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ALMOST PSEUDO CONTACT STRUCTURE

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Abstract. A new kind of structure is introduced in an even dimensional differentiable Riemannian manifold and some basic properties of this structure is discussed. Also the existence of such structure is shown with an example.

Introduction

The notion of an almost contact structure has been initiated by Boothby and Wang [1]. According to them an odd dimensional differentiable manifold M_{2n+1} is said to admit a $(\psi,\ \xi,\ \eta)$ structure if it admits a field ψ of endomorphism of the tangent spaces, i.e., a tensor field of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\psi^2(X) = -X + \eta(X)\xi, \quad \psi(\xi) = 0$$

for any vector field X. This structure is known as an almost contact structure. An odd dimensional manifold is usually called almost contact manifold which admits an almost contact structure $(\psi,\ \xi,\ \eta)$.

In this paper, a new type of structure in an even dimensional manifold M_{2n}, named as almost pseudo contact structure with two associated vector fields has been introduced .

Let $M_{2n}(n \geq 2)$ be an even dimensional manifold and let ϕ be a tensor field of type $(1,1)$, U and V be two linearly independent vector fields and A and B be two non-zero 1-forms respectively. If the system (ϕ, U, V, A, B) satisfies the conditions

$$\phi(U) = 0, \quad \phi(V) = 0 \tag{0.1}$$

and

$$\phi^2(X) = -X + A(X)U + B(X)V \tag{0.2}$$

for any vector field X, then $M_{2n}, (n \geq 2)$ is said to admit an almost pseudo contact structure (ϕ, U, V, A, B) and such a manifold $M_{2n}(n \geq 2)$ is called an almost pseudo contact manifold.

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So, we see that, if the sum of the two associated vector fields U and V becomes zero, i.e., if $U + V = 0$, then obviously the dimension of the manifold becomes $2n - 1$ and it becomes an almost contact manifold with $\eta = (A - B)$ and $\xi = U$.

1. Preliminaries

We have the almost pseudo contact structure (ϕ, U, V, A, B) as

$$\phi^2(X) = -X + A(X)U + B(X)V. \tag{1.1}$$

Pre-operating ϕ on the equation (1.1) and applying (0.1), we get

$$\phi^3(X) + \phi(X) = 0. \tag{1.2}$$

Again putting ϕX for X in the equation (1.1), we get

$$\phi^3(X) + \phi(X) = A(\phi X)U + B(\phi X)V. \tag{1.3}$$

From equations (1.2) and (1.3), we get

$$A(\phi X)U + B(\phi X)V = 0. \tag{1.4}$$

Now, we know that U and V are two linearly independent vector fields. Thus, by the equation (1.4), we have

$$A(\phi X) = B(\phi X) = 0$$

for every vector field X. So, we have the following:

Theorem 1.1. *In an almost pseudo contact manifold the structure tensor ϕ and the 1-forms A and B satisfies the relation $A \circ \phi = B \circ \phi = 0$.*

Now, we are going to prove that:

Theorem 1.2. *In an almost pseudo contact manifold the 1-forms A, B and the linearly independent vector fields U, V satisfies the relation $A(U) = B(V) = 1$ and $A(V) = B(U) = 0$.*

Proof. Putting $X = U$ in the equation (1.1), we get

$$[A(U) - 1]U + B(U)V = 0.$$

Since U and V are linearly independent, we get

$$A(U) = 1, \quad B(U) = 0.$$

Similarly, putting $X = V$ in the equation (1.1), we get

$$A(V) = 0, \quad B(V) = 1.$$

□

Now, we have $\phi(U) = 0$ and $\phi(V) = 0$ in an almost pseudo contact manifold and U and V are linearly independent. Thus

$$\text{rank } \phi \leq 2n - 2.$$

Let W be another vector field such that $\phi(W) = 0$. Thus by the equation (1.1) we get

$$W = A(W)U + B(W)V.$$

So, we see that W, U and V are linearly dependent. Therefore $\ker\phi$ is generated by U and V only and thus rank $\phi = 2n - 2$. Hence, we can state that:

Theorem 1.3. *In an almost pseudo contact manifold M_{2n},* rank $\phi = 2n - 2$.

We will now show that the almost pseudo contact structure in an almost pseudo contact manifold is not unique. Let f be a non singular vector valued linear function on M_{2n}.

Let us define the $(1,1)$ tensor field ϕ^*, the 1-forms A^*, B^* and the vector fields U^*, V^* as

$$f \circ \phi^* = \phi \circ f, \tag{1.5}$$

$$A^* = A \circ f, \tag{1.6}$$

$$B^* = B \circ f, \tag{1.7}$$

$$fU^* = U, \tag{1.8}$$

$$fV^* = V. \tag{1.9}$$

Now, post multiplying equation (1.5) by ϕ^* and again using (1.5), we get

$$\begin{aligned} f \circ {\phi^*}^2 &= \phi \circ f \circ \phi^* = \phi \circ (f \circ \phi^*) \\ &= \phi^2 \circ f \\ &= (-I + A \otimes U + B \otimes V) \circ f. \end{aligned}$$

Applying equations (1.6) and (1.7), we get

$$f \circ {\phi^*}^2 = -f + A^* \otimes U + B^* \otimes V,$$

using equation (1.8) and (1.9), we get

$$\begin{aligned} f \circ {\phi^*}^2 &= \ -f + fA^* \otimes U^* + fB^* \otimes V^* \\ &= f \circ (-I + A^* \otimes U^* + B^* \otimes V^*). \end{aligned}$$

Since f is non-singular, we have

$${\phi^*}^2 = -I + A^* \otimes U^* + B^* \otimes V^*. \tag{1.10}$$

Now, $f \circ \phi^* U^* = \phi \circ fU^* = \phi(U) = 0$, since f is non-singular $\phi^* U^* = 0$ and similarly $\phi^* V^* = 0$, i.e.,

$$\phi^* U^* = \phi^* V^* = 0. \tag{1.11}$$

Therefore, with the help of equations (1.10) and (1.11), we can state the following theorem:

Theorem 1.4. *The almost pseudo contact structure in an almost pseudo contact manifold is not unique.*

Next, let us prove:

Theorem 1.5. *The eigenvalues of* ϕ *are the roots of the equation* $\alpha(\alpha^2+1) = 0$.

Proof. Let α be the eigenvalue of ϕ and ζ be the corresponding eigenvector. Then

$$\phi(\zeta) = \alpha\zeta, \quad \phi^2(\zeta) = \alpha^2\zeta.$$

Now, putting in the equation (1.1), we get

$$(\alpha^2 + 1)\zeta = A(\zeta)U + B(\zeta)V, \tag{1.12}$$

again operating ϕ on the equation (1.12), we get

$$\alpha(\alpha^2 + 1) = 0. \tag{1.13}$$

Thus, the eigenvalues of ϕ are the roots of the equation $\alpha(\alpha^2 + 1) = 0$. □

Corollary 1.5.1. *The eigenvalues of* ϕ *are* 0, i *and* $-i$.

Corollary 1.5.2. *If the vectors* ζ, U *and* V *are linearly independent, then*

$$A(\zeta) = B(\zeta) = 0.$$

Proof. The proof is clear from the equation (1.12). □

Now, let us define a vector valued $(1,1)$ tensor field F such that

$$F^2(X) = \phi^2(X) - A(X)U - B(X)V$$

for all vector fields X in M_{2n}. Then we see that

$$F^2(X) = -X$$

for all vector fields X in M_{2n}. Thus, we have the following:

Theorem 1.6. *Every almost pseudo contact structure induces an almost complex structure in an almost pseudo contact manifold.*

If the Nijenhuis tensor, $N(X,Y)$ of the induced almost complex structure F, where

$$N(X,Y) = [FX, FY] - F[FX, Y] - F[X, FY] - [X, Y]$$

vanishes, i.e., the almost complex structure becomes integrable, we will call the almost pseudo contact structure a pseudo contact structure.

Though the almost pseudo contact structure always induces an almost complex structure but the basic difference is that, almost pseudo contact structure is singular but almost complex structure is non-singular [2].

Remark. Apparently it seems that an almost contact structure [1] ψ defined in $2n+1$ dimensional manifolds, such that

$$\psi^2(X) = -X + \eta(X)\xi, \quad \psi(\xi) = 0,$$

where ψ is a tensor field of type $(1,1)$, η is a 1-form and ξ is a vector field, also induces a complex structure (which is defined in even dimensions only) if we define

$$F^2(X) = \psi^2(X) - \eta(X)\xi,$$

since we have $F^2(X) = -X$ for all X. But we know that ψ is real and thus trace of ψ, which is the sum of its eigenvalues, is also real. Since the eigenvalues of ψ are $0, i, -i$, of which 0 is of multiplicity one and rank $\psi = 2n$ (as ξ is the only vector for which $\psi(\xi) = 0$), the eigenvalues $(i, -i)$ occur in pairs, the dimension of the almost contact manifold must be odd. So a complex structure in this manner can not be defined in an odd dimensional manifold, otherwise the complex structure will not be a real transformation because the eigenvalues of F are $i, -i$ and the trace F will become complex.

But in case of an almost pseudo contact structure defined in M_{2n}, since rank $\phi = 2n - 2$ and the eigenvalues are $0, i, -i$, of which 0 is of multiplicity 2, no ambiguity arises.

2. Metric on almost pseudo contact manifold

Now, we are going to show that:

Theorem 2.1. *Every almost pseudo contact manifold M_{2n} admits a Riemannian metric tensor field g such that*

$$g(\phi X, \phi Y) = g(X, Y) - A(X)A(Y) - B(X)B(Y) \tag{2.1}$$

and

$$g(X, U) = A(X), \quad g(X, V) = B(X). \tag{2.2}$$

Proof. To prove this, we first prove the following lemma:

Lemma 2.1.1. *Every almost pseudo contact manifold M_{2n} admits a Riemannian metric tensor field h such that $h(X, U) = A(X)$ and $h(X, V) = B(X)$ for every vector field X on M_{2n}.*

Proof. Since M_{2n} admits a metric tensor field f (which exists provided M_{2n} is paracompact), we set h as

$$\begin{aligned} h(X, Y) = {} & f[X - A(X)U - B(X)V, Y - A(Y)U - B(Y)V] \\ & + A(X)A(Y) + B(X)B(Y). \end{aligned}$$

Now, putting $Y = U$ and $Y = V$, we get

$$h(X, U) = A(X), \; h(X, V) = B(X).$$ □

Now, let us put

$$g(X, Y) = \frac{1}{2}[h(X, Y) + h(\phi X, \phi Y) + A(X)A(Y) + B(X)B(Y)],$$

so, we get

$$g(X, U) = A(X), \quad g(X, V) = B(X).$$

Again

$$g(\phi X, \phi Y) = \frac{1}{2}[h(\phi X, \phi Y) + h(\phi^2 X, \phi^2 Y) + A(\phi X)A(\phi Y) + B(\phi X)B(\phi Y)].$$

Now, by $A \circ \phi = B \circ \phi = 0$ and the equation (1.1), we get

$$g(\phi X, \phi Y) = \frac{1}{2}[h(\phi X, \phi Y) + h(X, Y) - A(X)A(Y) - B(X)B(Y)], \text{i.e.,}$$
$$g(\phi X, \phi Y) = g(X, Y) - A(X)A(Y) - B(X)B(Y). \qquad \square$$

Putting ϕY for Y in the equation (2.1) and using the equation (1.1), we have

$$g(\phi X, -Y + A(Y)U + B(Y)V) = g(X, \phi Y) - A(X)A(\phi Y) - B(X)B(\phi Y).$$

Since $A \circ \phi = B \circ \phi = 0$, we get

$$g(\phi X, Y) + g(X, \phi Y) = 0.$$

Thus, we state the following:

Theorem 2.2. *The structure tensor ϕ of the almost pseudo contact structure is skew-symmetric with respect to the metric tensor field g.*

3. Necessary and sufficient condition for being an almost pseudo contact manifold

Theorem 3.1. *The necessary and sufficient condition that a manifold M_{2m+2} will be an almost pseudo contact manifold is that, at each point of the manifold M_{2m+2}, it contains a tangent bundle Π_m of dimension m, a tangent bundle $\widetilde{\Pi}_m$, complex conjugate to Π_m and a tangent bundle Π_2 of dimension 2 such that $\Pi_m \cap \widetilde{\Pi}_m = \Pi_m \cap \Pi_2 = \widetilde{\Pi}_m \cap \Pi_2 = \{\Phi\}$ (where $\{\Phi\}$ is the null set) and $\Pi_m \cup \widetilde{\Pi}_m \cup \Pi_2 =$ a tangent bundle of dimension $2m+2$, projection L, M, N on $\Pi_m, \widetilde{\Pi}_m$ and Π_2 respectively being given by*

a) $2L = -\phi^2 - i\phi$ b) $2M = -\phi^2 + i\phi$ c) $N = \phi^2 + I_{2m+2} = A \otimes U + B \otimes V$.

Proof. Let P_x be m linearly independent eigenvectors corresponding to the eigenvalue i of ϕ, Q_x be m linearly independent eigenvectors corresponding to $-i$ and R_y be two linearly independent eigenvectors corresponding to the eigenvalue 0 respectively where $x = 1, 2, \ldots, m$ and $y = 1, 2$. Then we have
$a^x P_x = 0 \Rightarrow a^x = 0$, $b^x Q_x = 0 \Rightarrow b^x = 0$ and $c^y R_y = 0 \Rightarrow c^y = 0$ for all x and y.
Now, let us consider the equation

$$a^x P_x + b^x Q_x + c^y R_y = 0, \tag{3.1}$$

where a^x, b^x and c^y are scalars, $x = 1, 2, \ldots, m$ and $y = 1, 2$ and Einstein's summation convention is used. Applying ϕ on the equation (3.1), we get

$$a^x \phi(P_x) + b^x \phi(Q_x) = 0$$

$$\Rightarrow\ a^x P_x - b^x Q_x = 0. \tag{3.2}$$

Applying ϕ once again, we get

$$a^x P_x + b^x Q_x = 0. \tag{3.3}$$

Now, using equations (3.1), (3.2) and (3.3), we get

$$a^x P_x = b^x Q_x = c^y R_y = 0$$

for $x = 1, 2, \ldots, m$ and $y = 1, 2$. So, we get

$$a^x = b^x = c^y = 0$$

for $x = 1, 2, \ldots, m$ and $y = 1, 2$.
Therefore, $\{P_x, Q_x, R_y\}$ is a linearly independent set.

Now, if L, M, N are the projection maps on $\Pi_m, \widetilde{\Pi}_m$ and Π_2 respectively, then we must have

$$\begin{array}{lll} LP_x = P_x & LQ_x = 0 & LR_y = 0 \\ MP_x = 0 & MQ_x = Q_x & MR_y = 0 \\ NP_x = 0 & NQ_x = 0 & NR_y = R_y \end{array}$$

and we see that

a) $2L = -\phi^2 - i\phi$ b) $2M = -\phi^2 + i\phi$ c) $N = \phi^2 + I_{2m+2} = A \otimes U + B \otimes V$

satisfies all the conditions as

$$\begin{aligned} 2LP_x =& -\phi^2 P_x - i\phi P_x \\ =& -(i)^2 P_x - i(i) P_x \\ =& \ 2P_x \end{aligned}$$

i.e., we get $LP_x = P_x$. Similarly, other results can be proved.

Thus, we prove that an almost pseudo contact Manifold M_{2m+2}, at each of its point contains a tangent bundle Π_m of dimension m, a tangent bundle $\widetilde{\Pi}_m$, complex conjugate to Π_m and a tangent bundle Π_2 of dimension 2 such that $\Pi_m \cap \widetilde{\Pi}_m = \Pi_m \cap \Pi_2 = \widetilde{\Pi}_m \cap \Pi_2 = \{\Phi\}$ (where $\{\Phi\}$ is the null set) and $\Pi_m \cup \widetilde{\Pi}_m \cup \Pi_2 =$ a tangent bundle of dimension $2m+2$, projection L, M, N on $\Pi_m, \widetilde{\Pi}_m$ and Π_2 respectively being given by

a) $2L = -\phi^2 - i\phi$ b) $2M = -\phi^2 + i\phi$ c) $N = \phi^2 + I_{2m+2} = A \otimes U + B \otimes V$.

Conversely, suppose that, there is a tangent bundle Π_m of dimension m, a tangent bundle $\widetilde{\Pi}_m$, complex conjugate to Π_m and a tangent bundle Π_2 of dimension 2 such that $\Pi_m \cap \widetilde{\Pi}_m = \Pi_m \cap \Pi_2 = \widetilde{\Pi}_m \cap \Pi_2 = \{\Phi\}$ (where $\{\Phi\}$ is the null set) and $\Pi_m \cup \widetilde{\Pi}_m \cup \Pi_2 =$ a tangent bundle of dimension $2m+2$. Let P_x be m linearly independent vectors in Π_m, Q_x be complex conjugate to P_x, be m linearly independent vectors in $\widetilde{\Pi}_m$ and $U,\ V$ be two linearly independent vectors in Π_2. Let, $\{P_x, Q_x, U, V\}$ span a tangent bundle of dimension $2m+2$. Then $\{P_x, Q_x, U, V\}$ is a linearly independent set.

Let us define the inverse set $\{p^{'x}, q^{'x}, A, B\}$ such that

$$I_{2m+2} = p^{'x} \otimes P_x + q^{'x} \otimes Q_x + A \otimes U + B \otimes V. \tag{3.4}$$

We also define

$$\phi = i[p^{'x} \otimes P_x - q^{'x} \otimes Q_x]. \tag{3.5}$$

Therefore

$$\phi^2 = -[p^{'x} \otimes P_x + q^{'x} \otimes Q_x].$$

Thus, by (3.4), we have

$$\phi^2 = -I_{2m+2} + A \otimes U + B \otimes V.$$

So, we see that M_{2m+2} admits an almost pseudo contact structure. Hence the condition is sufficient. □

4. Existence of an almost pseudo contact structure in 4-dimensional Euclidean space

Let R_4 be any 4-dimensional Euclidean space and let us define

$$\phi = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.1}$$

So, we have

$$\phi^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, let us choose $U = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $V = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and correspondingly $A = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$.

Thus

$$A \otimes U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } B \otimes V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore

$$\phi^2 = -I_4 + A \otimes U + B \otimes V. \tag{4.2}$$

Again,

$$\phi(U) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\phi(V) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We also see that U and V are linearly independent. Thus, we conclude that the structure defined by the equation (4.1) is an almost pseudo contact structure and R_4 is an almost pseudo contact manifold.

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