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INTEGRABILITY OF DISTRIBUTIONS IN *GCR*-LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KAEHLER MANIFOLDS

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Abstract. In present paper we establish conditions for the integrability of various distributions of GCR-lightlike submanifolds and obtain conditions for the distributions to define totally geodesic foliations in GCR-lightlike submanifolds.

1. Introduction

The study of geometry of Cauchy-Riemann (CR)-submanifolds in Kaehler manifolds was initiated by Bejancu [2], which include holomorphic and totally real submanifolds as special cases and has been further studied by Bejancu [3, 4], Chen [6], Duggal [7, 8], Yano and Kon [17, 18] and others.

The geometry of lightlike submanifolds was initiated by Kupeli and further developed by Bejancu and Duggal [9] and they also introduced the notion of CR-lightlike submanifolds of indefinite Kaehler manifolds. But this class of submanifolds exclude the complex and totally real submanifolds as subcases. Later on, Duggal and Sahin [11] introduced SCR-lightlike submanifolds of indefinite Kaehler manifolds. Since there was no inclusion relation between CR and SCR cases therefore Duggal and Sahin [12] introduced a new class called GCR-lightlike submanifolds of indefinite Kaehler manifolds which is an umbrella for all these types of submanifolds. In present paper we establish conditions for the integrability of various distributions of GCR-lightlike submanifolds and obtain conditions for the distributions to define totally geodesic foliations in GCR-lightlike submanifolds.

2. Lightlike submanifolds

We recall notations and fundamental equations for lightlike submanifolds, which are due to the book [9] by Duggal and Bejancu.

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an

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m-dimensional submanifold of $\bar{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M

$$TM^{\perp} = \cup\{u \in T_x\bar{M} : \bar{g}(u, v) = 0, \forall v \in T_xM, x \in M\}, \tag{1}$$

is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $\mathrm{Rad}T_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping

$$\mathrm{Rad}TM : x \in M \longrightarrow \mathrm{Rad}T_xM, \tag{2}$$

defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $\mathrm{Rad}TM$ is called the radical distribution on M.

Screen distribution $S(TM)$ is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, that is

$$TM = \mathrm{Rad}TM \perp S(TM) \tag{3}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $\mathrm{Rad}TM$ in $TM^{\perp}$. Let $\mathrm{tr}(TM)$ and $\mathrm{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $\mathrm{Rad}TM$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$\mathrm{tr}(TM) = \mathrm{ltr}(TM) \perp S(TM^{\perp}), \tag{4}$$

$$T\bar{M}\mid_M = TM \oplus \mathrm{tr}(TM) = (\mathrm{Rad}TM \oplus \mathrm{ltr}(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{5}$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as $\{\xi_1, \ldots, \xi_r, W_{r+1}, \ldots, W_n, N_1, \ldots, N_r, X_{r+1}, \ldots, X_m\}$, where $\{\xi_1, \ldots, \xi_r\}, \{N_1, \ldots, N_r\}$ are local lightlike bases of $\Gamma(\mathrm{Rad}TM\mid_u)$, $\Gamma(\mathrm{ltr}(TM)\mid_u)$ and $\{W_{r+1}, \ldots, W_n\}, \{X_{r+1}, \ldots, X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})\mid_u)$ and $\Gamma(S(TM)\mid_u)$ respectively. For this quasi-orthonormal fields of frames, we have:

Theorem 2.1 ([9])**.** *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle* $\mathrm{ltr}(TM)$ *of* $\mathrm{Rad}TM$ *in* $S(TM^{\perp})^{\perp}$ *and a basis of* $\Gamma(\mathrm{ltr}(TM)\mid_u)$ *consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_u$, where u is a coordinate neighborhood of M such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0 \ \textit{ for any } \ i, j \in \{1, 2, \ldots, r\}, \tag{6}$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\Gamma(\mathrm{Rad}(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ then according to the decomposition (5), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall\ X, Y \in \Gamma(TM), \tag{7}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \quad \forall\ X \in \Gamma(TM), U \in \Gamma(\mathrm{tr}(TM)), \tag{8}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(\mathrm{tr}(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, A_U is a linear operator on M and known as shape operator.

According to (4), considering the projection morphisms L and S of $\mathrm{tr}(TM)$ on $\mathrm{ltr}(TM)$ and $S(TM^{\perp})$, respectively, then (7) and (8) become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{9}$$

$$\bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{10}$$

where we put $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$.

As h^l and h^s are $\Gamma(\mathrm{ltr}(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called as the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{11}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{12}$$

where $X \in \Gamma(TM), N \in \Gamma(\mathrm{ltr}(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Using (9)-(12) we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{13}$$

$$\bar{g}(h^l(X,Y),\xi) + \bar{g}(Y, h^l(X,\xi)) + g(Y, \nabla_X \xi) = 0, \tag{14}$$

$$\bar{g}(A_N X, N') + \bar{g}(N, A_{N'} X) = 0 \tag{15}$$

for any $\xi \in \Gamma(\mathrm{Rad}TM)$, $W \in \Gamma(S(TM^{\perp}))$ and $N, N' \in \Gamma(\mathrm{ltr}(TM))$.

Let P be the projection morphism of TM on $S(TM)$ then using (3), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \tag{16}$$

$$\nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi \tag{17}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}TM)$, where $\{\nabla^*_X PY, A^*_{\xi} X\}$ and $\{h^*(X, PY), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(\mathrm{Rad}TM)$, respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $\mathrm{Rad}TM$, respectively. h^* and A^* are $\Gamma(\mathrm{Rad}TM)$-valued and $\Gamma(S(TM))$-valued bilinear forms and called as the second fundamental forms of distributions $S(TM)$ and $\mathrm{Rad}TM$, respectively.

Using (9), (10), (16) and (17), we obtain

$$\bar{g}(h^l(X,PY),\xi) = g(A^*_{\xi} X, PY), \tag{18}$$

$$\bar{g}(h^*(X,PY),N) = \bar{g}(A_N X, PY) \tag{19}$$

for any $X, Y \in \Gamma(TM), \xi \in \Gamma(\mathrm{Rad}(TM))$ and $N \in \Gamma(\mathrm{ltr}(TM))$.

From the geometry of Riemannian submanifolds and non degenerate submanifolds, it is known that the induced connection ∇ on a non degenerate submanifold is a metric connection. Unfortunately, this is not true for a lightlike submanifold. Indeed, considering $\bar{\nabla}$ a metric connection then we have

$$(\nabla_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y) \tag{20}$$

for any $X, Y, Z \in \Gamma(TM)$. From [9] page 171, using the properties of linear connections we have

$$(\nabla_X h^l)(Y, Z) = \nabla^l_X(h^l(Y, Z)) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z), \tag{21}$$

$$(\nabla_X h^s)(Y, Z) = \nabla^s_X(h^l(Y, Z)) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z). \tag{22}$$

Barros and Romero [1] defined indefinite Kaehler manifolds as:

Definition 2.2. Let $(\bar{M}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then $\bar{M}$ is called an indefinite Kaehler manifold if $\bar{J}$ is parallel with respect to $\bar{\nabla}$, that is

$$(\bar{\nabla}_X \bar{J})Y = 0, \quad \forall \quad X, Y \in \Gamma(T\bar{M}). \tag{23}$$

3. Generalized Cauchy-Riemann lightlike submanifolds

Definition 3.1 ([12])**.** Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g}, \bar{J})$ then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied:

(A) There exist two subbundles D_1 and D_2 of $\mathrm{Rad}(TM)$ such that

$$\mathrm{Rad}(TM) = D_1 \oplus D_2, \quad \bar{J}(D_1) = D_1, \quad \bar{J}(D_2) \subset S(TM). \tag{24}$$

(B) There exist two subbundles D_0 and D' of $S(TM)$ such that

$$S(TM) = \{\bar{J}D_2 \oplus D'\} \perp D_0, \quad \bar{J}(D_0) = D_0, \quad \bar{J}(D') = L_1 \perp L_2, \tag{25}$$

where D_0 is a non degenerate distribution on M, L_1 and L_2 are vector subbundle of $\mathrm{ltr}(TM)$ and $S(TM)^\perp$, respectively.

Then the tangent bundle TM of M is decomposed as

$$TM = D \perp D', \quad D = \mathrm{Rad}(TM) \oplus D_0 \oplus \bar{J}D_2. \tag{26}$$

M is called a proper GCR-lightlike submanifold if $D_1 \neq \{0\}, D_2 \neq \{0\}, D_0 \neq \{0\}$ and $L_2 \neq \{0\}$.

Let Q, P_1 and P_2 be the projections on D, $\bar{J}(L_1) = M_1 \subset D'$ and $\bar{J}(L_2) = M_2 \subset D'$, respectively. Then for any $X \in \Gamma(TM)$ we have

$$X = QX + P_1X + P_2X, \tag{27}$$

applying $\bar{J}$ to (27) we obtain

$$\bar{J}X = TX + wP_1X + wP_2X, \tag{28}$$

and we can write (28) as

$$\bar{J}X = TX + wX, \tag{29}$$

where TX and wX are the tangential and transversal components of $\bar{J}X$, respectively.

Similarly

$$\bar{J}V = BV + CV \tag{30}$$

for any $V \in \Gamma(\mathrm{tr}(TM))$, where BV and CV are the sections of TM and $\mathrm{tr}(TM)$, respectively.

Differentiating (28) and using (9)-(12) and (30) we have

$$D^s(X, wP_1Y) = -\nabla^s_X wP_2Y + wP_2\nabla_X Y - h^s(X, TY) + Ch^s(X, Y), \tag{31}$$

$$D^l(X, wP_2Y) = -\nabla^l_X wP_1Y + wP_1\nabla_X Y - h^l(X, TY) + Ch^l(X, Y). \tag{32}$$

Using Kaehlerian property of $\bar{\nabla}$ with (11) and (12), we have the following lemmas.

Lemma 3.2. *Let M be a GCR-lightlike submanifold of an indefinite Kaehlerian manifold $\bar{M}$. Then we have*

$$(\nabla_X T)Y = A_{wY}X + Bh(X, Y) \tag{33}$$

and

$$(\nabla^t_X w)Y = Ch(X, Y) - h(X, TY), \tag{34}$$

where $X, Y \in \Gamma(TM)$ and

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{35}$$

$$(\nabla^t_X w)Y = \nabla^t_X wY - w\nabla_X Y. \tag{36}$$

Lemma 3.3. *Let M be a GCR-lightlike submanifold of an indefinite Kaehlerian manifold $\bar{M}$. Then we have*

$$(\nabla_X B)V = A_{CV}X - TA_V X \tag{37}$$

and

$$(\nabla^t_X C)V = -wA_V X - h(X, BV), \tag{38}$$

where $X \in \Gamma(TM)$, $V \in \Gamma(\mathrm{tr}(TM))$ and

$$(\nabla_X B)V = \nabla_X BV - B\nabla^t_X V, \tag{39}$$

$$(\nabla^t_X C)V = \nabla^t_X CV - C\nabla^t_X V. \tag{40}$$

4. Integrability of the distributions

Theorem 4.1. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. If the distribution D is integrable, then the following assertions hold*

(i) $\bar{g}(D^l(\bar{J}X,W),Y)=\bar{g}(D^l(X,W),\bar{J}Y) \Leftrightarrow \bar{g}(A_W\bar{J}X,Y)=\bar{g}(A_WX,\bar{J}Y)$.

(ii) $\bar{g}(D^l(\bar{J}X,W),\xi)=\bar{g}(A_WX,\bar{J}\xi)$.

(iii) $\bar{g}(D^l(X,W),\xi)=\bar{g}(A_W\bar{J}X,\bar{J}\xi)$ *for any* $X,Y\in\Gamma(D),\xi\in\Gamma(D_2)$ *and* $W\in\Gamma(S(TM^\perp))$.

Proof. Let the distribution D be integrable then for $X,Y\in\Gamma(D)$ and $W\in\Gamma(S(TM^\perp))$ we have

$$\begin{aligned}\bar{g}(D^l(\bar{J}X,W),Y) &= \bar{g}(\bar{\nabla}_{\bar{J}X}W+A_W\bar{J}X-\nabla^s_{\bar{J}X}W,Y)\\ &= -\bar{g}(W,\bar{\nabla}_{\bar{J}X}Y)+\bar{g}(A_W\bar{J}X,Y)\\ &= -\bar{g}(W,h(\bar{J}X,Y))+\bar{g}(A_W\bar{J}X,Y)\\ &= -\bar{g}(W,h(X,\bar{J}Y))+\bar{g}(A_W\bar{J}X,Y)\\ &= -\bar{g}(W,\bar{\nabla}_X\bar{J}Y)+\bar{g}(A_W\bar{J}X,Y)\\ &= \bar{g}(\bar{\nabla}_XW,\bar{J}Y)+\bar{g}(A_W\bar{J}X,Y)\\ &= -\bar{g}(A_WX,\bar{J}Y)+\bar{g}(D^l(X,W),\bar{J}Y)+\bar{g}(A_W\bar{J}X,Y), \quad (41)\end{aligned}$$

or

$$\bar{g}(D^l(\bar{J}X,W),Y)-\bar{g}(D^l(X,W),\bar{J}Y)=\bar{g}(A_W\bar{J}X,Y)-\bar{g}(A_WX,\bar{J}Y). \quad (42)$$

Therefore part (i) of the assertion follows. Substituting $Y=\xi$ and $Y=\bar{J}\xi$ in (42), for any $\xi\in\Gamma(D_2)$, we obtain (ii) and (iii), respectively. □

Theorem 4.2. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then D' is integrable if and only if $\nabla_X\bar{J}Y=\nabla_Y\bar{J}X$ for any $X,Y\in\Gamma(D')$.*

Proof. For any $X,Y\in\Gamma(D')$ we have

$$h(X,\bar{J}Y)=\bar{\nabla}_X\bar{J}Y-\nabla_X\bar{J}Y \quad (43)$$

and

$$h(\bar{J}X,Y)=\bar{\nabla}_Y\bar{J}X-\nabla_Y\bar{J}X. \quad (44)$$

Subtracting (44) from (43), we obtain

$$\begin{aligned}h(X,\bar{J}Y)-h(\bar{J}X,Y) &= \bar{J}\bar{\nabla}_XY-\bar{J}\bar{\nabla}_YX-\nabla_X\bar{J}Y+\nabla_Y\bar{J}X\\ &= \bar{J}[X,Y]-\nabla_X\bar{J}Y+\nabla_Y\bar{J}X\\ &= T[X,Y]+w[X,Y]-\nabla_X\bar{J}Y+\nabla_Y\bar{J}X.\end{aligned}$$

Equating tangential parts of above equation, we obtain

$$T[X,Y]=\nabla_X\bar{J}Y-\nabla_Y\bar{J}X. \quad (45)$$

Hence from (45) the result follows. □

Theorem 4.3. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then D_0 is integrable if and only if*

(i) $\bar{g}(h^*(X,Y),N) = \bar{g}(h^*(Y,X),N)$,
(ii) $\bar{g}(h^*(X,\bar{J}Y),N') = \bar{g}(h^*(Y,\bar{J}X),N')$,
(iii) $h^s(X,\bar{J}Y) = h^s(\bar{J}X,Y)$,
(iv) $\bar{g}(\nabla^*_X Y,\bar{J}\xi) = \bar{g}(\nabla^*_Y X,\bar{J}\xi)$ *for any* X, $Y \in \Gamma(D_0)$, $N \in \Gamma(\mathrm{ltr}(TM))$, $N' \in \Gamma(L_1)$ *and* $\xi \in \Gamma(D_2)$.

Proof. Using the definition of GCR-lightlike submanifold, D_0 is integrable if and only if

$$\bar{g}([X,Y],N) = \bar{g}([X,Y],\bar{J}N') = \bar{g}([X,Y],\bar{J}W) = \bar{g}([X,Y],\bar{J}\xi) = 0$$

for any $X,Y \in \Gamma(D_0)$, $N \in \Gamma(\mathrm{ltr}(TM))$, $N' \in \Gamma(L_1)$, $W \in \Gamma(L_2)$ and $\xi \in \Gamma(D_2)$.

Using (9) and (16) we have

$$\bar{g}([X,Y],N) = \bar{g}(h^*(X,Y),N) - \bar{g}(h^*(Y,X),N) \tag{46}$$

and

$$\begin{aligned}\bar{g}([X,Y],\bar{J}N') &= \bar{g}(\bar{\nabla}_X Y,\bar{J}N') - \bar{g}(\bar{\nabla}_Y X,\bar{J}N')\\ &= -\bar{g}(\bar{\nabla}_X \bar{J}Y,N') + \bar{g}(\bar{\nabla}_Y \bar{J}X,N'),\end{aligned}$$

using (16) we have

$$\bar{g}([X,Y],\bar{J}N') = -\bar{g}(h^*(X,\bar{J}Y),N') + \bar{g}(h^*(Y,\bar{J}X),N'). \tag{47}$$

Again using (9) we obtain

$$\begin{aligned}\bar{g}([X,Y],\bar{J}W) &= \bar{g}(\bar{\nabla}_X Y,\bar{J}W) - \bar{g}(\bar{\nabla}_Y X,\bar{J}W)\\ &= -\bar{g}(\bar{\nabla}_X \bar{J}Y,W) + \bar{g}(\bar{\nabla}_Y \bar{J}X,W)\\ &= -\bar{g}(h^s(X,\bar{J}Y),W) + \bar{g}(h^s(\bar{J}X,Y),W). \end{aligned}\tag{48}$$

Finally from (16) we obtain

$$\bar{g}([X,Y],\bar{J}\xi) = \bar{g}(\nabla^*_X Y,\bar{J}\xi) - \bar{g}(\nabla^*_Y X,\bar{J}\xi). \tag{49}$$

Thus from (46)-(49) the result follows. □

Corollary 4.4. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then D_0 is integrable if and only if*

(i) $g(X,A_N Y) = g(Y,A_N X)$,
(ii) $g(\bar{J}X,A_{N'}Y) = g(\bar{J}Y,A_{N'}X)$,
(iii) $h^s(X,\bar{J}Y) = h^s(\bar{J}X,Y)$,
(iv) $\bar{g}(h^l(X,\bar{J}Y),\xi) = \bar{g}(h^l(Y,\bar{J}X),\xi)$ *for any* $X,Y \in \Gamma(D_0)$, $N \in \Gamma(\mathrm{ltr}(TM))$, $N' \in \Gamma(L_1)$ *and* $\xi \in \Gamma(D_2)$.

Theorem 4.5. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then* $\mathrm{Rad}(TM)$ *is integrable if and only if*

(i) $\bar{g}(h^l(\xi,\bar{J}\xi''),\xi') = \bar{g}(h^l(\xi',\bar{J}\xi''),\xi)$,
(ii) $\bar{g}(h^l(\xi,Z),\xi') = \bar{g}(h^l(\xi',Z),\xi)$,

(iii) $h^s(\xi', \bar{J}\xi) = h^s(\bar{J}\xi', \xi)$,
(iv) $\bar{g}(\xi, h^l(\xi', \bar{J}N)) = \bar{g}(\xi', h^l(\xi, \bar{J}N))$ *for any* $Z \in \Gamma(D_0)$, $\xi'' \in \Gamma(D_2)$, ξ, $\xi' \in \Gamma(\mathrm{Rad}(TM))$, $W \in \Gamma(L_2)$ *and* $N \in \Gamma(L_1)$.

Proof. Using the definition of GCR-lightlike submanifolds, $\mathrm{Rad}(TM)$ is integrable if and only if

$$\bar{g}([\xi, \xi'], Z) = \bar{g}([\xi, \xi'], \bar{J}N) = \bar{g}([\xi, \xi'], \bar{J}W) = \bar{g}([\xi, \xi'], \bar{J}\xi'') = 0$$

for any $Z \in \Gamma(D_0)$, $\xi'' \in \Gamma(D_2)$, ξ, $\xi' \in \Gamma(\mathrm{Rad}(TM))$, $W \in \Gamma(L_2)$ and $N \in \Gamma(L_1)$.

Using (9) and (16), we obtain

$$\begin{aligned} \bar{g}([\xi, \xi'], \bar{J}\xi'') &= \bar{g}(\bar{\nabla}_\xi \xi', \bar{J}\xi'') - \bar{g}(\bar{\nabla}_{\xi'} \xi, \bar{J}\xi'') \\ &= -\bar{g}(\xi', \bar{\nabla}_\xi \bar{J}\xi'') + \bar{g}(\xi, \bar{\nabla}_{\xi'} \bar{J}\xi'') \\ &= -\bar{g}(h^l(\xi, \bar{J}\xi''), \xi') + \bar{g}(h^l(\xi', \bar{J}\xi''), \xi) \end{aligned} \tag{50}$$

and using (18)

$$\begin{aligned} \bar{g}([\xi, \xi'], Z) &= \bar{g}(\nabla_\xi \xi', Z) - \bar{g}(\nabla_{\xi'} \xi, Z) \\ &= -\bar{g}(A^*_{\xi'} \xi, Z) + \bar{g}(A^*_\xi \xi', Z) \\ &= -\bar{g}(h^l(\xi, Z), \xi') + \bar{g}(h^l(\xi', Z), \xi) \end{aligned} \tag{51}$$

and

$$\begin{aligned} \bar{g}([\xi, \xi'], \bar{J}W) &= \bar{g}(\bar{\nabla}_\xi \xi', \bar{J}W) - \bar{g}(\bar{\nabla}_{\xi'} \xi, \bar{J}W) \\ &= -\bar{g}(W, \bar{\nabla}_\xi \bar{J}\xi') + \bar{g}(W, \bar{\nabla}_{\xi'} \bar{J}\xi) \\ &= -\bar{g}(h^s(\xi, \bar{J}\xi'), W) + \bar{g}(h^s(\xi', \bar{J}\xi), W) \end{aligned} \tag{52}$$

and

$$\begin{aligned} \bar{g}([\xi, \xi'], \bar{J}N) &= \bar{g}(\bar{\nabla}_\xi \xi', \bar{J}N) - \bar{g}(\bar{\nabla}_{\xi'} \xi, \bar{J}N) \\ &= -\bar{g}(\xi', \bar{\nabla}_\xi \bar{J}N) + \bar{g}(\xi, \bar{\nabla}_{\xi'} \bar{J}N) \\ &= -\bar{g}(\xi', h^l(\xi, \bar{J}N)) + \bar{g}(\xi, h^l(\xi', \bar{J}N)). \end{aligned} \tag{53}$$

Thus from (50)-(53), the result follows. □

Corollary 4.6. *Let* M *be a* GCR*-lightlike submanifold of an indefinite Kaehler manifold* $\bar{M}$. *Then* $\mathrm{Rad}(TM)$ *is integrable if and only if*
(i) $A^*_\xi \xi' \notin \Gamma(M_1)$,
(ii) $g(\nabla^*_{\xi'} \bar{J}Z, \bar{J}\xi) = g(\nabla^*_\xi \bar{J}Z, \bar{J}\xi')$,
(iii) $g(A_W \xi', \bar{J}\xi) = g(A_W \xi, \bar{J}\xi')$,
(iv) $g(A_N \xi', \bar{J}\xi) = g(A_N \xi, \bar{J}\xi')$ *for any* $Z \in \Gamma(D_0)$, ξ, $\xi' \in \Gamma(\mathrm{Rad}(TM))$ *and* $N \in \Gamma(\mathrm{ltr}(TM))$.

Theorem 4.7. *Let* M *be a* GCR*-lightlike submanifold of an indefinite Kaehler manifold* $\bar{M}$. *Then the distribution* D_1 *is integrable if and only if*
(i) $\nabla^{*t}_X \bar{J}Y - \nabla^{*t}_Y \bar{J}X \in \Gamma(D_1)$,
(ii) $A^*_{\bar{J}Y} X = A^*_{\bar{J}X} Y$,

(iii) $Bh(X, \bar{J}Y) = Bh(Y, \bar{J}X)$ *for any* $X, Y \in \Gamma(D_1)$.

Proof. Since $\bar{J}$ is the almost complex structure on M therefore for any $X, Y \in \Gamma(D_1)$ we have

$$\bar{\nabla}_X Y = -\bar{\nabla}_X \bar{J}^2 Y = -\bar{J}\bar{\nabla}_X \bar{J}Y,$$

using (9), we have

$$\nabla_X Y + h(X, Y) = -\bar{J}(\nabla_X \bar{J}Y + h(X, \bar{J}Y)).$$

Now, using (17), (29)-(30) and then equating the tangential components of the resulting equation both sides we obtain

$$\nabla_X Y = TA^*_{\bar{J}Y}X - T\nabla^{*t}_X \bar{J}Y - Bh(X, \bar{J}Y), \tag{54}$$

replacing X by Y and then subtracting the resulting equation from (54) we obtain

$$[X, Y] = T(A^*_{\bar{J}Y}X - A^*_{\bar{J}X}Y) - T(\nabla^{*t}_X \bar{J}Y - \nabla^{*t}_Y \bar{J}X) - Bh(X, \bar{J}Y) + Bh(Y, \bar{J}X). \tag{55}$$

Hence from (55) the result follows. □

Corollary 4.8. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then D_1 defines a totally geodesic foliation in M if and only if*

(i) $\nabla^{*t}_X \bar{J}Y \in \Gamma(D_1)$,
(ii) $A^*_{\bar{J}Y}X = 0$,
(iii) $Bh(X, \bar{J}Y) = 0$ *for any* $X, Y \in \Gamma(D_1)$.

Theorem 4.9. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D_2 is integrable if and only if*

(i) $\nabla^*_X \bar{J}Y - \nabla^*_Y \bar{J}X \in \Gamma(\bar{J}D_2)$,
(ii) $h^*(X, \bar{J}Y) = h^*(Y, \bar{J}X)$,
(iii) $Bh(X, \bar{J}Y) = Bh(Y, \bar{J}X)$ *for any* $X, Y \in \Gamma(D_2)$.

Proof. Since $\bar{J}$ is the almost complex structure on M therefore for any $X, Y \in \Gamma(D_2)$ we have

$$\nabla_X Y + h(X, Y) = -\bar{J}(\nabla_X \bar{J}Y + h(X, \bar{J}Y)),$$

using (16) and (29)-(30) and then equating tangential components of the resulting equation both sides we obtain

$$\nabla_X Y = -T\nabla^*_X \bar{J}Y - Th^*(X, \bar{J}Y) - Bh(X, \bar{J}Y), \tag{56}$$

replacing X and Y and then subtracting the resulting equation from (56) we obtain

$$\begin{aligned} [X, Y] &= -T(\nabla^*_X \bar{J}Y - \nabla^*_Y \bar{J}X) - T(h^*(X, \bar{J}Y) - h^*(Y, \bar{J}X)) \\ &\quad -Bh(X, \bar{J}Y) + Bh(Y, \bar{J}X). \end{aligned} \tag{57}$$

Hence from (57) the theorem follows. □

Corollary 4.10. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D_2 defines a totally geodesic foliation in M if and only if*

(i) $\nabla^*_X \bar{J}Y \in \Gamma(\bar{J}D_2)$,
(ii) $h^*(X, \bar{J}Y) = 0$,
(iii) $Bh(X, \bar{J}Y) = 0$ *for any* $X, Y \in \Gamma(D_2)$.

Theorem 4.11. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then $\bar{J}D_2$ is integrable if and only if*

(i) $\bar{g}(\bar{J}\xi, A_N \bar{J}\xi') = \bar{g}(\bar{J}\xi', A_N \bar{J}\xi)$,
(ii) $h^s(\bar{J}\xi, \xi') = h^s(\bar{J}\xi', \xi)$,
(iii) $\bar{g}(h^l(\bar{J}\xi, \xi'), \xi'') = \bar{g}(h^l(\xi, \bar{J}\xi'), \xi'')$,
(iv) $\bar{g}(\bar{J}Z, A^*_\xi \bar{J}\xi') = \bar{g}(\bar{J}Z, A^*_{\xi'} \bar{J}\xi)$ *for any* $Z \in \Gamma(D_0)$, ξ, ξ', $\xi'' \in \Gamma(D_2)$, $W \in \Gamma(L_2)$ *and* $N \in \Gamma(\mathrm{ltr}(TM))$.

Proof. Using the definition of GCR-lightlike submanifolds, $\bar{J}D_2$ is integrable if and only if

$$\bar{g}([\bar{J}\xi', \bar{J}\xi], N) = \bar{g}([\bar{J}\xi', \bar{J}\xi], \bar{J}W) = \bar{g}([\bar{J}\xi', \bar{J}\xi], \bar{J}\xi'') = \bar{g}([\bar{J}\xi', \bar{J}\xi], Z) = 0$$

for any $Z \in \Gamma(D_0)$, ξ, ξ', $\xi'' \in \Gamma(D_2)$, $W \in \Gamma(L_2)$ and $N \in \Gamma(\mathrm{ltr}(TM))$. Using (11) we have

$$\begin{aligned} \bar{g}([\bar{J}\xi', \bar{J}\xi], N) &= \bar{g}(\bar{\nabla}_{\bar{J}\xi'} \bar{J}\xi, N) - \bar{g}(\bar{\nabla}_{\bar{J}\xi} \bar{J}\xi', N) \\ &= -\bar{g}(\bar{J}\xi, \bar{\nabla}_{\bar{J}\xi'} N) + \bar{g}(\bar{J}\xi', \bar{\nabla}_{\bar{J}\xi} N) \\ &= \bar{g}(\bar{J}\xi, A_N \bar{J}\xi') - \bar{g}(\bar{J}\xi', A_N \bar{J}\xi) \end{aligned} \tag{58}$$

and

$$\begin{aligned} \bar{g}([\bar{J}\xi', \bar{J}\xi], \bar{J}W) &= \bar{g}(\bar{\nabla}_{\bar{J}\xi'} \bar{J}\xi, \bar{J}W) - \bar{g}(\bar{\nabla}_{\bar{J}\xi} \bar{J}\xi', \bar{J}W) \\ &= \bar{g}(\bar{\nabla}_{\bar{J}\xi'} \xi, W) - \bar{g}(\bar{\nabla}_{\bar{J}\xi} \xi', W) \\ &= \bar{g}(h^s(\bar{J}\xi', \xi), W) - \bar{g}(h^s(\bar{J}\xi, \xi'), W) \end{aligned} \tag{59}$$

and

$$\begin{aligned} \bar{g}([\bar{J}\xi', \bar{J}\xi], \bar{J}\xi'') &= \bar{g}(\bar{\nabla}_{\bar{J}\xi'} \bar{J}\xi, \bar{J}\xi'') - \bar{g}(\bar{\nabla}_{\bar{J}\xi} \bar{J}\xi', \bar{J}\xi'') \\ &= \bar{g}(\bar{\nabla}_{\bar{J}\xi'} \xi, \xi'') - \bar{g}(\bar{\nabla}_{\bar{J}\xi} \xi', \xi'') \\ &= \bar{g}(h^l(\bar{J}\xi', \xi), \xi'') - \bar{g}(h^l(\bar{J}\xi, \xi'), \xi'') \end{aligned} \tag{60}$$

and finally

$$\begin{aligned} \bar{g}([\bar{J}\xi', \bar{J}\xi], Z) &= \bar{g}(\bar{\nabla}_{\bar{J}\xi'} \bar{J}\xi, Z) - \bar{g}(\bar{\nabla}_{\bar{J}\xi} \bar{J}\xi', Z) \\ &= -\bar{g}(\bar{\nabla}_{\bar{J}\xi'} \xi, \bar{J}Z) + \bar{g}(\bar{\nabla}_{\bar{J}\xi} \xi', \bar{J}Z) \\ &= \bar{g}(\bar{J}Z, A^*_\xi \bar{J}\xi') - \bar{g}(\bar{J}Z, A^*_{\xi'} \bar{J}\xi). \end{aligned} \tag{61}$$

Hence from (58)-(61) the result follows. ∎

Lemma 4.12. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. If M is D-geodesic, then D defines a totally geodesic foliation in M.*

Proof. For $X, Y \in \Gamma(D)$, using (31) and (32), we obtain $w\nabla_X Y = h(X, TY) - Ch(X, Y)$, then using the hypothesis, we obtain $w\nabla_X Y = 0$. Thus the proof is complete. □

Lemma 4.13. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. If T is parallel, then $Bh(X, Y) = 0$ for any $Y \in \Gamma(D)$.*

Proof. For any $X, Y \in \Gamma(TM)$, from (33) we have

$$(\nabla_X T)Y = A_{wY}X + Bh(X, Y). \tag{62}$$

From the hypothesis of lemma, the proof is complete. □

Theorem 4.14 ([12]). *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D defines a totally geodesic foliation in M if and only if $Bh(X, Y) = 0\ \forall\ X, Y \in \Gamma(D)$.*

Theorem 4.15. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. If T is parallel, then the distribution D defines a totally geodesic foliation in M.*

Proof. Let T is parallel, then from Lemma 4.13, we obtain $Bh(X, Y) = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$. Then using Theorem 4.14, D defines a totally geodesic foliation in M. □

Theorem 4.16. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D' is parallel if and only if $A_{\bar{J}Y}X$ has no components in holomorphic distribution for any $X, Y \in \Gamma(D')$.*

Proof. From (33), we have

$$-T\nabla_X Y = A_{wY}X + Bh(X, Y)$$

for any $X, Y \in \Gamma(D')$. Hence the proof follows. □

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