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NOTES ON (σ,τ)-DERIVATIONS OF LIE IDEALS IN PRIME RINGS

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Abstract. Let R be a prime ring with center Z and characteristic different from two, U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$ and d be a nonzero (σ,τ)-derivation of R. We prove the following results: (i) If $[d(u),u]_{\sigma,\tau} = 0$ or $[d(u),u]_{\sigma,\tau} \in C_{\sigma,\tau}$ for all $u \in U$, then $U \subseteq Z$. (ii) If $a \in R$ and $[d(u),a]_{\sigma,\tau} = 0$ for all $u \in U$, then $U \subseteq Z$ or $a \in Z$. (iii) If $d([u,v]) = \pm[u,v]_{\sigma,\tau}$ for all $u \in U$, then $U \subseteq Z$.

1. Introduction

Let R denote an assosiative ring with center Z. Recall that a ring R is prime if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol xoy denotes the anticommutator $xy + yx$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U, r \in R$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. Let S be a nonempty subset of R. A mapping F from R to R is called centralizing on S if $[F(x), x] \in Z$ for all $x \in S$ and is called commuting on S if $[F(x), x] = 0$ for all $x \in S$. In [15], Posner showed that if a prime ring has a nontrivial derivation which is centralizing on the entire ring, then the ring must be commutative (Posner's second theorem). In [13] and [16] the same results is proved for a prime ring with a nontrivial centralizing automorphism. A number of authors have generalized these results by considering mappings which are only assumed to be centralizing on an appropriate ideal of the ring.

In [3], Awtar considered centralizing derivations on Lie and Jordan ideals. For prime rings, Awtar showed that a nontrivial derivation which is centralizing on Lie ideal implies that the ideal is contained in the center if the ring is not of characteristic two or three. In [12], Lee and Lee obtained the result while

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removing the restriction of characteristic not three. This result is extended in [14] where it is shown that if R is any prime ring with a nontrivial centralizing automorphism on a Lie ideal U, then U is contained in the center of R. Bell and Martindale have proved similar results assuming that the ring is semiprime in [6].

Inspired by the definition derivation, the notion of (σ,τ)-derivation was extended as follow: Let σ and τ be any two automorphisims of R. An additive mapping $d: R \to R$ is called a (σ,τ)-derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in R$. Of course a $(1,1)$-derivation where 1 is the identity map on R is a derivation. For any $x, y \in R$, we set $[x,y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$. We set $C_{\sigma,\tau} = \{c \in R \mid c\sigma(x) = \tau(x)c \text{ for all } x \in R\}$ and call this set the (σ,τ)-center of R. In particular $C_{1,1} = Z$. It can be given (σ,τ)-centralizing (resp. (σ,τ)-commuting) on R by the similarly definition centralizing (resp. commuting).

In attempt to generalize Posner's second theorem Ashraf and Rehman proved that if R is a 2-torsion free prime ring and d is a nonzero (σ,τ)-derivation of R such that the map $x \to [d(x),x]_{\sigma,\tau}$ is (σ,τ)-commuting on R, then R is commutative in [2]. In [4], Aydin showed that the conclusion of the above theorem holds for a (σ,τ)-derivation d the mapping $x \to d(x)$ is (σ,τ)-centralizing on R. In the present paper, our objective is to generalize this result for a nonzero Lie ideal U of R such that $u^2 \in U$ for all $u \in U$.

A famous result due to Herstein [10] satates that if R is a prime ring of characteristic not 2 which admits a nonzero derivation d such that $[d(x),a] = 0$ for all $x \in R$, then $a \in Z$. This result proved for a nonzero Lie ideal of R in [7]. Aydin and Kaya showed that d be a nonzero (σ,τ)-derivation and U an ideal of a prime ring R such that $[d(u),a]_{\sigma,\tau} = 0$ for all $u \in U$, then $a \in Z$ in [5]. Güven proved that $\alpha, \beta \in \mathrm{Aut}R$, $I \neq (0)$ be an ideal, d be a nonzero (σ,τ)-derivation of R such that $d\sigma = \sigma d, d\tau = \tau d$ and $[a, d(I)]_{\alpha,\beta} = 0$ then $a \in C_{\alpha,\beta}$ or R is a commutative ring in [9]. In this paper, we shall prove Herstein's theorem for a nonzero Lie ideal U of R such that $u^2 \in U$ for all $u \in U$.

On the other hand, in [8], Daif and Bell showed that if a semiprime ring R has a derivation d satisfiying the following condition, then I is a central ideal; there exists a nonzero ideal I of R such that

$$d([x,y]) = [x,y] \text{ or } d([x,y]) = -[x,y] \text{ for all } x, y \in I.$$

In [1], Argaç proved this result for semiprime rings with derivation. Our second aim is to show this result for a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$ and a (σ,τ)-derivation d.

2. Preliminaries

Throughout the present paper, we shall make some extensive use of the basic commutator identities:

$[x, yz] = y[x, z] + [x, y]z,$

$[xy, z] = [x, z]y + x[y, z],$

$[xy,z]_{\sigma,\tau} = x[y,z]_{\sigma,\tau} + [x,\tau(z)]y = x[y,\sigma(z)] + [x,z]_{\sigma,\tau}y$,
$[x,yz]_{\sigma,\tau} = \tau(y)[x,z]_{\sigma,\tau} + [x,y]_{\sigma,\tau}\sigma(z)$, and
$[x,[y,z]]_{\sigma,\tau} + [[x,z]_{\sigma,\tau},y]_{\sigma,\tau} - [[x,y]_{\sigma,\tau},z]_{\sigma,\tau} = 0$.

Moreover, we shall require the following lemmas.

Lemma 1 ([10, Lemma 1]). *Let R be a semiprime, 2-torsion free ring and U a nonzero Lie ideal of R. Suppose that $[U,U] \subset Z$, then $U \subseteq Z$.*

Lemma 2 ([7, Lemma 4]). *Let R be a prime ring with characteristic not two, $a,b \in R$. If U is a noncentral Lie ideal of R and $aUb = 0$, then $a = 0$ or $b = 0$.*

Lemma 3 ([7, Theorem 1]). *Let R be a prime ring with characteristic not two and U a nonzero Lie ideal of R. If d is a nonzero derivation of R such that $d^2(U) = 0$, then $U \subseteq Z$.*

Lemma 4 ([12, Lemma 1.1]). *Let R be a prime ring with characteristic not two and U a nonzero Lie ideal of R. If d is a nonzero (σ,τ)-derivation of R such that $d(U) = 0$, then $U \subseteq Z$.*

Lemma 5 ([11, Lemma 1.2]). *Let R be a prime ring with characteristic not two, U a nonzero Lie ideal of R and $a \in R$. If d is a nonzero (σ,τ)-derivation of R such that $ad(U) = 0$ ($d(U)a = 0$), then $U \subseteq Z$ or $a = 0$.*

Lemma 6 ([11, Lemma 1.4]). *Let R be a prime ring with characteristic not two and $a \in R$. If $[U,a] \in Z$, then $a \in Z$ or $U \subseteq Z$.*

3. Results

The following theorem gives a generalization of Posner's well known result [15, Theorem 2] and a extension of [2, Theorem 1].

Theorem 1. *Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If R admits a nonzero (σ,τ)-derivation such that $[d(u),u]_{\sigma,\tau} = 0$ for all $u \in U$, then $U \subseteq Z$.*

Proof. By the hypothesis, we have

$$[d(u),u]_{\sigma,\tau} = 0 \quad \text{for all } u \in U. \tag{3.1}$$

A linearization of (3.1) yields that

$$[d(u),v]_{\sigma,\tau} + [d(v),u]_{\sigma,\tau} = 0 \quad \text{for all } u,v \in U. \tag{3.2}$$

Notice that $uv + vu = (u+v)^2 - u^2 - v^2$ for all $u,v \in U$. Since $u^2 \in U$ for all $u \in U$, $uv + vu \in U$. Also $uv - vu \in U$ for all $u,v \in U$. Hence, we get $2uv \in U$ for all $u,v \in U$. Replacing v by $2vu$ in this equation and using the hypothesis and (3.2), we obtain that

$$2[\tau(v),\tau(u)]d(u) = 0 \quad \text{for all } u,v \in U.$$

Since R is a 2-torsion free ring and τ is an automorphism of R, the above relation yields that

$$\tau([v,u])d(u) = 0 \quad \text{for all } u,v \in U.$$

Taking $2vw, w \in U$ instead of v and using R is a 2-torsion free ring, we get

$$\tau([v,u])\tau(w)\,d(u) = 0 \quad \text{for all } u, v, w \in U.$$

Since τ is an automorphism of R, we see that

$$[v,u]\,U\tau^{-1}(d(u)) = 0 \quad \text{for all } u, v \in U.$$

By Lemma 2, we get either $[v,u] = 0$ or $d(u) = 0$ for each $u \in U$. Let $K = \{u \in U \mid d(u) = 0\}$ and $L = \{u \in U \mid [v,u] = 0 \text{ for all } v \in U\}$ of additive subgroups of U. Morever, U is the set-theoretic union of K and L. But a group can not be the set-theoretic union of two proper subgroups, hence $K = U$ or $L = U$. In the former case, we get $U \subseteq Z$ by Lemma 4. In the latter case, $[U,U] = (0)$. That is $U \subseteq Z$ by Lemma 1. This completes the proof. □

Theorem 2. *Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If R admits a nonzero (σ,τ)-derivation such that $[d(u),u]_{\sigma,\tau} \subset C_{\sigma,\tau}$ for all $u \in U$, then $U \subseteq Z$.*

Proof. Linearizing $[d(u),u]_{\sigma,\tau} \in C_{\sigma,\tau}$, we get

(3.3) $$[d(u),v]_{\sigma,\tau} + [d(v),u]_{\sigma,\tau} \in C_{\sigma,\tau} \quad \text{for all } u, v \in U.$$

On the other hand, we have

$$[d(u),[v,u]]_{\sigma,\tau} = [[d(u),v]_{\sigma,\tau},u]_{\sigma,\tau} - [[d(u),u]_{\sigma,\tau},v]_{\sigma,\tau}$$

and so

(3.4) $$[d(u),[v,u]]_{\sigma,\tau} = [[d(u),v]_{\sigma,\tau},u]_{\sigma,\tau} \quad \text{for all } u, v \in U.$$

Replacing v by $[v,u]$ in (3.3), we see that

$$[d(u),[v,u]]_{\sigma,\tau} + [d([v,u]),u]_{\sigma,\tau} \in C_{\sigma,\tau} \text{ for all } u, v \in U.$$

Since $d([v,u]) = [d(v),u]_{\sigma,\tau} - [d(u),v]_{\sigma,\tau}$, we can write the last equation

$$[d(u),[v,u]]_{\sigma,\tau} + [[d(v),u]_{\sigma,\tau},u]_{\sigma,\tau} - [[d(u),v]_{\sigma,\tau},u]_{\sigma,\tau} \in C_{\sigma,\tau} \text{ for all } u, v \in U.$$

Using (3.4) and this in the last equation, we obtain that

(3.5) $$[[d(v),u]_{\sigma,\tau},u]_{\sigma,\tau} \in C_{\sigma,\tau} \text{ for all } u, v \in U.$$

Now, commutting (3.3) with u, we have

$$[[d(u),v]_{\sigma,\tau},u]_{\sigma,\tau} + [[d(v),u]_{\sigma,\tau},u]_{\sigma,\tau} = 0.$$

Using (3.5) in this equation, we arrive at

(3.6) $$[[d(u),v]_{\sigma,\tau},u]_{\sigma,\tau} \in C_{\sigma,\tau} \text{ for all } u, v \in U.$$

Again using (3.6) in (3.4), we obtain

(3.7) $$[d(u),[v,u]]_{\sigma,\tau} \in C_{\sigma,\tau} \text{ for all } u, v \in U.$$

Replacing v by $2vu$ in (3.7) and using this, we find that

$$\begin{aligned} &2[d(u),[v,u]u]_{\sigma,\tau} \\ &= 2\tau([v,u])[d(u),u]_{\sigma,\tau} + 2[d(u),[v,u]]_{\sigma,\tau}\sigma(u) \in C_{\sigma,\tau} \text{ for all } u, v \in U. \end{aligned}$$

Commutting this term with u, we have

$$2\tau([v,u])[[d(u),u]_{\sigma,\tau},u]_{\sigma,\tau}+2[\tau([v,u]),\tau(u)][d(u),u]_{\sigma,\tau}$$
$$+2[[d(u),[v,u]]_{\sigma,\tau},u]_{\sigma,\tau}\sigma(u)+2[d(u),[v,u]]_{\sigma,\tau}[\sigma(u),\sigma(u)]=0$$

and so

$$\tau([v,u],u])[d(u),u]_{\sigma,\tau}=0 \text{ for all } u,v\in U. \tag{3.8}$$

Multipliying (3.8) with $\sigma(w)$, we get

$$\tau([v,u],u])[d(u),u]_{\sigma,\tau}\sigma(w)=0 \text{ for all } u,v,w\in U.$$

By the hyphothesis, we have $[d(u),u]_{\sigma,\tau}\sigma(w)=\tau(w)[d(u),u]_{\sigma,\tau}$ for all $u,w\in U$. Appliying this in the last equation, we obtain that

$$\tau([v,u],u])\tau(w)[d(u),u]_{\sigma,\tau}=0 \text{ for all } u,v,w\in U.$$

Since τ is an automorphism of R, we get

$$[[v,u],u]U\tau^{-1}([d(u),u]_{\sigma,\tau})=0 \text{ for all } u,v,w\in U.$$

By the application of Lemma 2 yields that $[[v,u],u]=0$ or $[d(u),u]_{\sigma,\tau}=0$ for each $u\in U$. If $[d(u),u]_{\sigma,\tau}=0$ for all $u\in U$, then $U\subseteq Z$ by Theorem 1. Now let $[[v,u],u]=0$ for all $u,v\in U$. We define $I_u(x)=[x,u]$ an inner derivation determined by u. Hence we have $I_u^2(U)=(0)$, and so $U\subseteq Z$ by Lemma 3. □

Theorem 3. *Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2\in U$ for all $u\in U$ and $a\in R$. If R admits a nonzero (σ,τ)-derivation such that $[d(u),a]_{\sigma,\tau}=0$ for all $u\in U$, then $a\in Z$ or $U\subseteq Z$.*

Proof. Let $u,v\in U$. Then

$$0=[d(2uv),a]_{\sigma,\tau}=2[d(u)\sigma(v)+\tau(u)d(v),a]_{\sigma,\tau}$$
$$=2[d(u),a]_{\sigma,\tau}\sigma(v)+2d(u)[\sigma(v),\sigma(a)]+2\tau(u)[d(v),a]_{\sigma,\tau}+2[\tau(u),\tau(a)]$$

and so

$$d(u)\sigma([v,a])=\tau([a,u])d(v) \text{ for all } u,v\in U. \tag{3.9}$$

Replacing v by $2vw$ in (3.9) and using (3.9), we arrive at

$$d(u)\sigma(v)\sigma([w,a])=\tau([a,u])\tau(v)d(w) \text{ for all } u,v,w\in U. \tag{3.10}$$

Let in (3.10) v be $[v,a]$ and again using (3.9) we have

$$d(u)\sigma([v,a])\sigma([w,a])=\tau([a,u])\tau([v,a])d(w),$$
$$\tau([a,u])d(v)\sigma([w,a])=\tau([a,u])\tau([v,a])d(w)$$

and so

$$\tau([a,u])\tau([a,v])d(w))=\tau([a,u])\tau([v,a])d(w) \text{ for all } u,v,w\in U.$$

That is $2\tau([a,u])\tau([a,v])d(w))=0$. Since R is 2-torsion free, we get

$$\tau([a,u][a,v])d(w)=0 \text{ for all } u,v,w\in U.$$

By Lemma 5, we arrive at

$$[a,u][a,v]=0 \text{ for all } u,v\in U.$$

Again replacing v by $2vu$ in the last equation and using this, we have

$$[a,u]U[a,w]=0 \text{ for all } u,w\in U.$$

By the application of Lemma 2 yields that $[a,u]=0$ for all $u\in U$, and so, $a\in Z$ or $U\subseteq Z$ by Lemma 6. This completes the proof. □

Theorem 4. *Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2\in U$ for all $u\in U$. If R admits a nonzero (σ,τ)-derivation d such that $d([u,v])=0$ for all $u,v\in U$, then $U\subseteq Z$.*

Proof. We assume that

$$d([u,v])=0 \text{ for all } u,v\in U. \tag{3.11}$$

Replacing v by $2vu$ in (3.11) and using R is 2-torsion free, we get

$$d([u,v])\sigma(u)+\tau\ ([u,v])d(u)=0 \text{ for all } u,v\in U.$$

Appliying (3.11), we have

$$\tau([u,v])d(u)=0 \text{ for all } u,v\in U. \tag{3.12}$$

Writing $2vw$ in (3.12) instead of v and using this, we have

$$2(\tau([u,v])\tau(w)d(u)+\tau(v)\tau([u,w])d(u))=0$$

and so

$$\tau([u,v])\tau(w)d(u)=0 \text{ for all } u,v,w\in U$$

That is

$$[u,v]U\tau^{-1}(d(u))=0 \text{ for all } u,v\in U.$$

By the application of Lemma 2 yields that $[u,v]=0$ or $d(u)=0$ for each $u\in U$. Using the same arguments in the proof of Theorem 1, we get the required result. □

Theorem 5. *Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2\in U$ for all $u\in U$. If R admits a nonzero (σ,τ)-derivation d such that $d([u,v])=\pm[u,v]_{\sigma,\tau}$ for all $u,v\in U$, then $U\subseteq Z$.*

Proof. By the hypothesis, we get

$$d([u,v])=[u,v]_{\sigma,\tau} \text{ for all } u,v\in U. \tag{3.13}$$

Substituting $2vu$ for v in (3.13) and using R is 2-torsion free, we arrive at

$$d([u,v])\sigma(u)+\tau([u,v])d(u)=\tau(v)[u,u]_{\sigma,\tau}+[u,v]_{\sigma,\tau}\sigma(u) \text{ for all } u,v\in U.$$

Using the equation (3.13), we have

$$\tau([u,v])d(u)=\tau(v)[u,u]_{\sigma,\tau} \text{ for all } u,v\in U. \tag{3.14}$$

Replacing v by $2wv, w\in U$ in (3.14), we find that

$$\tau([u,w])\tau(v)d(u)+\tau(w)\tau[u,v]d(u)=\tau(w)\tau(v)[u,u]_{\sigma,\tau} \text{ for all } u,v,w\in U.$$

Using (3.14), we see that

$$\tau([u,w])\,\tau(v)\,d(u)=0 \quad \text{for all } u,v,w\in U$$

and so,

$$[u,w]U\tau^{-1}(d(u))=0 \quad \text{for all } u,w\in U.$$

We get the required result appliying similar arguments in the proof of Theorem 1.

Let assume that $d([u,v])=-[u,v]_{\sigma,\tau}$ for all $u,v\in U$. It can be proved using the same techniques above. This completes the proof. □

Remark 6. Since every ideal in a ring R is a Lie ideal of R, conclusion of the above theorems hold even if U is assumed to be an ideal of R. Though the assumption that $u^2\in U$ for all $u\in U$ seems close to assuming that U is an ideal of the ring, but there exist Lie ideals with this property which are no ideals. For example, let $R=\{\left(\begin{smallmatrix} a & b\\ 0 & c\end{smallmatrix}\right)\mid a,b,c\in Z\}$. Then it can be easily seen that $U=\{\left(\begin{smallmatrix} a & b\\ 0 & a\end{smallmatrix}\right)\mid a,b\in Z\}$ is a Lie ideal of R satisfiying $u^2\in U$ for all $u\in U$. However, U is not an ideal of R.

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