation $(\bullet)^*_{\mathcal{P}} : \mathcal{P}(X) \to \mathcal{P}(X)$ is called a *star operator* [5] on the space X.

It is obvious that $(A)^*_{\mathcal{P}}, [A] \subset \overline{A}$.

Question 1.1. How to characterize a standard network $\mathcal{P}$ for a space X such that $(A)^*_{\mathcal{P}} = \overline{A}$, or $(A)^*_{\mathcal{P}} = [A]$ for each $A \subset X$?

In this paper, certain relations between $(A)^*_{\mathcal{P}}$ and $[A]$ are discussed.

2. Star operators

Let X be a space, $x \in X$, and $P \subset X$. P is said to be a *sequential neighborhood* at x in X if each sequence converging to x is eventually in P. P is said to be *sequentially open* if P is a sequential neighborhood at x in X for each $x \in P$. X is called a *sequential space* [3] if each sequentially open subset in X is open.

Definition 2.1. Let $\mathcal{P}$ be a standard network of a space X. $\mathcal{P}$ is called an *sn-network* [7] for X if each element of $\mathcal{P}_x$ is a sequential neighborhood at x in X. $\mathcal{P}$ is called a *weak base* [1] for X if $G \subset X$ is open in X whenever for each $x \in G$, $P \subset G$ for some $P \in \mathcal{P}_x$.

A space is called *weakly first-countable* [1] (respectively, *sn-first-countable*) if X has a weak base (respectively, an *sn*-network) $\mathcal{P}$ such that each $\mathcal{P}_x$ is countable.

Every weakly first-countable space is sequential [9].

Lemma 2.1 ([7])**.** *Let X be a space, and $\mathcal{P}$ a family of subsets of X.*

(1) *If $\mathcal{P}$ is a weak base for X, it is an sn-network.*

(2) *If X is a sequential space and $\mathcal{P}$ is an sn-network for X, $\mathcal{P}$ is a weak base.*

Theorem 2.1. *Let X be a space, and $\mathcal{P}$ a standard network for X. Then*

(1) *$\mathcal{P}$ is a neighborhood base for X if and only if $(A)^*_{\mathcal{P}} = \overline{A}$ for each $A \subset X$.*

(2) *$\mathcal{P}$ is an sn-network for X if and only if $[A] \subset (A)^*_{\mathcal{P}}$ for each $A \subset X$.*

Proof. Put $\mathcal{P} = \cup_{x \in X} \mathcal{P}_x$.

(1) If $\mathcal{P}_x$ is a neighborhood base at x in X for each $x \in X$, it is obvious that $(A)^*_{\mathcal{P}} = \overline{A}$ for each $A \subset X$. Conversely, for each $x \in X$ and $P \in \mathcal{P}_x$, since

$P \cap (X - P) = \emptyset$, $x \notin (X - P)^*_{\mathcal{P}} = \overline{X - P} = X - P^\circ$, thus $x \in P^\circ$. Hence, $\mathcal{P}_x$ is a neighborhood base at x in X.

(2) If there is $x \in X$ such that $\mathcal{P}_x$ is not an sn-network at x in X, then P is not a sequential neighborhood at x in X for some $P \in \mathcal{P}_x$, thus there is a sequence $\{x_n\}$ in $X - P$ converging to x, hence $x \in [X - P] \subset (X - P)^*_{\mathcal{P}}$, so $P \cap (X - P) \neq \emptyset$, which is a contradiction. Conversely, suppose that $\mathcal{P}_x$ is an sn-network at x in X for each $x \in X$. Let $A \subset X$. If $x \in [A]$, there is a sequence $\{x_n\}$ in A converging to x. Let $P \in \mathcal{P}_x$, then the sequence $\{x_n\}$ is eventually in P, and thus $P \cap A \neq \emptyset$, which follows that $x \in (A)^*_{\mathcal{P}}$. Therefore, $[A] \subset (A)^*_{\mathcal{P}}$. □

Theorem 2.2. *Let $\mathcal{P}$ be a standard network for a space X. Then $(A)^*_{\mathcal{P}} \subset [A]$ for each $A \subset X$ if and only if there is $P_x \in \mathcal{P}_x$ such that $P_x \subset U$ whenever U is a sequential neighborhood at a point x in X.*

Proof. Suppose that $(A)^*_{\mathcal{P}} \subset [A]$ for each $A \subset X$. If there are $x \in X$ and a sequential neighborhood U at x in X such that $P \not\subset U$ for each $P \in \mathcal{P}_x$, then it can take a point $x(P) \in P - U$ for each $P \in \mathcal{P}_x$. Put $A = \{x(P) : P \in \mathcal{P}\}$. Then $x \in (A)^*_{\mathcal{P}} \subset [A]$, and hence there is a sequence $\{x_n\}$ in A such that $x_n \to x$. Then the $\{x_n\}$ is eventually in U, which is a contradiction.

Conversely, let $A \subset X$. If there is $x \in (A)^*_{\mathcal{P}} - [A]$, then $X - A$ is a sequential neighborhood at x in X, thus $P_x \subset X - A$ for some $P_x \in \mathcal{P}_x$, and $P_x \cap A = \emptyset$, which is a contradiction. □

Corollary 2.1. *If a space X has a standard network $\mathcal{P}$ with $(A)^*_{\mathcal{P}} = [A]$ for each $A \subset X$, then $\mathcal{P}$ is a weak base for X if and only if X is sequential.*

Proof. Since $(A)^*_{\mathcal{P}} = [A]$ for each $A \subset X$, $\mathcal{P}$ is an sn-network by Theorem 2.1. If X is sequential, then $\mathcal{P}$ is a weak base for X by Lemma 2.1. Conversely, let $\mathcal{P}$ be a weak base for X. If a subset U of X is a sequential neighborhood at x in X for each $x \in U$, then U is open by Theorem 2.2 and Definition 2.1. Hence, X is sequential. □

Theorem 2.3. *Let $\mathcal{P} = \cup_{x \in X} \mathcal{P}_x$ be a standard network for a space X such that each $\mathcal{P}_x$ is countable. Then $(A)^*_{\mathcal{P}} \subset [A]$ for each $A \subset X$.*

Proof. Let $A \subset X$. If $x \in (A)^*_{\mathcal{P}}$, there is a decreasing sequence $\{P_n\}_{n \in \mathbb{N}} \subset \mathcal{P}_x$ such that $\{P_n\}_{n \in \mathbb{N}}$ is a network at x in X. Take a sequence $\{x_n\}_{n \in \mathbb{N}}$ with each $x_n \in P_n \cap A$. Then the sequence $\{x_n\}$ converges to x, so $x \in [A]$. Therefore, $(A)^*_{\mathcal{P}} \subset [A]$. □

If $\mathcal{P}$ is an sn-network for X such that each $\mathcal{P}_x$ is countable, then $(A)^*_{\mathcal{P}} = [A]$ for each $A \subset X$ by Theorems 2.1 and 2.3, which is shown for weakly first-countable spaces by W. C. Hong [6]. But, the converse of Theorem 2.3 is not hold.

Recall that a space X is called a *Fréchet space* [4] if $x \in \overline{A} \subset X$ there is a sequence in A which converges to x. Every first-countable space is Fréchet, and every Fréchet space is sequential.

If $\mathcal{P}$ is a standard network for a Fréchet space X, then $(A)^*_{\mathcal{P}} \subset \overline{A} = [A]$ for each $A \subset X$.

Example 2.1. The sequential fan S_ω is defined as follows. Let T_n be a sequence converging to $a_n \notin T_n$ for each $n \in \mathbb{N}$. Put $T_0 = \{a_n : n \in \mathbb{N}\}$. Let T be the topological sum of $\{T_n \cup \{a_n\} : n \in \mathbb{N}\}$.

$$S_\omega = \{s\} \cup \big(\bigcup\{T_n : n \in \mathbb{N}\}\big)$$

is a quotient space obtained from the T by identifying T_0 to a point s.

Let X be the sequential fan S_ω. Then X is a non-first-countable, Fréchet space. If $\mathcal{P}$ is a base for X, then $(A)^*_{\mathcal{P}} = \overline{A} = [A]$ for each $A \subset X$.

3. The induced topology and Fréchet spaces

In this section, we shall discuss some conditions in which $(A)^*_{\mathcal{P}} \subset [A]$ for each $A \subset X$. A new topology $\tau_{\mathcal{P}}$ from a topology τ on a space X is induced by a standard network $\mathcal{P}$. Some relations of sequential neighborhoods and neighborhoods in τ or $\tau_{\mathcal{P}}$ are obtained.

Let $\mathcal{P}$ be a standard network for a topological space (X, τ). Set $\mathcal{P} = \cup_{x \in X} \mathcal{P}_x$, and put

$$\tau_{\mathcal{P}} = \{U \subset X : \text{ there is } P \in \mathcal{P}_x \text{ such that } P \subset U \text{ for each } x \in U\}.$$

It is easy to see that

(1) $\tau_{\mathcal{P}}$ is a topology on X;
(2) $\tau \subset \tau_{\mathcal{P}}$;
(3) $\mathcal{P}$ is always a weak base for $(X, \tau_{\mathcal{P}})$;
(4) $\mathcal{P}$ is a weak base for (X, τ) if and only if $\tau = \tau_{\mathcal{P}}$.

The $\tau_{\mathcal{P}}$ is called a *topology induced by* $\mathcal{P}$.

Lemma 3.1 ([8, Lemma 1.4.7])**.** *A space X is Fréchet if and only if each sequential neighborhood at x in X is a neighborhood at x in X for each $x \in X$.*

Each sequential neighborhood at x in (X, τ) is a sequential neighborhood at x in $\tau_{\mathcal{P}}$ for each $x \in X$ because $\tau \subset \tau_{\mathcal{P}}$. Consider the following conditions.

(F): Each sequential neighborhood at x in (X, τ) is a neighborhood at x in $\tau_{\mathcal{P}}$ for each $x \in X$.

(G): Each sequential neighborhood at x in $\tau_{\mathcal{P}}$ is a sequential neighborhood at x in τ for each $x \in X$.

By Theorem 2.2, (F) $\Longrightarrow (A)^*_{\mathcal{P}} \subset [A]$ for each $A \subset X$.

Theorem 3.1. *If (X, τ) or $(X, \tau_{\mathcal{P}})$ is Fréchet, then X has the* (F)*.*

Proof. If (X, τ) is Fréchet and U is a sequential neighborhood at $x \in X$ in (X, τ), U is a neighborhood at x in τ by Lemma 3.1, then U is a neighborhood at x in $\tau_{\mathcal{P}}$ by $\tau \subset \tau_{\mathcal{P}}$. If $(X, \tau_{\mathcal{P}})$ is Fréchet and U is a sequential neighborhood at x in (X, τ), U is a sequential neighborhood at x in $\tau_{\mathcal{P}}$, then U is a neighborhood at x in $\tau_{\mathcal{P}}$ by Lemma 3.1. □

Let (X, τ) be a space. Since $\mathcal{P}$ is a weak base for $(X, \tau_{\mathcal{P}})$, $\mathcal{P}$ is always an sn-network for $(X, \tau_{\mathcal{P}})$ by Lemma 2.1. Put $\mathcal{P} = \cup_{x\in X}\mathcal{P}_x$, where each $\mathcal{P}_x = \{\{x\}\}$ for each $x \in X$. Then $\mathcal{P}$ is a standard network for X, $\tau_{\mathcal{P}}$ is a discrete topology on X, and X has the (F). But, X has not the (G) if X is the unit closed interval with the usual topology. Hence, (F)$\not\Rightarrow$(G).

Lemma 3.2. *Let $\mathcal{P}$ be a standard network for X. Consider the following conditions*:

(1) *$\mathcal{P}$ is an sn-network for (X, τ);*

(2) *X has the (G), i.e., each sequential neighborhood at x in $\tau_{\mathcal{P}}$ is a sequential neighborhood at x in τ for each $x \in X$;*

(3) *Each neighborhood at x in $\tau_{\mathcal{P}}$ is a sequential neighborhood at x in τ for each $x \in X$.*

Then $(1) \Leftrightarrow (2) \Rightarrow (3)$*, and* $(3) \Rightarrow (1)$ *if X is Hausdorff.*

Proof. $(1) \Rightarrow (2)$ Let $\mathcal{P}$ be an sn-network for (X, τ). For each $x \in X$, let U be a sequential neighborhood at x in $\tau_{\mathcal{P}}$. Let S be a sequence converging to x in τ, and $x \in V \in \tau_{\mathcal{P}}$. Then $P_x \subset V$ for some $P_x \in \mathcal{P}_x$, S is eventually in P_x because P_x is a sequential neighborhood at x in τ, S is eventually in V, and S is a convergent sequence in $\tau_{\mathcal{P}}$. Thus S is eventually in U. Hence U is a sequential neighborhood at x in τ.

$(2) \Rightarrow (1)$ Since $\mathcal{P}$ is always a weak base for $(X, \tau_{\mathcal{P}})$, $\mathcal{P}$ is an sn-network for $(X, \tau_{\mathcal{P}})$ by Lemma 2.1. Then $\mathcal{P}$ is an sn-network for (X, τ) by (2).

$(2) \Rightarrow (3)$ is obvious. Next, show that $(3) \Rightarrow (1)$ if X is Hausdorff. Let $\mathcal{P} = \cup_{x\in X}\mathcal{P}_x$. If there are $x_0 \in X$ and $P_0 \in \mathcal{P}_{x_0}$ such that P_0 is not a sequential neighborhood at x_0, take a sequence $\{x_n\}$ in $X - P_0$ such that $x_n \to x_0$ in τ. Put $V = X - \{x_n : n \in \mathbb{N}\}$. Then V is not a sequential neighborhood at x_0 in τ. For each $x \in V$, if $x = x_0$, take $P_x = P_0$, then $P_x \subset V$; if $x \neq x_0$, then $x \in X - (\{x_0\} \cup \{x_n : n \in \mathbb{N}\})$, thus $P_x \subset X - (\{x_0\} \cup \{x_n : n \in \mathbb{N}\}) \subset V$ for some $P_x \in \mathcal{P}_x$ (X is Hausdorff). Hence, $x_0 \in V \in \tau_{\mathcal{P}}$, which is a contradiction. Thus, $\mathcal{P}$ is an sn-network for (X, τ). □

Corollary 3.1. *Let $\mathcal{P}$ be a standard network for X. If X has* (F) *and* (G)*, then $(X, \tau_{\mathcal{P}})$ is Fréchet.*

Proof. Let U be a sequential neighborhood at $x \in X$ in $\tau_{\mathcal{P}}$. Then U is a sequential neighborhood at x in τ by Lemma 3.2, thus U is a neighborhood at x in $\tau_{\mathcal{P}}$ by (F). Hence $(X, \tau_{\mathcal{P}})$ is a Fréchet space by Lemma 3.1. □

By Theorems 2.1, 2.2 and Corollary 3.1, the following result holds.

Theorem 3.2. *Let $\mathcal{P}$ be a standard network for X. Then $(X, \tau_{\mathcal{P}})$ is a Fréchet space and X has the* (G) *if and only if X has the* (F) *and $(A)^*_{\mathcal{P}} = [A]$ for each $A \subset X$.*

Example 3.1. Stone-Čech compactification $\beta\mathbb{N}$ is not a sequential space, but there is an sn-network $\mathcal{P}$ for $\beta\mathbb{N}$ such that $(A)^*_{\mathcal{P}} = [A]$ for each $A \subset \beta\mathbb{N}$.

Since $\beta\mathbb{N}$ contains no non-trivial convergent sequence [4], every one-point set of $\beta\mathbb{N}$ is sequential open, and $\beta\mathbb{N}$ is not a sequential space. Let $\mathcal{P} = \cup_{x\in X}\mathcal{P}_x$, where $\mathcal{P}_x = \{\{x\}\}$ for each $x \in X$. Then $\mathcal{P}$ is an sn-network for $\beta\mathbb{N}$, $(\beta\mathbb{N}, \tau_{\mathcal{P}})$ is a discrete space and $(A)^*_{\mathcal{P}} = A = [A]$ for each $A \subset \beta\mathbb{N}$.

Example 3.2. Let (X, τ) be the sequential fan S_ω. There is a standard network $\mathcal{P}$ for X such that $(X, \tau_{\mathcal{P}})$ is not sequential.

Continue the Example 2.1. For each $x \in X$, let $\mathcal{P}_x = \{\{x\}\}$ if $x \neq s$; let $\mathcal{P}_x = \{\{s\} \cup (\cup_{n\geq m} L_n) : m \in \mathbb{N}, L_n \subset T_n, |T_n - L_n| < \aleph_0\}$ if $x = s$. Then $\mathcal{P}$ is a standard network for X. $\{s\}$ is sequentially open in $\tau_{\mathcal{P}}$, it is not open in $\tau_{\mathcal{P}}$, thus $(X, \tau_{\mathcal{P}})$ is not sequential.

Example 3.3. Let (X, τ) be the Arens' space S_2. There are weak bases $\mathcal{P}$ and $\mathcal{Q}$ for X such that

(1) $\mathcal{P}$ is countable, X has the (G), but it has not the (F);
(2) $(A)^*_{\mathcal{Q}} \not\subset [A]$ for some $A \subset X$.

The Arens' space S_2 [2] is defined as follows. Let $T_0 = \{a_n : n \in \mathbb{N}\}$ be a sequence converging to $s \notin T_0$ and let each T_n $(n \in \mathbb{N})$ be a sequence converging to $a_n \notin T_n$. Let T be the topological sum of $\{T_n \cup \{a_n\} : n \in \mathbb{N}\}$. Thus

$$S_2 = \{s\} \cup (\bigcup\{T_n : n \geq 0\})$$

is a quotient space obtained from the topological sum of T_0 and T by identifying each $a_n \in T_0$ with $a_n \in T$. For each $x \in S_2 - \{s\}$, let $\mathcal{P}_x$ be a countable neighborhood base at x in S_2, and $\mathcal{P}_s = \{\{s\} \cup \{a_m : m \geq n\} : n \in \mathbb{N}\}$. Put $\mathcal{P} = \cup_{x\in S_2}\mathcal{P}_x$. It is easy to see that $\mathcal{P}$ is a weak base for S_2, S_2 is weakly first-countable, and $(A)^*_{\mathcal{P}} = [A]$ for each $A \subset X$. Since $\mathcal{P}$ is a weak base for S_2, $\tau = \tau_{\mathcal{P}}$, thus X has the (G). Put $S = \{s\} \cup \{a_n : n \in \mathbb{N}\}$. Then S is a sequential neighborhood at s in τ, but it is not a neighborhood at s in $\tau_{\mathcal{P}}$. Thus X has not the (F).

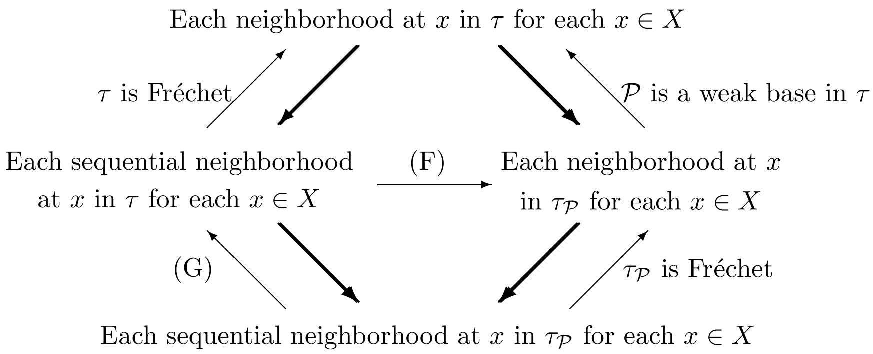
Let $\mathcal{Q}$ be a neighborhood base of S_2, and $A = \cup\{T_n : n \in \mathbb{N}\}$. Then $(A)^*_{\mathcal{Q}} = S_2 \not\subset S_2 - \{s\} = [A]$.

Question 3.1. How to characterize a standard network $\mathcal{P}$ for a space X such that $(X, \tau_{\mathcal{P}})$ is sequential?

It is well known that a space X is Fréchet if and only if it is a pesudo-open image of a metric space, and a space X is snf-countable if and only if it is an 1-sequence-covering image of a metric space [8]. The following question is raised.

Question 3.2. Let $\mathcal{P}$ be a standard network for a space X. How to characterize a space X in which $(A)^*_{\mathcal{P}} \subset [A]$ for each $A \subset X$ by a nice image of a metric space?

In the following, the relations of neighborhoods are obtained in this section.



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