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Mathematical Thinking and Developing Mathematical Structure[1]

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The mathematical thinking which transforms important mathematical content and developed into mathematical structure is a vital process in building up mathematical ability as mathematical knowledge based on structure. Such process based on students' recognition of mathematical concept. Developing mathematical thinking into mathematical structure happens when different cognitive units are connected and compressed to form schema of solution, which could happen through some guided problems. The effort of arithmetic approach in problem solving did not necessarily provide students the structure schema of solution. The using of equation to solve the problem is based on the schema of building equation, and is not necessary recognizing the structure of the solution, as the recognition of structure may be lost in the process of simplification of algebraic expressions, leaving only the final numeric answer of the problem.

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MESC Classification: C32, C52, F32
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INTRODUCTION

English (1998) studied how students used analogy in mathematical thinking by comparing problems and their solutions. Alexander, White, & Daugherty (1997) proposed that early mathematics learning is based on analogical reasoning, and such reasoning is through comparing problem and their mathematical structure. Barnard and Tall (1997)

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and Dubinsky (1991) had proposed the concepts of cognitive units and compression on mathematics thinking. Two factors are important in building a powerful thinking structure. These are 1) the ability to compress information to fits into cognitive units, and 2) the ability to make connections between cognitive units and such relevant information.

Vergnaud (1983) proposed that multiplicative structure is the basic foundation of proportion. And such proportional knowledge helps to develop mathematics structure and related knowledge in mathematics. The development of mathematical thinking is based on these multiplicative fields. Krutetskii (1976) proposed that there are three kinds of students thinking; analytic thinker, geometric thinker, or harmonic thinker (using both analytic and geometric thinking). In this paper, it is found that students develop their mathematical thinking through manipulating the numbers (arithmetic approach), forming a mathematical expression (structure) and some through abstracting into linear equation (algebraic approach). And using problem construction can help both groups of students to develop their thinking into structure.

This paper discusses how students develop their mathematical thinking in solving problems into a suitable mathematical structure. The discussion is through working an old Chinese problem (Guo & Liu, 2001) and an extended problem. The original version of the problem appeared in ancient Chinese mathematics problem book (Guo & Liu, 2001). The task stated: "There was a woman doing dish washing at the river. An officer asked why there were so many dishes. The woman answered there was a gathering. The officer asked how many guests were there. The woman replied: two persons shared a dish of rice, three persons shared a dish of soup, and four persons shared a dish of meat, and there are a total of 65 dishes, but I do not know the number of guests." The text then continued that the answer is 60 persons and the method for calculation is multiply 65 with 12, which gives the number 780. Divide 780 by 13 and you get the number of guests is 60. However, the text did not explain why the method is correct or how it worked. In fact, the answer is

$$65\div\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=65\div\left(\frac{6+4+3}{12}\right)=65\times\frac{12}{13}=60.$$

The text skipped the record of the thinking process and left only with the expression $65\times12\div13=60$.

The question has certain importance in the course of mathematical thinking. First, a schema of solution can be obtained through exploration of solving this problem by trial and error; then students can relate and compress the process and the cognitive units to obtain an arithmetic structure. Also, by using algebra, students not only can solve this problem, but also they can relate the structure of the solution with the answer. If we look at the solution

$$65 \div \left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)$$

is actually the number of dishes divided by number of dishes consumed by one person.

Hence the understanding of the schema "(Number of persons) = (Number of boxes of food) ÷ (number of boxes per person)", should enable students to obtain the answer. And from this schema, students deduce the structure expression through abstraction.

THE RESEARCH QUESTION

The paper will investigate how a class of primary 5 students formulate their mathematical thinking through exploration of solving the problem; develop their thinking into solution with mathematical structure. The structure will be extended and used to solve a more generalized problem (Final Task 1 and Final Task 2).

The sequence of lessons focuses on the development of the thinking process and the structure correspondence; this involved discussion and exploration of the solutions of sets of related and simplified problems. The purpose is to allow students to establish a schema of solution of the problem through connection of these cognitive units. The schema is then transformed to a structure through abstraction of the relation of numeric values; this involved a structure of division by a sum of 2 unit-fractions and then extended to division by a sum of 3 unit-fractions in Final Task 1. The discussion is then moved to a new structure, which involved division of sum of 3 non-unit fractions (in Final Task 2).

The two tasks used in the study are as follow:

Final Tasks	Solution Structure
Final Task 1: There are 65 boxes of food, and 2 persons shared a box of vegetables and 3 persons shared a box of fish, and 4 persons shared a box of chicken. Find the number of persons.	$65 \div \left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) = 60$ {3 unit-fractions}
Final Task 2 (Extension of Final Task 1): There are 46 boxes of food, and 2 persons shared a box of vegetables and 3 persons shared 2 boxes of fish, and 4 persons shared 3 boxes of chicken. Find the number of persons.	$46 \div \left(\frac{1}{2}+\frac{2}{3}+\frac{3}{4}\right) = 24$ {3 non unit-fractions}

The schema of the solution of both tasks is "(Number of persons) = (Number of boxes of food) ÷ (number of boxes per person)", and the closed form of solution structure for Final Task 1 is

$$65 \div \left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=60,$$

And, that for Final Task 2 is

$$46 \div \left(\frac{1}{2}+\frac{2}{3}+\frac{3}{4}\right)=24.$$

Two sets of questions: "simplified tasks (A)" and "intermediate tasks (B)" are used. The sets of simplified and intermediate tasks aimed as hints, students are expected to transfer the solution structure and knowledge. Finally, the solution structure of the intermediate question is extended and generalized to solve the Final Task 1. The solving of the Final Task 1 and 2 also included using backward thinking process, so that students learn the structure by constructing their own questions.

1. INVESTIGATION OF MATHEMATICAL THINKING INTO STRUCTURE FOR FINAL TASK 1

The following is the investigation of mathematical thinking into structure through task 1.

The process of solving the Task 1				
Set of Simplified Tasks A	⇨	Set of Intermediate Tasks B	⇨	Final Task 1
There are 10 boxes of food, and each person has 2 boxes of vegetables and 3 boxes of fish. Find the number of persons.		There are 10 boxes of food, and 2 persons shared a box of vegetables and every 3 persons shared a box of fish. Find the number of persons.		There are 65 boxes of food, and 2 persons shared a box of vegetables and 3 persons shared a box of fish, and 4 persons shared a box of chicken. Find the number of persons.
$10 \div (2+3)=2$		$10 \div \left(\frac{1}{2}+\frac{1}{3}\right)=12$		$65 \div \left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=60$

Process of mathematical thinking – connection, transformation and compression

The first stage is to develop mathematical thinking through solving a set of simplified tasks, so that mapping of structure can occurred, where the division by "sum of integers" is replaced by division of "sum of integer and simple decimal." For example, in the following 3 related simplified tasks, the structure "10÷(2+3)" is transferred to the structure "5÷(2+0.5)" or "5÷(1+0.25)", so that the change from integers in the divisor to a decimal makes sense for the students in mapping the structure.

For the set of simplified tasks, students can devise the schema "(number of persons) = (number of boxes of food) ÷(number of boxes per person)" and represent it by a structure. However, students may not be successful in transferring the structure of the questions A1 and A2 to Simplified Task A. Some students are lost between the integers and fractions even though there exists a correspondence. Students found it tricky to change some of the number in the divisor into a decimal. For example, students can arrive at "10÷(2+3) = 5" easily but are not too sure if "5÷(2+0.5)" really expresses the answer. The "0.5" appeared in the expression gives students difficult times in the calculation, though the answer is still an integer "5÷(2+0.5) = 2". Only by doing a few problems of the similar format, do they arrive at the conclusion that the expression is a correct answer and the solution structure is the same?

<table>
<tr><td colspan="2">Set of Simplified Task A
There are 10 boxes of food, and each person has 2 box of vegetables and 3 boxes of fish. Find the number of persons.
Answer: 10÷(2+3) = 2.</td></tr>
<tr><td colspan="2" align="center">Structure
⇩</td></tr>
<tr><td>Simplified Task A1:
There are 5 boxes of food, and each person has 2 boxes of vegetables and half a box of fish. Find the number of persons.</td><td>Simplified Task A2:
There are 5 boxes of food, and each person has 1 box of vegetables and a quarter of a box of fish. Find the number of persons.</td></tr>
<tr><td>Answer: 5÷(2+0.5) = 2 (persons)</td><td>Answer: 5÷(1+0.25) = 4 (persons)</td></tr>
</table>

2. MATHEMATICAL THINKING THROUGH WORKING ON INTERMEDIATE TASKS

After solving simplified tasks, students worked on the intermediate tasks. The tasks are a set of questions in which the numeral quantity in the questions is related in the same structure and appeared as a certain kind of progression and ratio. Students can solve the problem through numeral pattern recognition of these numeric values. Through the solving of the simplest case of the question and deducing the solution of the related question, students can consolidate the knowledge of the structure of the question. The sets of questions allow students to perform structure mapping from the simplified tasks to the intermediate ones.

In the course of investigation, the numeric values appeared in the intermediate task and the related tasks are changed from 10 to 5 and 15, so that through simplified cases, students can obtain a schema of solution by using analogy and compression.

Intermediate Task B: There are 10 boxes of food, and 2 persons shared a box of vegetables and 3 persons shared a box of fish. Find the number of persons. Answer: $10 \div (\frac{1}{2}+\frac{1}{3})$ Structure: $N \div (\frac{1}{a}+\frac{1}{b})$	⇔	**Related Task B1:** There are 5 boxes of food, and 2 persons shared a box of vegetables and 3 persons shared a box of fish.
		Related Task B2: There are 15 boxes of food, and 2 persons shared a box of vegetables and 3 persons shared a box of fish.

The set of intermediate tasks involves the structure

$$\text{“}N\div(\frac{1}{a}+\frac{1}{b})\text{”},$$

which involves two unit fraction

$$\frac{1}{2} \text{ and } \frac{1}{3}.$$

Not many students recognize the structure at once. However, the use of language in questions can hints students on the values of the fractions. In the question, for example, if the phrase "2 persons shared a box of vegetables and every 3 persons shared a box of fish" is replaced by "each persons has half a box of vegetables and 1/3 box of fish", more students are convinced that the expression

$$\text{“}10\div(\frac{1}{2}+\frac{1}{3})\text{”}$$

is the structure they are looking for. At this stage, the discussion on the effect of the size of the fraction and its relation to the number of persons involved helps to conceptualize the schema.

2A. Answer developed in Intermediate tasks by using Trial and Error

Trial and error 1

Students know that the number of persons should be a multiple of 2 and 3, which is 6.

So they tried the number 6 persons. However, 6 persons only come up with 5 boxes and they multiple everything together to get 2×2×3 = 12. Which happen 12 is the number of persons.

After checking, they get the answer 12.

$$2 \times \boxed{6} = \boxed{12} = 3 \times \boxed{4}$$

Trial and error 2

Some students make use of the results of multiplication and least common multiple. The students know that the number of persons must be a multiple of 2 and 3. So they try the number 6. By "6 ÷2 = 3 and 6÷3 = 2", they get "2+3 = 5", which they thought is the number of boxes of food. But 5 boxes of food did not comply with the conditions of 10 boxes of food in the task. So students know that the answer is a multiple of 5 and obtain the answer 6×2 = 12.

Trial and error 3

Some students divide the number of boxes by the sum of the group in people. That is 10 ÷(2+3) and get 2, but 2 persons obviously is not the answer and they look for the relation of the number 2 and the problem. Finally they agreed that "2" means "groups" and come up with 5 boxes of food and 6 persons. And using ratio, 10 boxes of food corresponds to 12 persons.

Trial and error 4

Some students try to find two numbers which, when they multiply with 3 and 2, will give the same product. Used their own representation as follow, the common number "12" is the answer. Students used the following format.

$$\begin{array}{cccc} 2 & \times & \boxed{} & = \boxed{} \\ & & + & \\ 3 & \times & \boxed{} & = \boxed{} \\ & & = & \\ & & 10 & \end{array} \qquad \begin{array}{cccc} 2 & \times & \boxed{6} & = \boxed{12} \\ & & + & \\ 3 & \times & \boxed{4} & = \boxed{12} \\ & & = & \\ & & 10 & \end{array}$$

Trial and error 5

Some students try to play around the fractions

$$\frac{1}{2}$$

(refers to 2 persons) and

$$\frac{1}{3}$$

(refers to 3 persons), and obtained

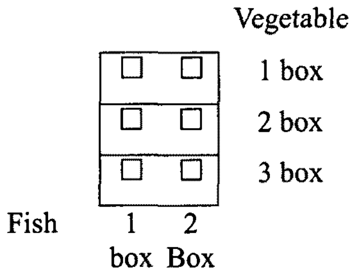
$$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}.$$

They looked at the relation of the three numbers "5, 6 and 10" and then used ratio to verify the answer

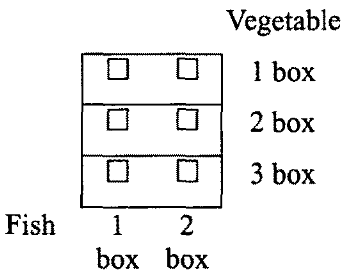
$$12.\ (10\times\frac{6}{5}=12)$$

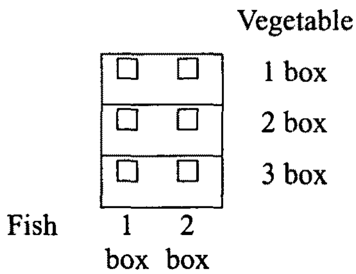
Trial and error 6

Students use the following diagram to find out the relation of the conditions, where a square represents a person.



From the table they draw, they observe that there are 6 persons. And if they repeat the same diagram, they got the following results.





However, all the process and effort during trial and error process did not come out with the structure of the question

$$\text{“}10\div(\frac{1}{2}+\frac{1}{3})\ =\text{number of person”.}$$

2B. Guided Arithmetic approach

As round up exercise, students are given the chance to share what they have explored and connect possible link of cognitive units. This will develop into a schema and such schema can also be introduced in the following procedure through guided arithmetic approach. Three tasks are given so that progressively, students can solve the tasks and recognize the same structure through the recognition of number pattern.

Using progressive number of fractions in Guided Tasks	Schema and structure
Guided Task 1: There are 3 boxes of food, and each person shared a box of vegetables and 2 persons shared a box of fish.	$3 \div (1+0.5) = 6$ ⇩
Guided Task 2 There are 8 boxes of food, and each person shared a box of vegetables and 3 persons shared a box of fish.	$8 \div (1+\frac{1}{3}) = 6$ ⇩
Guided Task 3: There are 10 boxes of food, and 2 persons shared a box of vegetables and 3 persons shared a box of fish.	$10 \div (\frac{1}{2}+\frac{1}{3}) = 12$

This is the schema of "(number of persons) = (number of boxes of food) ÷(number of boxes each persons get)" together with the structure of

$$\text{"number of boxes} \div (\frac{1}{a}+\frac{1}{b}) = \text{number of persons".}$$

Students are given two unit fractions

$$(\frac{1}{a},\frac{1}{b}), \text{ such as } \frac{1}{4} \text{ and } \frac{1}{5}$$

and use them to construct a new problem of the same format.

For example,

$$\text{using "}\frac{1}{4}+\frac{1}{5} = \frac{9}{20}\text{",}$$

students finally proposed a question and a statement of solution: "there are 9 boxes of food, and 4 share a box of one kind, and 5 share a box of another kind, the number of persons in the group is

$$9 \div (\frac{1}{4}+\frac{1}{5}) = 20\text{".}$$

2C. Using Equation to solve Intermediate Tasks

Some students set up a linear equation, supposing x be the number of boxes of fish and $15-x$ be the number of vegetable. This give them the equation $3(15-x) = 2x$.

Solving the equation, the students get $x = 3$ and hence the total number of person is $3x = 6$.

However, some students use x equal to number of boxes of fish and y be the number of boxes of vegetables and end up with equation $x + y = 15$. But then, they are not able to get the equation $3y = 2x$, and could not obtain any further results.

Obviously, students used two equations are not able to compress the knowledge into a cognitive unit. In this case, is the compression of two equations into one? This hindered their successful rate of getting the answer.

Other students let k be the number of persons, and obtain the equation

$$\frac{k}{2}+\frac{k}{3} = 5,$$

and simplify to get

$$\frac{3k+2k}{6} = 5,$$

hence $5k = 30$ and $k = 6$.

The method of using only one variable in the equation is more successful in getting the correct answer; however, the calculation involved also simplified the process of finding the solution and also the structure. Only when algebraic transformation is performed in the original expression

$$\text{“}\frac{k}{2}+\frac{k}{3} = 5 \Rightarrow k\left(\frac{1}{2}+\frac{1}{3}\right) = 5\text{”},$$

the structure expression

$$\text{“}k = 5\div\left(\frac{1}{2}+\frac{1}{3}\right)\text{”}$$

can be obtained.

3. DEVELOPMENT OF MATHEMATICAL THINKING THROUGH THE STRUCTURE OF THE FINAL TASK 1

In Final Task 1, the conditions of three kinds of foods and three unit fractions are an extension of the intermediate tasks. Some students use trial and error when they first meet the problem. However, using diagram to solve this task is not possible as the task needs a 3-dimensional diagram, not a 2-dimensional diagram as in the intermediate task.

Final Task 1:
There are 65 boxes of food, and 2 persons shared a box of vegetables and 3 persons shared a box of fish, and 4 persons shared a box of chicken. Find the number of persons.

The following table gives the information on how students tackle the Final Task 1.

Approaches taken by students in solving Final Task 1	
Trial and Error	6
Trial and Error plus schema	9
Schema	10
Equation	9
Total number of students	34

3A. Using trial and Error

Students use the three numbers 2, 3 and 4 to obtain the least common multiple 12. Using the number 12 as a test number, $12 \div 2 = 6$, $12 \div 3 = 4$, $12 \div 4 = 3$. The total number of persons is $6 + 4 + 3 = 13$. Since 65 is 5 times of 13, so the number is 5 times of 12, which is 60.

Some students use similar table as in intermediate tasks, however, when dealing with the Final Task, students are not able to produce the following representation.

$$\begin{array}{ccccc} 2 & \times & \boxed{30} & = & 60 \\ & & + & & \\ 3 & \times & \boxed{20} & = & 60 \\ & & + & & \\ 4 & \times & \boxed{15} & = & 60 \\ & & = & & \\ & & 65 & & \end{array}$$

3B. Using Equation

By using equation to solve Final Task, all students use the number of persons as the variable and obtain the equation

$$\frac{k}{2}+\frac{k}{3}+\frac{k}{4}=65.$$

Hence

$$\frac{6k+4k+3k}{12}=65 \Rightarrow \frac{13k}{12}=65 \text{ and } k=60.$$

Again, the above simplification process wiped out the expression of structure

$$\text{“}k=65\div\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\text{”}$$

when they solve to find the value of k by the expression

$$“k\,(\frac{1}{2}+\frac{1}{3}+\frac{1}{4})=65”.$$

By using equation approach, students can obtain the structure of the expression of the problem, the schema

$$“\frac{k}{2}+\frac{k}{3}+\frac{k}{4}=65”,$$

but may not the structure

$$“k=65\div(\frac{1}{2}+\frac{1}{3}+\frac{1}{4})”$$

that we expected.

3C. Constructing problem to develop thinking into structure

Discussion was made on how to construct tasks similar to the Final Task. The development in extending structure to three kinds of food, say using fractions

$$\frac{1}{2},\frac{1}{3},\frac{1}{4}$$

to construct a similar question. Such construction arrive at

$$“\frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{13}{12}”$$

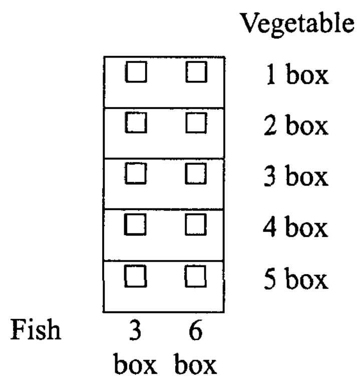
and it corresponds to that 13 boxes of foods are shared by 12 people, so that 2 share 1 box of a kind, 3 share one box of second kind, and 4 shared one box of the third kind. When students come back to the Final Task, they see that the answer should be a multiple of 13, and as 65 is 5 times 13, the number of persons is 5 times the number 12, which is 60. This exercise helps students to play around the structure and combine the schema into structure.

4. INVESTIGATION OF MATHEMATICAL THINKING AND STRUCTURE THROUGH FINAL TASK 2

Coming to Final Task 2, students are given the task directly. After some discussion, most students can transfer the schema to the task and obtain the relationship $46\div(2+3+4)$, though they may not be able to calculate the division of fractions correctly. Two sets of Simplified Tasks (Task C) and Intermediate Tasks (Task D) are used as working examples and hints or connecting question in case students could not solve Final Task 2. The following table describes the process of solving the Task 2.

The process of solving the Final Task 2				
Final Task 2	⇔	Set of Simplified Tasks C	⇔	Set of Intermediate Tasks D
There are 46 boxes of food, and 2 persons shared a box of vegetables and 3 persons shared 2 boxes of fish, and 4 persons shared 3 box of chicken. Find the number of persons.		There are 7 boxes of food, and each person get a box of vegetables and 4 persons shared three boxes of fish. Find the number of persons.		There are 11 boxes of food, and 2 persons shared a box of vegetables and 5 persons shared 3 boxes of fish. Find the number of persons.
$46\div(\frac{1}{2}+\frac{2}{3}+\frac{3}{4})=24$ (sum of 3 fractions)		$7\div(1+\frac{3}{4})=4$ (1 + fraction)		$11\div(\frac{1}{2}+\frac{3}{5})=10.$ (fraction + fraction)

In Final Task 2, students spend quite some time to understand the relation of the conditions and the fractional values implied. Some students still use their trial and error approach and diagram, however, the structure of "1 unit fraction and 1 non unit fraction" in the intermediate tasks requires knowledge of proportion (this example, the boxes of fishes). Some students use the following diagram to find out the relation of the conditions.



The following table indicated the approaches taken by students in tackling Final Task 1 and 2.

Approaches taken by students	Final Task 1	Final Task 2
Trial and Error	6	3
Trial and Error plus schema	9	6
Schema	10	13
Equation	9	12
Total number of students	34	34

More students tackled Task 2 directly using schema and equation. It indicates that they are more comfortable and confident in transferring the structure they acquired in Task 1.

4A. Non-unit fractions correspondence in Intermediate Tasks

Some Intermediate Tasks use fraction that can be expressed as simple decimal values so as to allow students to connect the structure. For example, "2 persons shared a box of vegetables" corresponds to fraction

$\frac{1}{2}$ as well as decimal value 0.5,

and

$\frac{3}{5}$ as 0.6 in the condition "5 persons shared 3 boxes of fish".

This is illustrated in the following Intermediate Task.

Intermediate Task:
There are 11 boxes of food, and 2 persons shared a box of vegetables and 5 persons shared 3 boxes of fish. Find the number of persons.

Based on their earlier schema, students arrive at the expression "$11 \div (0.5 + 0.6) = 10$". Again, students can add up the fraction to obtain

$$“\frac{1}{2}+\frac{3}{5}=\frac{11}{10}”,$$

and check that

$$11\times\frac{10}{11}=10,$$

verifying that 10 persons is the answer.

The use of the fraction

$$\frac{3}{5}$$

in the question allows students to change it to a decimal value 0.6. This "easy calculation" helps students to obtain answer, but not the necessary structure. Hence, fractional values such as

$$\frac{2}{7}$$

is used, corresponds to the condition "7 persons shared 2 boxes of fish", so that the fraction could not be written in decimal. These focus students to concentrate on the structure expression

$$\text{“}11\div(0.5+\frac{2}{7})\text{”, or “}11\div(\frac{1}{2}+\frac{2}{7})\text{”.}$$

4B. Using equation to solve the Final Task 2

Students who are solving Final Task 2 by equation use k as number of persons, and establish the following equation

$$\text{“}(\frac{k}{2}+\frac{2k}{3}+\frac{3k}{4})=46\text{”},$$

and obtain

$$k=46\div(\frac{1}{2}+\frac{2}{3}+\frac{3}{4})=24.$$

This time, some students can relate the expression of structure

$$\text{“}k=46\div(\frac{1}{2}+\frac{2}{3}+\frac{3}{4})\text{”}$$

with the equation they established.

Using equation and simplification	Relating schema and the structure
Let k = number of persons $\frac{k}{2}+\frac{2k}{3}+\frac{3k}{4}=46$ $\Rightarrow \frac{6k+8k+9k}{12}=46$ $\Rightarrow \frac{23k}{12}=24$ $\Rightarrow k=24.$	Let k = number of persons $(\frac{k}{2}+\frac{2k}{3}+\frac{3k}{4})=46$ $\Rightarrow k=46\div(\frac{1}{2}+\frac{2}{3}+\frac{3}{4})$ $\Rightarrow k=24.$

4C. Learning the structure by forming new problem

Constructing new problem through using different fractions is used as a strategy in helping students to establish the structure. For example, using the fractions

$$\frac{3}{4},\ \frac{2}{3}$$

and

$$\frac{3}{4}+\frac{2}{3}=\frac{17}{12},$$

a question such as "17 boxes of food (fishes and vegetables) are shared among a class of people, so that a group of 4 people will receive 3 boxes of fishes and a group of 3 people will receive 2 boxes of vegetables. Similarly, using fractions

$$\frac{2}{3}, \frac{1}{4}, \frac{3}{5}$$

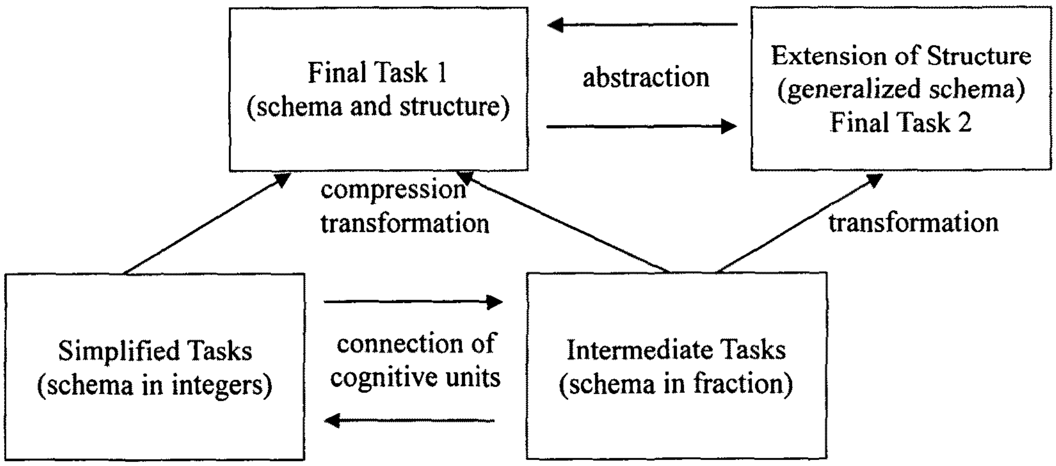
gives

$$\text{“}\left(\frac{2}{3}+\frac{1}{4}+\frac{3}{5}\right)= \frac{40+15+36}{60} = \frac{91}{60}\text{”},$$

allowing students to use the number 91, or the multiple of 91 as number of boxes to propose a new question.

5. CONCLUSION: FROM MATHEMATICS THINKING TO MATHEMATICAL STRUCTURE

The following is the theoretical process on how students develop their thinking and structure.



In solving the mathematics tasks, students are encouraged to share and discuss different solutions proposed by students and by teachers. This process helps to develop the schema of the structure. Schema provides the context of the thinking, to allow pupils to understand structure. In return, schema helps to discover the development of *mathematics structure, and structure mapping in turn* helps to formulate schema and reinforce the concept of the associated structure.

The different cognitive units appeared during the problem solving process are related and then compressed into a schema in solving the problem. The aim of solving individual cases of the problem is to generalize a schema for solving this problem. Such schema is more useful when they obtain the mathematical structure of the solution through abstraction, and extend such abstraction to related problems and more generalized problem.

The set of related tasks helps students to compress the meaning of cognitive units and develop it into a structure. The understanding of structure can be achieved by manipulating the numeric values in different forms such as fraction and decimals and seeing that they are in correspondence. Without structure correspondence, there is no structure recognition. The following describes the process and framework of solving the tasks and the extended tasks, and develop thinking to structure.

Stages	Remark
1. Formulation of schema through connecting and compression cognitive units	This is the intuition stage; the formation of thinking process, where students try out their solution and thinking through solving the simplified task, forming schema and a simple structure. This happens by connecting individual cognitive unit and compressing the thinking.
2. Structure mapping through pattern recognition of numbers.	The second stage is where students formulate part of the solution of the final tasks through using simple number or related numeric values and observe the similarities among questions. Structure mapping is performed between simplified task and the intermediate task to check their conjecture of the structure format.
3. Extended Structure Mapping	In the third stage, students build up a formal structure, mapping the structure and extended the mapping to Final Task 1.
4. Extension and Generalization	At the fourth stage, pupils can extend the results obtained to related tasks, and extend the structure of the intermediate task or the Final Task 1 to Final Task 2.

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