SEVERAL STABILITY PROBLEMS OF A QUADRATIC FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we investigate the stability using shadowing property in Abelian metric group and the generalized Hyers-Ulam-Rassias stability in Banach spaces of a quadratic functional equation,

$$
\begin{aligned}
f(x_1 + x_2 + x_3 + x_4) &+ f(-x_1 + x_2 - x_3 + x_4) \\
&+ f(-x_1 + x_2 + x_3) + f(-x_2 + x_3 + x_4) + f(-x_3 + x_4 + x_1) \\
&+ f(-x_4 + x_1 + x_2) = 5 \sum_{i=1}^{4} f(x_i).
\end{aligned}
$$

Also, we study the stability using the alternative fixed point theory of the functional equation in Banach spaces.

1. Introduction

In 1940, the problem of stability of the above functional equations was originated by Ulam [15] as follows: Under what condition does there exist an additive mapping near an approximately additive mapping?

The first partial solution to Ulam's question was provided by D. H. Hyers [7]. Let X and Y are Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Hyers showed that if a function $f : X \to Y$ satisfies the following inequality

$$
\|f(x+y) - f(x) - f(y)\| \leq \epsilon
$$

for all $\epsilon \geq 0$ and for all $x, y \in X$, then the limit

$$
a(x) = \lim_{n \to \infty} 2^{-n} f(2^n x)
$$

exists for each $x \in X$ and $a : X \to Y$ is the unique additive function such that

$$
\|f(x) - a(x)\| \leq \epsilon
$$

for any $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then a is linear.

Hyers' Theorem was generalized in various directions. In particular, thirty seven years after Hyers' Theorem, Th. M. Rassias provided a generalization of Hyers's result by allowing the Cauchy difference to be unbounded; see [10]. He proved the following theorem: if a function $f : X \to Y$ satisfies the following inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for some $\theta \geq 0$, $0 \leq p < 1$, and for all $x, y \in X$, then there exists a unique additive function such that

$$\|f(x) - a(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then a is linear.

Th. M. Rassias result provided a generalization of Hyers Theorem, a fact which rekindled interest in the study of stability of functional equations. Taking this fact into consideration the Hyers-Ulam stability is called Hyers-Ulam-Rassias stability. In 1990, Th. M. Rassias during the 27th International Symposium on Functional Equations asked the question whether an extension of his theorem can be proved for all positive real numbers p that are greater or equal to one. A year later in 1991, Gajda provided an affirmative solution to Rassias' question in the case the number p is greater than one; see [5].

During the last two decades several results for the Hyers-Ulam-Rassias stability of functional equations have been proved by several mathematicians worldwide in the study of several important functional equations of several variables. Găvruta [6] following Rassias's approach for the unbounded Cauchy difference provided a further generalization.

The quadratic function $f(x) = cx^2 (c \in \mathbb{R})$ satisfies the functional equation

(1.1) $$f(x+y) + f(x-y) = 2f(x) + 2f(y).$$

Hence this question is called the quadratic functional equation, and every solution of the quadratic equation (1.1) is called a quadratic function.

A Hyers-Ulam stability theorem for the quadratic functional equation (1.1) was proved by Skof [12] for functions $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [3], Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equation. Recently, Tabor proved the general stability result for functional equations in the case when the target space is a metric group (with some local divisibility condition); see [13].

Consider the following 3-dimension quadratic functional equation:

$$f(x+y+z) + f(x-y) + f(y-z) + f(x-z) = 3f(x) + 3f(y) + 3f(z).$$

Recently, the 3-dimensional quadratic functional equation was investigated by Bae and Jun [1]. Also, Najati and Park introduced new Euler-Lagrange type of functional equation; see [9].

In this paper, we will investigate the stability in metric group and the generalized Hyers-Ulam stability of a 4-dimensional quadratic functional equation as follows:

$$f(x_1 + x_2 + x_3 + x_4) + f(-x_1 + x_2 - x_3 + x_4) + f(-x_1 + x_2 + x_3)$$
$$+ f(-x_2 + x_3 + x_4) + f(-x_3 + x_4 + x_1) + f(-x_4 + x_1 + x_2)$$
$$= 5\sum_{i=1}^{4} f(x_i).$$

Also, we study the stability using the alternative fixed point of the functional equation in Banach spaces.

Lemma 1.1. *Let X, Y be vector spaces. If a mapping $f : X \to Y$ satisfying*

$$f(x_1 + x_2 + x_3 + x_4) + f(-x_1 + x_2 - x_3 + x_4) + f(-x_1 + x_2 + x_3)$$
$$+ f(-x_2 + x_3 + x_4) + f(-x_3 + x_4 + x_1) + f(-x_4 + x_1 + x_2)$$

(1.2)
$$= 5\sum_{i=1}^{4} f(x_i)$$

for all $x_1, x_2, x_3, x_4 \in X$, then f has the following properties:

(1) $f(0) = 0$.
(2) $f(x) = f(-x)$ *for all $x \in X$.*
(3) f *is a quadratic mapping.*

Proof. (1). Let $x_1 = x_2 = x_3 = x_4 = 0$. Then $6f(0) = 20f(0)$, that is, $f(0) = 0$.
(2). Let $x_1 = x$, and $x_2 = x_3 = x_4 = 0$. By (1), we have $3f(x) + 2f(-x) = 5f(x)$. Hence the desired result is obtained. (3). Let $x_1 = x, x_2 = y$, and $x_3 = x_4 = 0$. Then $2f(x + y) + 2f(x - y) = 4f(x) + 4f(y)$. Thus $f(x + y) + f(x - y) = 2f(x) + 2f(y)$, as desired. $\square$

2. Stability using shadowing property

In this section, we will investigate the stability of the given functional equation based on the ideas from dynamical systems. Before we proceed, we would like to introduce some basic definitions concerning shadowing and key concepts to establish the stability; see [13].

Let us fix some notations which will be used throughout this section. We denote $\mathbb{N}$ the set of all nonnegative integers, X a complete normed space and $B(x, s)$ the closed ball centered at x with radius s and let ϕ be given.

Definition 2.1. Let $\delta \geq 0$. We say that a sequence $(x_k)_{k \in \mathbb{N}}$ is a δ-pseudoorbit (for ϕ) if

$$d(x_{k+1}, \phi(x_k)) \leq \delta \text{ for } k \in \mathbb{N}.$$

A 0-pseudoorbit is called an *orbit*.

Definition 2.2. Let $s, R > 0$ be given. We say that $\phi : X \to X$ is locally (s, R)-invertible at $x_0 \in X$ if

$$\forall y \in B(\phi(x_0), R), \exists! x \in B(x_0, s) : \phi(x) = y.$$

If ϕ is locally (s, R)-invertible at each $x \in X$, then we say that ϕ is locally (s, R)-invertible.

For a locally (s, R)-invertible function ϕ, we define a function $\phi_{x_0}^{-1} : B(\phi(x_0), R) \to B(x_0, s)$ in such a way that $\phi_{x_0}^{-1}(y)$ denote the unique x from the above definition which satisfies $\phi(x) = y$. Moreover, we put

$$\text{lip}_R \phi^{-1} := \sup_{x_0 \in X} \text{lip}(\phi_{x_0}^{-1}),$$

where $\text{lip}(\phi_{x_0}^{-1})$ is the lipschitz constant of $\phi_{x_0}^{-1}$.

Theorem 2.3 ([14]). *Let $l \in (0, 1), R \in (0, \infty)$ be fixed and let $\phi : X \to X$ be locally (lR, R)-invertible. We assume additionally that $\text{lip}_R(\phi^{-1}) \leq l$. Let $\delta \leq (1 - l)R$ and let $(x_k)_{k \in \mathbb{N}}$ be an arbitrary δ-pseudoorbit. Then there exists a unique $y \in X$ such that*

$$d(x_k, \phi^k(y)) \leq lR \text{ for } k \in \mathbb{N}.$$

Moreover,

$$d(x_k, \phi^k(y)) \leq \frac{l\delta}{1 - l} \text{ for } k \in \mathbb{N}.$$

Let $(X, *)$ be a semigroup. We denote kx to be $\underbrace{x * \cdots * x}_{k}$, where $x \in X$ and $k \in \mathbb{N}$. Then the function $|| \cdot || : X \to \mathbb{R}$ is called a (*semigroup*) *norm* if it satisfies the following properties:

  (1) for all $x \in X, ||x|| \geq 0$.
  (2) for all $x \in X, k \in \mathbb{N}, ||kx|| = k||x||$.
  (3) for all $x, y \in X, ||x|| + ||y|| \geq ||x * y||$ and also the equality holds when $x = y$, where $*$ is the binary operation on X.

Note $(X, *, || \cdot ||)$ is called a *normed group* if X is a group with an identity e, and it additionally satisfies that $||x|| = 0$ if and only if $x = e$.

We say that $(X, *, || \cdot ||)$ is a *normed (semi)group* if X is a (semi)group with a norm $|| \cdot ||$. Now, given an Abelian group X and $n \in \mathbb{Z}$, we define the mapping $[n_X] : X \to X$ by the formula

$$[n_X](x) := nx \text{ for } x \in X.$$

Also, we are going to need the following result. In recent years, Lee et al. showed the next lemma by using Theorem 2.3.

Lemma 2.4 ([8]). *Let $l \in (0, 1), R \in (0, \infty), \delta \in (0, (1 - l)R), \varepsilon > 0, m \in \mathbb{N}, n \in \mathbb{Z}$. Let G be a commutative semigroup and X a complete Abelian metric*

group. We assume that the mapping $[n_X]$ is locally (lR, R)-invertible and that $\mathrm{lip}_R([n_X]^{-1}) \leq l$. Let $f : G \to X$ satisfy the following two inequalities

$$\left\| \sum_{i=1}^{N} a_i f(b_{i_1} x_1 + \cdots + b_{i_n} x_n) \right\| \leq \varepsilon \text{ for } x_1, \ldots, x_n \in G,$$

$$\|f(mx) - nf(x)\| \leq \delta \text{ for } x \in G,$$

where all a_i are endomorphisms in X and b_{i_j} are endomorphisms in G. We assume additionally that there exists $K \in \{1, \ldots, N\}$ such that

$$(2.1) \qquad \sum_{i=1}^{K} \mathrm{lip}(a_i)\delta \leq (1-l)R, \varepsilon + \sum_{i=K+1}^{N} \mathrm{lip}(a_i)\frac{l\delta}{1-l} \leq lR.$$

Then there exists a unique mapping $F : G \to X$ such that

$$F(mx) = nF(x) \text{ for } x \in G,$$

and

$$\|f(x) - F(x)\| \leq \frac{l\delta}{1-l} \text{ for } x \in G.$$

Moreover, F satisfies

$$\sum_{i=1}^{N} a_i F(b_{i_1} x_1 + \cdots + b_{i_n} x_n) = 0 \text{ for } x_1, \ldots, x_n \in G.$$

Now, we are ready to prove our functional equation as follows: for the given mapping $f : X \to Y$, we define

$$
\begin{aligned}
Df(x_1, x_2, x_3, x_4) := \ & f(x_1 + x_2 + x_3 + x_4) + f(-x_1 + x_2 - x_3 + x_4) \\
& + f(-x_1 + x_2 + x_3) + f(-x_2 + x_3 + x_4) \\
& + f(-x_3 + x_4 + x_1) + f(-x_4 + x_1 + x_2) \\
& - 5\sum_{i=1}^{4} f(x_i)
\end{aligned}
$$

(2.2)

for all $x_1, x_2, x_3, x_4 \in X$.

Theorem 2.5. *Let $R > 0$, let G be an Abelian group, and let X be a complete normed Abelian group. Let $\varepsilon \leq \frac{7R}{152}$ be arbitrary and let $f : G \to X$ be a mapping such that*

$$(2.3) \qquad \|Df(x_1, \ldots, x_4)\| \leq \varepsilon$$

for all $x_1, \ldots, x_4, x \in G$. Then there exists a unique mapping $F : G \to X$ such that

$$F(4x) = 16F(x),$$

$$F(x_1 + x_2 + x_3 + x_4) + F(-x_1 + x_2 - x_3 + x_4) + F(-x_1 + x_2 + x_3)$$
$$+ F(-x_2 + x_3 + x_4) + F(-x_3 + x_4 + x_1) + F(-x_4 + x_1 + x_2)$$

(2.4)
$$= 5 \sum_{i=1}^{4} F(x_i),$$

$$||F(x) - f(x)|| \leq \frac{1}{14}\varepsilon$$

for all $x_1, \ldots, x_4, x \in G$.

Proof. By letting $x_1 = \cdots = x_4 = 0$ in the equation (2.3), we have

$$||14f(0)|| \leq \varepsilon,$$

that is, $||f(0)|| \leq \frac{\varepsilon}{14}$. Now, by putting $x_1 = \cdots = x_4 = x$ in (2.3),

$$||f(4x) + f(0) - 16f(x)|| \leq \varepsilon.$$

Since $||f(0)|| \leq \frac{\varepsilon}{14}$, we have $||f(4x) - 16f(x)|| \leq \frac{15}{14}\varepsilon$ for all $x \in G$. To apply Lemma 2.4 for the function f, we may let

$$l = \frac{1}{16}, \delta = \frac{15}{14}\varepsilon,$$
$$a_1 = \cdots = a_6 = id_X, a_7 = \cdots = a_{11} = -5id_X,$$
$$K = 6, \text{ and } N = 11.$$

Then we have

(2.5) $$\delta \leq (1-l)R, \quad \sum_{i=1}^{K} \text{lip}(a_i)\delta \leq (1-l)R, \quad \varepsilon + \sum_{i=K+1}^{N} \text{lip}(a_i)\frac{l\delta}{1-l} \leq lR.$$

Hence all conditions of Lemma 2.4 are satisfied, and thus we conclude that there exists a unique mapping $F : G \to X$ such that

$$F(4x) = 16F(x),$$

$$F(x_1 + x_2 + x_3 + x_4) + F(-x_1 + x_2 - x_3 + x_4) + F(-x_1 + x_2 + x_3)$$
$$+ F(-x_2 + x_3 + x_4) + F(-x_3 + x_4 + x_1) + F(-x_4 + x_1 + x_2)$$
$$= 5 \sum_{i=1}^{4} F(x_i),$$

and also we have

$$||F(x) - f(x)|| \leq \frac{l\delta}{1-l} = \frac{1}{14}\varepsilon \text{ for all } x_1, \ldots, x_4, x \in G.$$ □

Theorem 2.6. *Let $R > 0$, let G be an Abelian group, let X be a complete normed Abelian group, and let $f : G \to X$ be a mapping. Suppose that $[14_X]$ is*

locally $\left(\frac{R}{14}, R\right)$*-invertible and* $[2_X]$ *is locally* $\left(\frac{R}{2}, R\right)$*-invertible. If* f *satisfies the following equation*

$$f(x_1 + x_2 + x_3 + x_4) + f(-x_1 + x_2 - x_3 + x_4) + f(-x_1 + x_2 + x_3)$$

$$+ f(-x_2 + x_3 + x_4) + f(-x_3 + x_4 + x_1) + f(-x_4 + x_1 + x_2)$$

(2.6)
$$= 5 \sum_{i=1}^{4} f(x_i)$$

for all $x_1, \ldots, x_4 \in G$*, then* f *is a quadratic even mapping.*

Proof. By letting $x_1 = \cdots = x_4 = 0$ in the equation (2.6), we have

$$14f(0) = 0.$$

By the uniqueness of the local division by 14, we get $f(0) = 0$. Also, setting $x_1 = x, x_k = 0$ $(k = 2, \ldots, 4)$ and by the uniqueness of the local division by 2, we have $f(x) = f(-x)$ for all $x \in G$, that is, f is even. By the uniqueness of the local division by 2 and letting $x_1 = x, x_2 = y, x_3 = x_4 = 0$, we have

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

for all $x, y \in G$, that is, f is a quadratic mapping, as desired. $\square$

The direct application of Theorems 2.5 and 2.6 yields the following corollary.

Corollary 2.7. *Let* $R > 0$*, let* G *be an Abelian group, and let* X *be a complete normed Abelian group. Let* $\varepsilon \leq \frac{7R}{152}$ *be arbitrary and let* $f : G \to X$ *be a function satisfying equation (2.3). Suppose that* $[14_X]$ *is locally* $\left(\frac{R}{14}, R\right)$*-invertible and* $[2_X]$ *is locally* $\left(\frac{R}{2}, R\right)$*-invertible. Then there exists a quadratic even mapping* $F : G \to X$ *such that*

$$\|F(x) - f(x)\| \leq \frac{1}{14}\varepsilon.$$

3. Hyers-Ulam-Rassias stability

Throughout in this section, let X be a normed vector space with norm $\|\cdot\|$ and Y be a Banach space with norm $\|\cdot\|$.

Theorem 3.1. *Let* $f : X \to Y$ *be an even mapping satisfying* $f(0) = 0$ *for which there exists a function* $\phi : X^4 \to [0, \infty)$ *such that*

$$\widetilde{\phi}(x_1, x_2, x_3, x_4) := \sum_{j=0}^{\infty} 4^{-j}\phi(2^j x_1, 2^j x_2, 2^j x_3, 2^j x_4) < \infty,$$

(3.1)
$$\|Df(x_1, x_2, x_3, x_4)\| \leq \phi(x_1, x_2, x_3, x_4)$$

for all $x_1, x_2, x_3, x_4 \in X$*. Then there exists a unique 4-dimensional quadratic mapping* $Q : X \to Y$ *such that*

(3.2)
$$\|f(x) - Q(x)\| \leq \frac{1}{8}\widetilde{\phi}(x, x, 0, 0)$$

for all $x \in X$*.*

Proof. Letting $x_1 = x_2 = x$ and $x_3 = x_4 = 0$ in the equation (3.2), we have

$$\|2f(2x) - 8f(x)\| \le \phi(x, x, 0, 0)$$

for all $x \in X$. Then we write

(3.3) $$\left\| f(x) - \frac{1}{4}f(2x) \right\| \le \frac{1}{8}\phi(x, x, 0, 0)$$

for all $x \in X$.

Then

$$\left\| \left(\frac{1}{4}\right)^d f\left(2^d x\right) - \left(\frac{1}{4}\right)^{d+1} f\left(2^{d+1}x\right) \right\| = \left(\frac{1}{4}\right)^d \left\| f\left(2^d x\right) - \frac{1}{4} f\left(2^{d+1}x\right) \right\|$$

$$\le \frac{1}{8}\left(\frac{1}{4}\right)^d \phi\left(2^d x, 2^d x, 0, 0\right)$$

for all $x \in X$ and all positive integer d. Hence we have

(3.4) $$\left\| \left(\frac{1}{4}\right)^s f\left(2^s x\right) - \left(\frac{1}{4}\right)^d f(2^d x) \right\| \le \frac{1}{8}\sum_{j=s}^{d-1}\left(\frac{1}{4}\right)^j \phi(2^j x, 2^j x, 0, 0)$$

for all $x \in X$ and all positive integers s, d with $s < d$.

Hence we may conclude that the sequence $\{\left(\frac{1}{4}\right)^s f\left(2^s x\right)\}$ is a Cauchy sequence. Since Y is complete, the sequence $\{\left(\frac{1}{4}\right)^s f\left(2^s x\right)\}$ converges in Y for all $x \in X$. Thus we may define a mapping $Q : X \to Y$ via

$$Q(x) = \lim_{s \to \infty} \left(\frac{1}{4}\right)^s f\left(2^s x\right)$$

for all $x \in X$. Then

$$\|DQ(x_1, x_2, x_3, x_4)\| = \lim_{s \to \infty} \left(\frac{1}{4}\right)^s \|Df\left(2^s x_1, 2^s x_2, 2^s x_3, 2^s x_4\right)\|$$

$$\le \lim_{s \to \infty} \left(\frac{1}{4}\right)^s \phi\left(2^s x_1, 2^s x_2, 2^s x_3, 2^s x_4\right)$$

$$= 0$$

for all $x_1, \ldots, x_4 \in X$. Lemma 1.1 induces that Q is a quadratic mapping. Also, by letting $s = 0$, and $d \to \infty$ in the equation (3.4), we have the equation (3.2).

Now, let $Q' : X \to Y$ be another quadratic mapping satisfying the equation (3.2). Then for all $x \in X$

$$\|Q(x) - Q'(x)\| = \left(\frac{1}{4}\right)^s \|Q\left(2^s x\right) - Q'\left(2^s x\right)\|$$

$$\le \left(\frac{1}{4}\right)^s \left(\|Q\left(2^s x\right) - f\left(2^s x\right)\| + \|Q'\left(2^s x\right) - f\left(2^s x\right)\|\right)$$

$$\le \frac{2 \cdot 4^{-s}}{8} \cdot \widetilde{\phi}(2^s x, 2^s x, 0, 0) \to 0$$

as $s \to \infty$. Thus we may conclude that such a quadratic mapping Q is unique.
$\square$

Theorem 3.2. *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^4 \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, x_2, x_3, x_4) := \sum_{j=1}^{\infty} 4^j \phi(2^{-j} x_1, 2^{-j} x_2, 2^{-j} x_3, 2^{-j} x_4) < \infty,$$

$$\|Df(x_1, x_2, x_3, x_4)\| \leq \phi(x_1, x_2, x_3, x_4)$$

for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique 4-dimensional quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{2} \widetilde{\phi}(x, x, 0, 0)$$

for all $x \in X$.

Proof. In the proof of Theorem 3.1, if x is inductively replaced by $\frac{1}{2} x$, then we have

$$\left\| f(x) - 4^r f(2^{-r} x) \right\| \leq \frac{1}{2} \sum_{j=1}^{r} 4^j \phi(2^{-j} x, 2^{-j} x, 0, 0)$$

for all $x \in X$. The remains follow from Therorem 3.1. $\square$

Corollary 3.3. *Let $p \neq 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and*

$$\|Df(x_1, x_2, x_3, x_4)\| \leq \theta \sum_{i=1}^{4} \|x_i\|^p$$

for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique 4-dimensional quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{\theta}{|4 - 2^p|} \|x\|^p$$

for all $x \in X$.

Proof. Let

$$\phi(x_1, x_2, x_3, x_4) = \theta \sum_{i=1}^{4} \|x_i\|^p .$$

Let $p < 2$. Applying to Theorem 3.1, we have the desired result. Now, let $p > 2$, similar to the previous case applying to Theorem 3.2. $\square$

Theorem 3.4. *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^4 \to [0, \infty)$ such that*

$$(3.5) \qquad \widetilde{\phi}(x_1, x_2, x_3, x_4) := \sum_{j=0}^{\infty} 4^{-2j} \phi(4^j x_1, 4^j x_2, 4^j x_3, 4^j x_4) < \infty,$$

(3.6) $\|Df(x_1, x_2, x_3, x_4)\| \leq \phi(x_1, x_2, x_3, x_4)$

for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique 4-dimensional quadratic mapping $Q : X \to Y$ such that

(3.7) $\|f(x) - Q(x)\| \leq \dfrac{1}{16}\widetilde{\phi}(x, x, x, x)$

for all $x \in X$.

Proof. Letting $x_1 = x_2 = x_3 = x_4 = x$ in the equation (3.6), since $f(0) = 0$, we have

$$\|f(4x) - 16f(x)\| < \phi(x, x, x, x)$$

for all $x \in X$. Hence we have

(3.8) $\left\| f(x) - \dfrac{1}{16}f(4x) \right\| \leq \dfrac{1}{16}\phi(x, x, x, x)$

for all $x \in X$. Then

$$\left\| \left(\frac{1}{4}\right)^{2d} f\left(4^d x\right) - \left(\frac{1}{4}\right)^{2(d+1)} f\left(4^{d+1} x\right) \right\| = \left(\frac{1}{4}\right)^{2d} \left\| f\left(4^d x\right) - \frac{1}{16}f\left(4^{d+1} x\right) \right\|$$

$$\leq \frac{1}{16}\left(\frac{1}{4}\right)^{2d} \phi\left(4^d x, 4^d x, 4^d x, 4^d x\right)$$

for all $x \in X$ and all positive integer d. Hence we have

(3.9) $\left\| \left(\frac{1}{4}\right)^{s} f\left(4^s x\right) - \left(\frac{1}{4}\right)^{d} f\left(4^d x\right) \right\| \leq \dfrac{1}{16} \displaystyle\sum_{j=s}^{d-1} \left(\frac{1}{4}\right)^{2j} \phi\left(4^j x, 4^j x, 4^j x, 4^j x\right)$

for all $x \in X$ and all positive integers s, d with $s < d$.

Hence we may conclude that the sequence $\{\left(\frac{1}{4}\right)^{2s} f(4^s x)\}$ is a Cauchy sequence. Since Y is complete, the sequence $\{\left(\frac{1}{4}\right)^{2s} f(4^s x)\}$ converges in Y for all $x \in X$. Thus we may define a mapping $Q : X \to Y$ via

$$Q(x) = \lim_{s \to \infty} \left(\frac{1}{4}\right)^{2s} f(4^s x)$$

for all $x \in X$. Then

$$\|DQ(x_1, x_2, x_3, x_4)\| = \lim_{s \to \infty} \left(\frac{1}{4}\right)^{2s} \|Df\left(4^s x_1, 4^s x_2, 4^s x_3, 4^s x_4\right)\|$$

$$\leq \lim_{s \to \infty} \left(\frac{1}{4}\right)^{2s} \phi\left(4^s x_1, 4^s x_2, 4^s x_3, 4^s x_4\right)$$

$$= 0$$

for all $x_1, \ldots, x_4 \in X$. Lemma 1.1 induces that Q is a quadratic mapping. Also, by letting $s = 0$, and $d \to \infty$ in the equation (3.9), we have the equation (3.7).

Now, let $Q' : X \to Y$ be another quadratic mapping satisfying the equation (3.7). Then for all $x \in X$

$$
\begin{aligned}
\|Q(x) - Q'(x)\| &= \left(\frac{1}{4}\right)^{2s} \|Q\left(4^s x\right) - Q'\left(4^s x\right)\| \\
&\leq \left(\frac{1}{4}\right)^{2s} \left(\|Q\left(4^s x\right) - f\left(4^s x\right)\| + \|Q'\left(4^s x\right) - f\left(4^s x\right)\|\right) \\
&\leq \frac{2 \cdot 4^{-2s}}{16} \cdot \widetilde{\phi}\left(4^s x, 4^s x, 4^s x, 4^s x\right) \to 0
\end{aligned}
$$

as $s \to \infty$. Thus we may conclude that such a quadratic mapping Q is unique. $\qquad\square$

Theorem 3.5. *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^4 \to [0, \infty)$ such that*

$$
\widetilde{\phi}(x_1, x_2, x_3, x_4) := \sum_{j=1}^{\infty} 4^{2j} \phi(4^{-j} x_1, 4^{-j} x_2, 4^{-j} x_3, 4^{-j} x_4) < \infty,
$$

$$
\|Df(x_1, x_2, x_3, x_4)\| \leq \phi(x_1, x_2, x_3, x_4)
$$

for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique 4-dimensional quadratic mapping $Q : X \to Y$ such that

$$
\|f(x) - Q(x)\| \leq \widetilde{\phi}(x, x, x, x)
$$

for all $x \in X$.

Proof. In the proof of Theorem 3.4, if x is inductively replaced by $\frac{1}{4}x$, then we have

$$
\left\|f(x) - 4^{2r} f(4^{-r} x)\right\| \leq \sum_{j=1}^{r-1} 4^{2j} \phi(4^{-j} x, 4^{-j} x, 4^{-j} x, 4^{-j} x)
$$

for all $x \in X$. Similar to Theorem 3.4, the proof follows. $\qquad\square$

Corollary 3.6. *Let $p \neq 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and*

$$
\|Df(x_1, x_2, x_3, x_4)\| \leq \theta \sum_{i=1}^{4} \|x_i\|^p
$$

for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique 4-dimensional quadratic mapping $Q : X \to Y$ such that

$$
\|f(x) - Q(x)\| \leq \frac{4\theta}{|16 - 4^p|} \|x\|^p
$$

for all $x \in X$.

Proof. Let

$$\phi(x_1, x_2, x_3, x_4) = \theta \sum_{i=1}^{4} \|x_i\|^p.$$

It is easy to check that when $p < 2$, apply to Theorem 3.4 or when $p > 2$, apply to Theorem 3.5 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Stability using alternative fixed point

In this section, we will investigate the stability of the given 4-dimensional quadratic functional equation (2.2) using the alternative fixed point. Before proceeding the proof, we will state theorem, the alternative of fixed point.

Theorem 4.1 (The alternative of fixed point [4], [11]). *Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty \text{ for all } n \geq 0,$$

or there exists a natural number n_0 such that

(1) *$d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;*
(2) *The sequence $(T^n x)$ is convergent to a fixed point y^* of T;*
(3) *y^* is the unique fixed point of T in the set*

$$\triangle = \{y \in \Omega \mid d(T^{n_0} x, y) < \infty\};$$

(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \triangle$.*

Now, let $\phi : X^n \to [0, \infty)$ be a function such that

$$\lim_{m \to \infty} \frac{\phi(\lambda_i^m x_1, \ldots, \lambda_i^m x_n)}{\lambda_i^{2m}} = 0$$

for all $x_1, \ldots, x_n \in X$, where $\lambda_i = 2$ if $i = 0$ and $\lambda_i = \frac{1}{2}$ if $i = 1$.

Theorem 4.2. *Suppose that an even mapping $f : X \to Y$ satisfies the functional inequality*

(4.1) $$\|Df(x_1, \ldots, x_4)\| \leq \phi(x_1, \ldots, x_4)$$

for all $x_1, \ldots, x_4 \in X$ and $f(0) = 0$. If there exists $L = L(i) < 1$ such that the function

(4.2) $$x \mapsto \psi(x) = \phi(x, x, 0, 0)$$

has the property

(4.3) $$\psi(x) \leq L \cdot \lambda_i^2 \cdot \psi\left(\frac{x}{\lambda_i}\right)$$

for all $x \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ such that the inequality

(4.4) $$\|f(x) - Q(x)\| \leq \frac{L^{1-i}}{1-L} \psi(x)$$

holds for all $x \in X$.

Proof. Consider the set

$$\Omega = \{g \mid g : X \to Y, g(0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(g, h) = d_\psi(g, h) = \inf\{K \in (0, \infty) | \, \|g(x) - h(x)\| \le K\psi(x), x \in X\}.$$

It is easy to show that (Ω, d) is complete. Now we define a function $T : \Omega \to \Omega$ by

$$Tg(x) = \frac{1}{\lambda_i^2} g(\lambda_i x)$$

for all $x \in X$. Note that for all $g, h \in \Omega$,

$$
\begin{aligned}
d(g, h) < K \quad &\Rightarrow \quad \|g(x) - h(x)\| \le K\psi(x) \text{ for all } x \in X, \\
&\Rightarrow \quad \left\| \frac{1}{\lambda_i^2} g(\lambda_i x) - \frac{1}{\lambda_i^2} h(\lambda_i x) \right\| \le \frac{1}{\lambda_i^2} K\psi(\lambda_i x) \text{ for all } x \in X, \\
&\Rightarrow \quad \left\| \frac{1}{\lambda_i^2} g(\lambda_i x) - \frac{1}{\lambda_i^2} h(\lambda_i x) \right\| \le LK\psi(x) \text{ for all } x \in X, \\
&\Rightarrow \quad d(Tg, Th) \le LK.
\end{aligned}
$$

Hence we have that

$$d(Tg, Th) \le Ld(g, h)$$

for all $g, h \in \Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant L. By setting $x_1 = x_2 = x$ and $x_3 = x_4 = 0$, we have the equation (3.3) as in the proof of Theorem 3.1 and we use the equation (4.3) with the case where $i = 0$, which is reduced to

$$(4.5) \qquad \left\| f(x) - \frac{1}{4} f(2x) \right\| \le \frac{1}{4}\psi(2x) \le L\psi(x)$$

for all $x \in X$, that is, $d(f, Tf) \le L = L^1 < \infty$. Now, replacing x by $\frac{1}{2}x$ in the equation (4.5), multiplying 4, and using the equation (4.3) with the case where $i = 1$, we have that

$$\left\| f(x) - 2^2 f\left(\frac{x}{2}\right) \right\| \le \psi(x)$$

for all $x \in X$, that is, $d(f, Tf) \le 1 = L^0 < \infty$. In both cases we can apply the fixed point alternative and since $\lim_{r \to \infty} d(T^r f, Q) = 0$, there exists a fixed point Q of T in Ω such that

$$(4.6) \qquad Q(x) = \lim_{n \to \infty} \frac{f(\lambda_i^r x)}{\lambda_i^{2r}}$$

for all $x \in X$. Letting $x_j = \lambda_i^r x_j$ for $j = 1, \ldots, 4$ in the equation (4.1) and dividing by λ_i^{2r},

$$
\begin{aligned}
\|DQ(x, \ldots, x_n)\| &= \lim_{r \to \infty} \frac{\|Df(\lambda_i^r x_1, \ldots, \lambda_i^r x_4)\|}{\lambda_i^{2r} x_1} \\
&\leq \lim_{r \to \infty} \frac{\|\phi(\lambda_i^r x_1, \ldots, \lambda_i^r x_4)\|}{\lambda_i^{2r} x_1} \\
&= 0
\end{aligned}
$$

for all $x_1, \ldots, x_4 \in X$; that is, it satisfies the equation (1.2). By Lemma 1.1, the Q is quadratic. Also, the fixed point alternative guarantees that such a Q is the unique mapping such that

$$
\|f(x) - Q(x)\| \leq K\psi(x)
$$

for all $x \in X$ and some $K > 0$. Again using the fixed point alternative, we have

$$
d(f, Q) \leq \frac{1}{1 - L} d(f, Tf).
$$

Hence we may conclude that

$$
d(f, Q) \leq \frac{L^{1-i}}{1 - L},
$$

which implies the equation (4.4). □

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