

SOME NEW BONNESEN-STYLE INEQUALITIES

JIAZU ZHOU, YUNWEI XIA, AND CHUNNA ZENG

ABSTRACT. By evaluating the containment measure of one domain to contain another, we will derive some new Bonnesen-type inequalities (Theorem 2) via the method of integral geometry. We obtain Ren's sufficient condition for one domain to contain another domain (Theorem 4). We also obtain some new geometric inequalities. Finally we give a simplified proof of the Bottema's result.

1. Introduction and preliminaries

A geometric inequality describes the relation among the invariants of a geometric object in space. Perhaps the isoperimetric inequality is the best known geometric inequality. The classical isoperimetric inequality says that: for a simple closed curve C of length L in the Euclidean plane \mathbb{R}^2 , the area A enclosed by C satisfies

$$(1.1) \quad L^2 - 4\pi A \geq 0.$$

The equality sign holds if and only if C is a circle. It follows that the circle is the only curve of constant length L enclosing maximum area.

One could find the traditional proofs by Hurwitz, Schmidt, Osserman, Minkowski and others. The analytic proofs of the isoperimetric inequality root back to centuries ago. One can find some simplified and beautiful proofs that lead to generalizations of higher dimensions, domains in a surface of constant curvature, and applications to other branches of mathematics ([1], [6], [7], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [20], [24], [21], [22], [25], [26], [27], [29], [30], [31], [32], [33], [34], [36], [39], [40]).

Let K be a domain with the boundary composing of the simple curve of length L and area A . Then the isoperimetric deficit of K is defined as

$$(1.2) \quad \Delta(K) = L^2 - 4\pi A.$$

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The isoperimetric deficit measures the deficit between the domain of length L and area A and a disc of radius $L/2\pi$. Perhaps Bonnesen was the first one to give a concrete lower isoperimetric deficit ([26], [25]):

Theorem 1. *Let K be a domain with a simple boundary ∂K of length L and area A . Denote by r_e and r_i , respectively, radius of the minimum circumscribed disc and radius of the maximum inscribed disc of K . Then*

$$(1.3) \quad L^2 - 4\pi A \geq \pi^2(r_e - r_i)^2,$$

where the equality holds if and only if K is a disc.

During the 1920's, Bonnesen proved a series of inequalities of the form

$$(1.4) \quad \Delta(K) \geq B,$$

where the equality B is an invariant of geometric significance having the following basic properties:

1. B is non-negative;
2. B is vanish only when K is a disc.

Many B s are found in the last century and mathematicians are still working on those unknown invariants of geometric significance. See references [2], [3], [4], [6], [10], [17], [22], [25], [26], [39], [40], [47] for more details.

A set of points K in \mathbb{R}^n is called *convex* if for each pair of points $x \in K$, $y \in K$ it is true that the line segment \overline{xy} joining x and y is contained in K . Let $D \neq \emptyset$ be a set; then the convex hull D^* is the intersection of all convex sets containing D . Since for any domain D in \mathbb{R}^2 , its convex hull D^* increases the area A^* and decreases the length L^* . Then we have $L^2 - 4\pi A \geq L^{*2} - 4\pi A^*$, that is $\Delta(D) \geq \Delta(D^*)$. Therefore the isoperimetric inequality and the Bonnesen-type inequality are valid for all domains in \mathbb{R}^2 if they are valid for convex domains.

One of our main results is the following Bonnesen-style inequalities:

Theorem 2. *Let D be a plane domain of area A and bounded by a simple closed curve of length L . Let D^* be the convex hull of D and its area and circumference be, respectively, A^* and L^* . Then we have the following Bonnesen-type isoperimetric inequalities:*

$$(1.5) \quad L^2 - 4\pi A \geq (L - L^*)^2; \quad L^2 - 4\pi A \geq 2\pi(A^* - A).$$

Each equality holds when D is a disc.

Another geometric question closely related to the isoperimetric inequality ask: given two domains D_k ($k = i, j$), when can one domain contain another? More precisely, we ask if there is an isometry (rigid motion) g of the Euclidean space \mathbb{R}^n so that $gD_j \subset D_i$ or $gD_j \supset D_i$. We wish to have an answer that depends only on the geometric invariants of domains involved, preferably on the "surface" areas A_k and the volumes V_k . One would ask the same question for domains of the space of constant curvature. But the invariants may involve

more curvature integrals. See [10], [18], [25], [35], [36], [39]-[43], [46], [47] for more references.

In this paper, we will discuss these geometric problems by method of integral geometry. By estimating the containment measure of one domain to contain another we obtain some new Bonnesen-style inequalities. We also obtain the Ren's sufficient condition for a plane domain to contain another domain. The sufficient condition of Ren is an analog of Hadwiger's sufficient condition and is characterized beautifully by the isoperimetric deficits of two domains. Finally, we give a simplified proof of the Bottema's isoperimetric deficit upper limit. The idea can be applied to higher dimensions ([39]-[45], [46]).

2. The Bonnesen-style inequalities

Let $g \in G$, the isometries in \mathbb{R}^2 . In integral geometry dg is called the kinematic density of G . For a domain D let $\chi(D)$ be the Euler-Poincaré characteristic of D . We regard the domain D_i as fixed and the domain gD_j as moving via the isometry g . Averaging the Euler-Poincaré characteristic of the intersection $D_i \cap gD_j$ we have the Blaschke kinematic formula ([25], [26])

$$(2.1) \quad \int_{\{g: D_i \cap gD_j \neq \emptyset\}} \chi(D_i \cap gD_j) dg = 2\pi(A_i + A_j) + L_i L_j.$$

Denote the number of points of the intersection $\partial D_i \cap g\partial D_j$ by $\#\{\partial D_i \cap g\partial D_j\}$. We have the Poincaré formula:

$$(2.2) \quad \int_{\{g: \partial D_i \cap g\partial D_j \neq \emptyset\}} \#\{\partial D_i \cap g\partial D_j\} dg = 4L_i L_j.$$

Since the domain D_k ($k = i, j$) are assumed to be connected and simply connected and bounded by simple curves, we have $\chi(D_i \cap gD_j) = n(g) =$ the number of connected components of the intersection $D_i \cap gD_j$. The fundamental kinematic formula of Blaschke can be rewritten as

$$(2.3) \quad \int_{\{g: D_i \cap gD_j \neq \emptyset\}} n(g) dg = 2\pi(A_i + A_j) + L_i L_j.$$

If μ denotes the set of all positions of D_j in which either $gD_j \subset D_i$ or $gD_j \supset D_i$, then the above formula of Blaschke can be rewritten as

$$(2.4) \quad \int_{\mu} dg + \int_{\{g: \partial D_i \cap g\partial D_j \neq \emptyset\}} n(g) dg = 2\pi(A_i + A_j) + L_i L_j.$$

When $\partial D_i \cap g\partial D_j \neq \emptyset$, each component of $D_i \cap gD_j$ is bounded by at least an arc of ∂D_i and an arc of $g\partial D_j$. Therefore $n(g) \leq \#\{\partial D_i \cap g\partial D_j\}/2$. Then by formulas of Poincaré and Blaschke we obtain

$$(2.5) \quad \int_{\mu} dg \geq 2\pi(A_i + A_j) - L_i L_j.$$

Therefore this inequality immediately gives ([25], [26]):

Theorem 3 (Hadwiger). *Let D_i and D_j be two domains in the euclidean plane \mathbb{R}^2 . A sufficient condition for D_j to contain, or to be contained in, D_i is*

$$(2.6) \quad 2\pi(A_i + A_j) - L_i L_j > 0.$$

Moreover, if $A_i \geq A_j$, then D_i contains D_j .

The following analog of Hadwiger's result is due to Ren [25].

Theorem 4. *Let D_k ($k = i, j$) be two domains in the euclidean plane \mathbb{R}^2 with areas A_k and circum-lengths L_k . Denote by $\Delta(D_k) = L_k^2 - 4\pi A_k$ the isoperimetric deficit of D_k . Then a sufficient condition for D_j to contain D_i is*

$$(2.7) \quad L_j - L_i > \sqrt{\Delta(D_i) + \Delta(D_j)}.$$

Proof. The sufficient condition (2.7) means $L_j > L_i$ and

$$(2.8) \quad 2\pi(A_i + A_j) - L_i L_j > 0.$$

By Theorem 1, we conclude that D_i either contain D_j or to be contained in D_j . Also from the inequality (2.7) we have

$$(2.9) \quad 2\pi(A_j - A_i) > L_i L_j - 4\pi A_i > L_i^2 - 4\pi A_i.$$

The isoperimetric inequality guarantees $A_j > A_i$. Then by the Hadwiger's Theorem 3 we complete the proof of the theorem. \square

If we let $D_i \equiv D_j \equiv D$, then D can not contain any copy of D itself. Then the left hand side integral of (2.5) vanishes and we immediately obtain the classical isoperimetric inequality (1.1).

If we let D_i be, respectively, the maximum inscribed disc of radius r_i of D_j ($\equiv D$) and the minimum circumscribed disc of radius r_e of D_j ($\equiv D$), then we have

$$(2.10) \quad \pi r_i^2 - L r_i + A \leq 0; \quad \pi r_e^2 - L r_e + A \leq 0.$$

From these inequalities and another known general inequality $x^2 + y^2 \geq (x + y)^2/2$, we obtain the Bonnesen's isoperimetric inequality:

$$L^2 - 4\pi A \geq \pi^2(r_e - r_i)^2.$$

Since

$$(2.11) \quad (L - 2\pi r_i)^2 - (L^2 - 4\pi A) = 4\pi(\pi r_i^2 - L r_i + A) \leq 0$$

and

$$(2.12) \quad (2\pi r_e - L)^2 - (L^2 - 4\pi A) = 4\pi(\pi r_e^2 - L r_e + A) \leq 0,$$

we immediately obtain the following Bonnesen-style inequalities:

Theorem 5. *Let D be a plane domain of area A and bounded by a simple closed curve of length L . Let r_i and r_e be, respectively, the radius of the maximum inscribed circle and the radius of the minimum circumscribed circle of D . Then we have*

$$(2.13) \quad L^2 - 4\pi A \geq (L - 2\pi r_i)^2; \quad L^2 - 4\pi A \geq (2\pi r_e - L)^2.$$

Each equality holds if and only if D is a disc.

Note that Bonnesen proved inequalities of (2.13) for convex domains only (see [22]).

If we let $D_j \equiv D$ and D_i be a disc of radius r satisfying the condition, $r_i \leq r \leq r_e$, then formula (2.5) immediately gives the following Bonnesen's inequality ([3], [26]).

Corollary 1. *Let D be a plane domain of area A and bounded by a simple closed curve of length L . Let r_i and r_e be, respectively, the radius of the maximum inscribed circle and the radius of the minimum circumscribed circle of D . Then we have*

$$(2.14) \quad \pi r^2 - Lr + A \leq 0; \quad r_i \leq r \leq r_e.$$

The above Bonnesen's inequality can be rewritten in several equivalent forms:

$$(2.15) \quad \begin{aligned} Lr &\geq A + \pi r^2; \\ L^2 - 4\pi A &\geq (L - 2\pi r)^2; \\ L^2 - 4\pi A &\geq \left(L - \frac{2A}{r}\right)^2; \\ L^2 - 4\pi A &\geq \left(\frac{A}{r} - \pi r\right)^2. \end{aligned}$$

Notice that the third inequality of (2.15) implies

$$(2.16) \quad \sqrt{L^2 - 4\pi A} \geq \frac{2A}{r_i} - L, \quad \sqrt{L^2 - 4\pi A} \geq L - \frac{2A}{r_e}.$$

Adding these two inequalities yields

$$(2.17) \quad L^2 - 4\pi A \geq A^2 \left(\frac{1}{r_i} - \frac{1}{r_e}\right)^2.$$

Again adding after multiplied by r_i and r_e gives the following Bonnesen-style inequality

$$(2.18) \quad L^2 - 4\pi A \geq L^2 \left(\frac{r_e - r_i}{r_e + r_i}\right)^2.$$

Notice that equation $\pi r^2 - Lr + A = 0$ has two roots $\frac{L - \sqrt{L^2 - 4\pi A}}{2\pi}$ and $\frac{L + \sqrt{L^2 - 4\pi A}}{2\pi}$ if $L^2 - 4\pi A > 0$. The inequality $\pi r^2 - Lr + A \leq 0$ holds for any r in the closed interval $[\frac{L - \sqrt{L^2 - 4\pi A}}{2\pi}, \frac{L + \sqrt{L^2 - 4\pi A}}{2\pi}]$. Hence we have the following Bonnesen-style inequalities ([6], [47]):

Corollary 2. *Let D be a plane domain of area A and bounded by a simple closed curve of length L . Let r_i and r_e be, respectively, the radius of the maximum inscribed circle and the radius of the minimum circumscribed circle of D . Then for any disc of radius r , $r_i \leq r \leq r_e$, we have the following Bonnesen-type isoperimetric inequalities:*

$$(2.19) \quad \begin{aligned} L^2 - 4\pi A &\geq A^2 \left(\frac{1}{r_i} - \frac{1}{r_e}\right)^2; & L^2 - 4\pi A &\geq L^2 \left(\frac{r_e - r_i}{r_e + r_i}\right)^2; \\ L^2 - 4\pi A &\geq A^2 \left(\frac{1}{r_i} - \frac{1}{r}\right)^2; & L^2 - 4\pi A &\geq L^2 \left(\frac{r - r_i}{r + r_i}\right)^2; \\ L^2 - 4\pi A &\geq A^2 \left(\frac{1}{r} - \frac{1}{r_e}\right)^2; & L^2 - 4\pi A &\geq L^2 \left(\frac{r_e - r}{r_e + r}\right)^2; \end{aligned}$$

$$(2.20) \quad \frac{L - \sqrt{L^2 - 4\pi A}}{2\pi} \leq r_i \leq r_e \leq \frac{L + \sqrt{L^2 - 4\pi A}}{2\pi}.$$

Each equality holds if and only if D is a disc.

The equation $\pi r^2 - Lr + A = 0$ has the unique root $\frac{L}{2\pi}$ if and only if $L^2 - 4\pi A = 0$. Therefore inequality (2.20) gives isoperimetric inequality another explanation. That is, D must be a disc.

Let D_r be a disc of radius r . Then by the containment measure inequality (2.5) we have:

Corollary 3. *Let D_r be a disc of radius r and D be a domain of area A and bounded by a simple closed curve of length L . A sufficient condition for disc D_r to contain, or to be contained in, domain D is*

$$(2.21) \quad \pi r^2 - Lr + A > 0.$$

Moreover, if $\text{diam}(D) > r$, then D contains D_r . And $\text{diam}(D) < r$, then D can be contained in D_r .

The proof of Theorem 2. Let D be a domain of area A and bounded by a simple closed curve of length L . Let D^* be the convex hull of D and its area and circum-length be, respectively, A^* and L^* . Since there is no rigid motion g such that $gD \subset D^*$ or $gD \supset D^*$, then the inequality (2.5) gives

$$(2.22) \quad 2\pi(A + A^*) - LL^* \leq 0.$$

This gives

$$(2.23) \quad 2\pi(A^* - A) \leq LL^* - 4\pi A \leq L^2 - 4\pi A,$$

and

$$(2.24) \quad \begin{aligned} L^2 - 4\pi A - (L - L^*)^2 &= 2LL^* - L^{*2} - 4\pi A \\ &\geq 2LL^* - LL^* - 4\pi A \\ &= LL^* - 4\pi A \\ &\geq LL^* - 2\pi(A + A^*) \geq 0. \end{aligned}$$

Then we obtain the new Bonnesen-style inequalities (1.5) and we complete the proof of Theorem 2. □

The discrete case of the first inequality in Theorem 2 is proved by X. Zhang (cf. [37], [38]) via a different approach.

3. The isoperimetric deficit upper limit

When mathematicians are mainly interested in and focus on the lower bound of the isoperimetric deficit, there is another question: is there invariant C of geometric significance such that

$$(3.1) \quad \Delta(K) \leq C?$$

Of course we expect that the upper bound be attained when K is a disc. This is a long standing unsolved problem in geometry. Unfortunately we are not aware of any upper bound up today except for few special convex domains ([5], [23], [26]).

Assume that the boundary ∂K of the convex set K has a continuous radius of curvature ρ . By the Bonnesen's inequality

$$(3.2) \quad \pi r^2 - Lr + A \leq 0; \quad r_i \leq r \leq r_e,$$

therefore if $r \leq r_i$ or $r \geq r_e$ we have

$$(3.3) \quad \pi r^2 - Lr + A \geq 0.$$

Since

$$(3.4) \quad \pi r^2 - Lr + A = \pi \left(\frac{L}{2\pi} - r \right)^2 - \left(\frac{L^2}{4\pi} - A \right),$$

therefore we have

$$(3.5) \quad \frac{L^2}{4\pi} - A \leq \pi \left(\frac{L}{2\pi} - r \right)^2 \quad \text{for } r \leq \rho_m \quad \text{or} \quad r \geq \rho_M.$$

In particular, for $r = \rho_m$ and $r = \rho_M$, we have

$$(3.6) \quad \frac{L^2}{4\pi} - A \leq \pi \left(\frac{L}{2\pi} - \rho_m \right)^2; \quad \frac{L^2}{4\pi} - A \leq \pi \left(\rho_M - \frac{L}{2\pi} \right)^2.$$

Since $\rho_m \leq \frac{L}{2\pi} \leq \rho_M$, then we have

$$(3.7) \quad \frac{L^2}{4\pi} - A \leq \pi \left(\frac{L}{2\pi} - \rho_m \right) \left(\rho_M - \frac{L}{2\pi} \right).$$

Finally by $4xy \leq (x+y)^2$ we obtain

$$(3.8) \quad \Delta(K) = L^2 - 4\pi A \leq \pi^2 (\rho_M - \rho_m)^2.$$

Therefore we obtain the following Bottema's upper isoperimetric deficit limit of K ([5], [26], [25]):

Theorem 6. *Let K be a convex domain of area A and length L with the continuous radius of curvature ρ of ∂K . Let ρ_m and ρ_M be, respectively, the minimum and the maximum of the curvature radius of ∂K . Then we have*

$$(3.9) \quad L^2 - 4\pi A \leq \pi^2(\rho_M - \rho_m)^2.$$

The equality sign holds if and only if $\rho_M = \rho_m$, that is, if K is a circle.

Pleijel (see [23], [26]) has an improvement of Bottema's result as follows:

$$(3.10) \quad L^2 - 4\pi A \leq \pi(4 - \pi)(\rho_M - \rho_m)^2.$$

We recently investigate convex sets in a plane E_ϵ^2 of constant curvature ϵ and obtain an upper limit for the isoperimetric deficit of a convex domain K in E_ϵ^2 with continuous radius ρ_ϵ of ∂K (see [19]). The upper limit can be seen as a generalization of the Bottema's result in E_ϵ^2 and involves the smallest radius ρ_m and the greatest radius ρ_M of curvature radius ρ_ϵ of ∂K . The upper limit is attained when and only when K is a geodesic disc.

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JIAZU ZHOU
SCHOOL OF MATHEMATICS AND STATISTICS
SOUTHWEST UNIVERSITY
CHONGQING 400715, P. R. CHINA
E-mail address: zhoujz@swu.edu.cn

YUNWEI XIA
SCHOOL OF MATHEMATICS AND STATISTICS
SOUTHWEST UNIVERSITY
CHONGQING 400715, P. R. CHINA
E-mail address: xiayunw@126.com

CHUNNA ZENG
SCHOOL OF MATHEMATICS AND STATISTICS
SOUTHWEST UNIVERSITY
CHONGQING 400715, P. R. CHINA
E-mail address: zengchn@swu.edu.cn