Commun. Korean Math. Soc. **24** (2009), No. 4, pp. 629–640
DOI 10.4134/CKMS.2009.24.4.629

CERTAIN RELATIONS FOR MOCK THETA FUNCTIONS OF ORDER EIGHT

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Abstract. The aim of the present paper is to establish certain relations for partial mock theta functions and mock theta functions of order eight with other partial mock theta functions and mock theta functions of order two, six, eight and ten respectively.

1. Introduction

The last gift of Ramanujan to the mathematical world "The Mock Theta Functions" about which Ramanujan informed Hardy in his last letter just three month before his death.

Ramanujan quoted that I discovered very interesting functions recently which I call 'Mock theta functions'. Ramanujan mentioned about certain mock theta functions of order three, five and seven respectively. In Ramanujan's Collected Papers mock theta functions are described as follows by Hardy [8]:
"Ramanujan... makes it clear that what he means by a 'mock theta function' is a function, defined by a q-series convergent for $\mid q \mid< 1$, for which we can calculate asymptotic formulae when q tend to a 'rational point' $e^{2i\pi r/s}$, of the same degree of precision as those furnished, for the ordinary θ-functions, by the theory of linear transformation. Thus he asserts, for example, that if

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$$

and $q = e^{-t} \to 1$ by positive values, then

$$f(q) + \sqrt{[\frac{\pi}{t}]} \exp\left[\frac{\pi^2}{24t^2} - \frac{t}{24}\right] \to 4."$$

The discovery of these functions was known to us only in 1935, through the famous presidential address of Watson to the London Mathematical Society published in 1936 in the J. London Math. Soc. A substantial portion of mock

Received February 5, 2009; Revised February 23, 2009.
2000 *Mathematics Subject Classification.* 33D15, 11B65, 11A55.
Key words and phrases. mock theta functions, continued fractions, basic hypergeometric series.

theta functions discovered by Ramanujan lay buried in the debris in the attic of Watson's house for almost half-a-century.

After these discovery eminent mathematicians working in the field of q-series like Andrews and Hickerson [4], Gordon and McIntosh [7], Y. S. Choi [5] have introduced new class of mock theta functions. Andrews and Hickerson [4] have introduced seven mock theta functions of order 6, Gordon and McIntosh [7] have introduced eight mock theta functions of order 8 and Y. S. Choi [5] provided four mock theta functions of order 10.

Mathematicians working in the field of q-series were interested for the relations among mock theta functions of different order and they succeeded in their goal. Remy Y. Denis et. al. [6] established results which provide relationship between any two mock theta functions of order 3, 5 and 7. They exclude the relationship involving the recently discovered mock theta functions of order 6, 8 and 10. Srivastava, Bhaskar [11] provided the relations among mock theta functions and partial mock theta functions of order 10, 3, 5 and 6. Recently, Srivastava, Bhaskar [12] provided relations between mock theta functions and partial mock theta functions of order 2, 3 and 6 and Ramanujan's function $\mu(q)$.

Mathematicians working on Ramanujan's mathematics have established relations among partial mock theta functions and mock theta functions of all order, but they are silent on the relations of mock theta functions of order 8. In this paper we have established relations among mock theta functions and partial mock theta functions of order 8 and mock theta functions of different order and an attempt has been made to develop continued fraction representation for the ratio of two mock theta functions of order 2. In §3 a numbers of results representing relations among partial mock theta functions and mock theta functions of order 8 have been established, which shall be interesting for further study of mock theta functions of order 8 and other.

2. Definitions and notations

We shall use the following q-symbols: For $\mid q \mid < 1$ and $\mid q^r \mid < 1$,

$$\begin{aligned}
(a;q)_n &= \Pi_{s=0}^{n-1}(1-aq^s),\ n \geq 1\\
(a;q^r)_n &= \Pi_{s=0}^{n-1}(1-aq^{rs}),\ n \geq 1\\
(a;q)_0 &= 1,\ (a;q^r)_0 = 1,\\
(a;q^r)_\infty &= \Pi_{s=0}^{\infty}(1-aq^{rs}),\\
(a)_n &= (a;q)_n.
\end{aligned}$$

A generalized basic hypergeometric series with base q is defined as:

$$_A\phi_{A-1}(a_1,a_2,\ldots,a_A;b_1,b_2,\ldots,b_{A-1};q,z) = \sum_{n=0}^{\infty}\frac{(a_1;q)_n\cdots(a_A;q)_n z^n}{(b_1;q)_n\cdots(b_{A-1};q)_n(q;q)_n},$$

where $|z|<1, |q|<1$.

A finite continued fraction is an expression of the type:

$$\frac{a_1}{a_2+}\ \frac{a_3}{a_4+}\ \frac{a_5}{a_6+}\ \frac{a_7}{a_8+}\ \frac{a_9}{a_{10}+}\ \frac{a_{11}}{a_{12}+}\cdots\frac{a_{n-1}}{a_n},$$

where $a_1, a_2, a_3, a_4, \ldots$ are real or complex numbers.

It is an infinite continued fraction when $n \to \infty$. If $F(q) = \sum_{n=0}^{\infty} f(q,n)$ is a mock theta function, then the corresponding partial mock theta function is denoted by the truncated series,

$$F_p(q) = \textstyle\sum_{n=0}^{p} f(q,n).$$

Definitions and notations of the mock theta functions that shall be used in our analysis are as:

Mock theta functions of order 2:

Andrews defined the second order mock theta functions as:

$$A(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^{(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}},$$

$$B(q) = \sum_{n=0}^{\infty} \frac{q^{(n^2+n)}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}}.$$

Mock theta functions of order 6:

Andrews and Hickerson defined the mock theta functions of order six in the Ramanujan's Lost Notebook VII as given below:

$$\phi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}},$$

$$\psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}},$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q;q)_n}{(q;q^2)_{n+1}},$$

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}(-q;q)_n}{(q;q^2)_{n+1}},$$

$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q;q)_n},$$

$$2\mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)}(1+q^n)(q;q^2)_n}{(-q;q)_{n+1}},$$

$$\gamma(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n}.$$

Mock theta functions of order 8:

Gordon and McIntosh found eight mock theta functions of order 8 which are given below as:

$$S_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n},$$

$$S_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n},$$

$$T_0(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},$$

$$T_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},$$

$$U_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n},$$

$$U_1(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}},$$

$$V_0(q)=-1+2\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}$$

$$=-1+2\sum_{n=0}^{\infty}\frac{q^{2n^2}(-q^2;q^4)_n}{(q;q^2)_{2n+1}},$$

$$V_1(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}}$$

$$=\sum_{n=0}^{\infty}\frac{q^{2n^2+2n+1}(-q^4;q^4)_n}{(q;q^2)_{2n+2}}.$$

Mock theta functions of order 10:
Tenth order mock theta functions defined by Y. S. Choi found in Ramanujan's Lost Notebook I, II, IV are given below as:

$$\phi_{LC}(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}},$$

$$\psi_{LC}(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)(n+2)/2}}{(q;q^2)_{n+1}},$$

$$X_{LC}(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(-q;q)_{2n}},$$

$$\chi_{LC}(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}.$$

Srivastava [10] has established the following identity:

$$\sum_{m=0}^{n}\delta_m\sum_{r=0}^{m}\alpha_r=\sum_{r=0}^{n}\alpha_r\sum_{m=0}^{n}\delta_m-\sum_{r=0}^{n-1}\alpha_{r+1}\sum_{m=0}^{r}\delta_m.$$

The identity can be written as:

$$\sum_{r=0}^{n}\alpha_r\sum_{m=0}^{n}\delta_m=\sum_{m=0}^{n}\delta_m\sum_{r=0}^{m}\alpha_r+\sum_{r=0}^{n-1}\alpha_{r+1}\sum_{m=0}^{r}\delta_m. \tag{1}$$

3. Main results

In this section we shall establish following results involving partial mock theta functions and mock theta functions of order two, six, eight and ten respectively.

(a) Relations between partial mock theta functions and mock theta functions of order eight and order two:

$$A_n(q)S_{0n}(q)=\sum_{m=0}^{n}\frac{q^{m^2}(-q;q^2)_m}{(-q^2;q^2)_m}A_m(q)+\sum_{r=0}^{n-1}\frac{q^{(r+1)^2}(-q;q^2)_{r+1}}{(q;q^2)^2_{r+2}}S_{0r}(q). \tag{2}$$

This provides relationship between partial mock theta functions of order 8 and 2.

$$A(q)S_0(q)=\sum_{m=0}^{\infty}\frac{q^{m^2}(-q;q^2)_m}{(-q^2;q^2)_m}A_m(q)+\sum_{r=0}^{\infty}\frac{q^{(r+1)^2}(-q;q^2)_{r+1}}{(q;q^2)^2_{r+2}}S_{0r}(q). \tag{3}$$

This provides relation between mock theta functions of order 8 and 2.

(4)

$$A_n(q)S_{1n}(q)=\sum_{m=0}^{n}\frac{q^{m(m+2)}(-q;q^2)_m}{(-q^2;q^2)_m}A_m(q)+\sum_{r=0}^{n-1}\frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(q;q^2)^2_{r+1}}S_{1r}(q).$$

This is the relation between partial mock theta functions of order 8 and 2.

$$A(q)S_1(q)=\sum_{m=0}^{\infty}\frac{q^{m(m+2)}(-q;q^2)_m}{(-q^2;q^2)_m}A_m(q)+\sum_{r=0}^{\infty}\frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(q;q^2)^2_{r+1}}S_{1r}(q). \tag{5}$$

This is the relation between mock theta functions of order 8 and 2.

$$\begin{aligned}B_n(q)T_{0n}(q)=&\sum_{m=0}^{n}\frac{q^{(m+1)(m+2)}(-q^2;q^2)_m}{(-q;q^2)_{m+1}}B_m(q)\\&+\sum_{r=0}^{n-1}\frac{q^{(r+1)(r+2)}(-q^2;q^2)_{r+1}}{(q;q^2)^2_{r+2}}T_{0r}(q).\end{aligned} \tag{6}$$

It is another relation between partial mock theta functions of order 8 and 2.

$$B(q)T_0(q) = \sum_{m=0}^{\infty} \frac{q^{(m+1)(m+2)}(-q^2;q^2)_m}{(-q;q^2)_{m+1}} B_m(q) + \sum_{r=0}^{\infty} \frac{q^{(r+1)(r+2)}(-q^2;q^2)_{r+1}}{(q;q^2)_{r+2}^2} T_{0r}(q). \tag{7}$$

It is another relation between mock theta functions of order 8 and 2.

$$B_n(q)T_{1n}(q) = \sum_{m=0}^{n} \frac{q^{m(m+1)}(-q^2;q^2)_m}{(-q;q^2)_{m+1}} B_m(q) + \sum_{r=0}^{n-1} \frac{q^{(r+1)(r+2)}(-q^2;q^2)_{r+1}}{(q;q^2)_{r+2}^2} T_{1r}(q) \tag{8}$$

which is the relation between partial mock theta functions of order 8 and 2.

(9)

$$B(q)T_1(q) = \sum_{m=0}^{\infty} \frac{q^{m(m+1)}(-q^2;q^2)_m}{(-q;q^2)_{m+1}} B_m(q) + \sum_{r=0}^{\infty} \frac{q^{(r+1)(r+2)}(-q^2;q^2)_{r+1}}{(q;q^2)_{r+2}^2} T_{1r}(q)$$

which is the relation between mock theta functions of order 8 and 2.

$$A_n(q)[1+V_{0n}(q)] = \sum_{m=0}^{n} \frac{2q^{m^2}(-q;q^2)_m}{(q;q^2)_m} A_m(q) + \sum_{r=0}^{n-1} \frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(q;q^2)_{r+2}^2} [1+V_{0r}(q)] \tag{10}$$

which provides relationship between partial mock theta functions of order 8 and 2.

(11)

$$A(q)[1+V_0(q)] = \sum_{m=0}^{\infty} \frac{2q^{m^2}(-q;q^2)_m}{(q;q^2)_m} A_m(q) + \sum_{r=0}^{\infty} \frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(q;q^2)_{r+2}^2} [1+V_{0r}(q)]$$

which gives relation between mock theta functions of order 8 and 2.

$$B_n(q)[1+V_{0n}(q)] = \sum_{m=0}^{n} \frac{2q^{m^2}(-q;q^2)_m}{(q;q^2)_m} B_m(q) + \sum_{r=0}^{n-1} \frac{q^{(r+1)(r+2)}(-q^2;q^2)_{r+1}}{(q;q^2)_{r+2}^2} [1+V_{0r}(q)]. \tag{12}$$

This provides relation between partial mock theta functions of order 8 and 2.

$$B(q)[1+V_0(q)] = \sum_{m=0}^{\infty} \frac{2q^{m^2}(-q;q^2)_m}{(q;q^2)_m} B_m(q) + \sum_{r=0}^{\infty} \frac{q^{(r+1)(r+2)}(-q^2;q^2)_{r+1}}{(q;q^2)_{r+2}^2}[1+V_{0r}(q)]. \tag{13}$$

This provides relation between mock theta functions of order 8 and 2.

Similarly we can establish the relations between other partial mock theta functions and mock theta functions of order 8 and 2.

(b) Relations between partial mock theta functions and mock theta functions of order eight and order six:

(14)

$$\lambda_n(q)S_{0n}(q) = \sum_{m=0}^{n} \frac{q^{m^2}(-q;q^2)_m}{(-q^2;q^2)_m}\lambda_m(q) + \sum_{r=0}^{n-1} \frac{(-1)^{r+1}q^{r+1}(q;q^2)_{r+1}}{(-q;q)_{r+1}} S_{0r}(q)$$

which provides relation between partial mock theta functions of order 8 and 6.

$$\lambda(q)S_0(q) = \sum_{m=0}^{\infty} \frac{q^{m^2}(-q;q^2)_m}{(-q^2;q^2)_m}\lambda_m(q) + \sum_{r=0}^{\infty} \frac{(-1)^{r+1}q^{r+1}(q;q^2)_{r+1}}{(-q;q)_{r+1}} S_{0r}(q) \tag{15}$$

which provides relation between mock theta functions of order 8 and 6.

(16)

$$\phi_n(q)S_{1n}(q) = \sum_{m=0}^{n} \frac{q^{m(m+2)}(-q;q^2)_m}{(-q^2;q^2)_m}\phi_m(q) + \sum_{r=0}^{n-1} \frac{(-1)^{r+1}q^{(r+1)^2}(q;q^2)_{r+1}}{(-q;q)_{2(r+1)}} S_{1r}(q)$$

which gives relation between partial mock theta functions of order 8 and 6.

(17)

$$\phi(q)S_1(q) = \sum_{m=0}^{\infty} \frac{q^{m(m+2)}(-q;q^2)_m}{(-q^2;q^2)_m}\phi_m(q) + \sum_{r=0}^{\infty} \frac{(-1)^{r+1}q^{(r+1)^2}(q;q^2)_{r+1}}{(-q;q)_{2(r+1)}} S_{1r}(q)$$

which gives relation between mock theta functions of order 8 and 6.

$$\psi_n(q)T_{0n}(q) = \sum_{m=0}^{n} \frac{q^{(m+1)(m+2)}(-q^2;q^2)_m}{(-q;q^2)_{m+1}}\psi_m(q) + \sum_{r=0}^{n-1} \frac{(-1)^{r+1}q^{(r+2)^2}(q;q^2)_{r+1}}{(-q;q)_{2r+3}} T_{0r}(q). \tag{18}$$

This is the relation between partial mock theta functions of order 8 and 6.

$$\psi(q)T_0(q) = \sum_{m=0}^{\infty} \frac{q^{(m+1)(m+2)}(-q^2;q^2)_m}{(-q;q^2)_{m+1}}\psi_m(q) + \sum_{r=0}^{\infty} \frac{(-1)^{r+1}q^{(r+2)^2}(q;q^2)_{r+1}}{(-q;q)_{2r+3}} T_{0r}(q). \tag{19}$$

This is the relation between mock theta functions of order 8 and 6.

(20)
$$\sigma_n(q)U_{1n}(q)=\sum_{m=0}^{n}\frac{q^{(m+1)^2}(-q;q^2)_m}{(-q^2;q^4)_{m+1}}\sigma_m(q)+\sum_{r=0}^{n-1}\frac{q^{(r+2)(r+3)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}}U_{1r}(q)$$

which provides relation between partial mock theta functions of order 8 and 6.

(21)
$$\sigma(q)U_1(q)=\sum_{m=0}^{\infty}\frac{q^{(m+1)^2}(-q;q^2)_m}{(-q^2;q^4)_{m+1}}\sigma_m(q)+\sum_{r=0}^{\infty}\frac{q^{(r+2)(r+3)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}}U_{1r}(q)$$

which provides relation between mock theta functions of order 8 and 6.

(22)
$$V_{1n}(q)\rho_n(q)=\sum_{m=0}^{n}\frac{q^{m(m+1)/2}(-q;q)_m}{(q;q^2)_{m+1}}V_{1m}(q)+\sum_{r=0}^{n-1}\frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(q;q^2)_{r+2}}\rho_r(q)$$

which gives relation between partial mock theta functions of order 8 and 6.

(23) $$V_1(q)\rho(q)=\sum_{m=0}^{\infty}\frac{q^{m(m+1)/2}(-q;q)_m}{(q;q^2)_{m+1}}V_{1m}(q)+\sum_{r=0}^{\infty}\frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(q;q^2)_{r+2}}\rho_r(q)$$

which gives relation between mock theta functions of order 8 and 6.

Similarly relations for other partial mock theta functions and mock theta functions of order 8 and 6 can be established.

(c) Relations between partial mock theta functions and mock theta functions of order eight and order ten:

(24)
$$T_{0n}(q)\phi_{LCn}(q)=\sum_{m=0}^{n}\frac{q^{m(m+1)/2}}{(q;q^2)_{m+1}}T_{0m}(q)+\sum_{r=0}^{n-1}\frac{q^{(r+2)(r+3)}(-q^2;q^2)_{r+1}}{(-q;q^2)_{r+2}}\phi_{LCr}(q).$$

It is the relation between partial mock theta functions of order 8 and 10.

(25)
$$T_0(q)\phi_{LC}(q)=\sum_{m=0}^{\infty}\frac{q^{m(m+1)/2}}{(q;q^2)_{m+1}}T_{0m}(q)+\sum_{r=0}^{\infty}\frac{q^{(r+2)(r+3)}(-q^2;q^2)_{r+1}}{(-q;q^2)_{r+2}}\phi_{LCr}(q)$$

which provides relation between mock theta functions of order 8 and 10.

(26)
$$T_{1n}(q)\psi_{LCn}(q)=\sum_{m=0}^{n}\frac{q^{(m+1)(m+2)/2}}{(q;q^2)_{m+1}}T_{1m}(q)+\sum_{r=0}^{n-1}\frac{q^{(r+1)(r+2)}(-q^2;q^2)_{r+1}}{(-q;q^2)_{r+2}}\psi_{LCr}(q).$$

This is the relation between partial mock theta functions of order 8 and 10.

(27)
$$T_1(q)\psi_{LC}(q)=\sum_{m=0}^{\infty}\frac{q^{(m+1)(m+2)/2}}{(q;q^2)_{m+1}}T_{1m}(q)+\sum_{r=0}^{\infty}\frac{q^{(r+1)(r+2)}(-q^2;q^2)_{r+1}}{(-q;q^2)_{r+2}}\psi_{LCr}(q).$$

This is the relation between mock theta functions of order 8 and 10.

(28)
$$\chi_{LCn}(q)S_{0n}(q) = \sum_{m=0}^{n} \frac{q^{m^2}(-q;q^2)_m}{(-q^2;q^2)_m}\chi_{LCm}(q) + \sum_{r=0}^{n-1} \frac{(-1)^{r+1}q^{(r+2)^2}}{(-q;q)_{2r+3}}S_{0r}(q).$$

This gives relation between partial mock theta functions of order 8 and 10.

$$\chi_{LC}(q)S_0(q) = \sum_{m=0}^{\infty} \frac{q^{m^2}(-q;q^2)_m}{(-q^2;q^2)_m}\chi_{LCm}(q) + \sum_{r=0}^{\infty} \frac{(-1)^{r+1}q^{(r+2)^2}}{(-q;q)_{2r+3}}S_{0r}(q). \tag{29}$$

This gives relation between mock theta functions of order 8 and 10.

(30)
$$\chi_{LCn}(q)S_{1n}(q) = \sum_{m=0}^{n} \frac{q^{m(m+2)}(-q;q^2)_m}{(-q^2;q^2)_m}\chi_{LCm}(q) + \sum_{r=0}^{n-1} \frac{(-1)^{r+1}q^{(r+2)^2}}{(-q;q)_{2r+3}}S_{1r}(q).$$

It is the relation between partial mock theta functions of order 8 and 10.

(31)
$$\chi_{LC}(q)S_1(q) = \sum_{m=0}^{\infty} \frac{q^{m(m+2)}(-q;q^2)_m}{(-q^2;q^2)_m}\chi_{LCm}(q) + \sum_{r=0}^{\infty} \frac{(-1)^{r+1}q^{(r+2)^2}}{(-q;q)_{2r+3}}S_{1r}(q).$$

It is the relation between mock theta functions of order 8 and 10.

Similarly relations for other partial mock theta functions and mock theta functions of order 8 and 10 can be established.

(d) Relations between partial mock theta functions and mock theta functions of order six and order two:

(32)
$$A_n(q)\psi_n(q) = \sum_{m=0}^{n} \frac{(-1)^m q^{(m+1)^2}(q;q^2)_m}{(-q;q)_{2m+1}}A_m(q) + \sum_{r=0}^{n-1} \frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(q;q^2)_{r+2}^2}\psi_r(q)$$

which provides relationship between partial mock theta functions of order 6 and 2.

(33)
$$A(q)\psi(q) = \sum_{m=0}^{\infty} \frac{(-1)^m q^{(m+1)^2}(q;q^2)_m}{(-q;q)_{2m+1}}A_m(q) + \sum_{r=0}^{\infty} \frac{q^{(r+2)^2}(-q;q^2)_{r+1}}{(q;q^2)_{r+2}^2}\psi_r(q)$$

which provides relationship between mock theta functions of order 6 and 2.

$$\begin{aligned} B_n(q)\psi_n(q) = &\sum_{m=0}^{n} \frac{(-1)^m q^{(m+1)^2}(q;q^2)_m}{(-q;q)_{2m+1}}B_m(q) \\ &+ \sum_{r=0}^{n-1} \frac{q^{(r+1)(r+2)}(-q^2;q^2)_{r+1}}{(q;q^2)_{r+2}^2}\psi_r(q). \end{aligned} \tag{34}$$

This is the relation between partial mock theta functions of order 6 and 2.
(35)

$$B(q)\psi(q) = \sum_{m=0}^{\infty} \frac{(-1)^m q^{(m+1)^2}(q;q^2)_m}{(-q;q)_{2m+1}} B_m(q) + \sum_{r=0}^{\infty} \frac{q^{(r+1)(r+2)}(-q^2;q^2)_{r+1}}{(q;q^2)^2_{r+2}} \psi_r(q).$$

This is the relation between mock theta functions of order 6 and 2.

Similarly we can provide the relations between other partial mock theta functions and mock theta functions of order 6 and 2.

(e) Relations between partial mock theta functions and mock theta functions of same order:

$$(36)\quad B_n(q)A_n(q) = \sum_{m=0}^{n} \frac{q^{(m+1)}(-q^2;q^2)_m}{(q;q^2)_{m+1}} B_m(q) + \sum_{r=0}^{n-1} \frac{q^{r+1}(-q;q^2)_{r+1}}{(q;q^2)_{r+2}} A_r(q)$$

which provides relationship between partial mock theta functions of order 2.

$$(37)\quad B(q)A(q) = \sum_{m=0}^{\infty} \frac{q^{(m+1)}(-q^2;q^2)_m}{(q;q^2)_{m+1}} B_m(q) + \sum_{r=0}^{\infty} \frac{q^{r+1}(-q;q^2)_{r+1}}{(q;q^2)_{r+2}} A_r(q)$$

which provides relationship between mock theta functions of order 2.
(38)

$$\rho_n(q)\phi_n(q) = \sum_{m=0}^{n} \frac{(-1)^m q^{m^2}(q;q^2)_m}{(-q;q)_{2m}} \rho_m(q) + \sum_{r=0}^{n-1} \frac{q^{(r+1)(r+2)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}} \phi_r(q).$$

This is the relation between partial mock theta functions of order 6.
(39)

$$\rho(q)\phi(q) = \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2}(q;q^2)_m}{(-q;q)_{2m}} \rho_m(q) + \sum_{r=0}^{\infty} \frac{q^{(r+1)(r+2)/2}(-q;q)_{r+1}}{(q;q^2)_{r+2}} \phi_r(q).$$

This is the relation between mock theta functions of order 6.
(40)

$$S_{1n}(q)S_{0n}(q) = \sum_{m=0}^{n} \frac{q^{m^2}(-q;q^2)_m}{(-q^2;q^2)_m} S_{1m}(q) + \sum_{r=0}^{n-1} \frac{q^{(r+1)(r+3)}(-q;q^2)_{r+1}}{(-q^2;q^2)_{r+1}} S_{0r}(q)$$

which provides relationship between partial mock theta functions of order 8.
(41)

$$S_1(q)S_0(q) = \sum_{m=0}^{\infty} \frac{q^{m^2}(-q;q^2)_m}{(-q^2;q^2)_m} S_{1m}(q) + \sum_{r=0}^{\infty} \frac{q^{(r+1)(r+3)}(-q;q^2)_{r+1}}{(-q^2;q^2)_{r+1}} S_{0r}(q)$$

which provides relationship between mock theta functions of order 8.

$$(42)\quad \psi_{LCn}(q)\phi_{LCn}(q) = \sum_{m=0}^{n} \frac{q^{m(m+1)/2}}{(q;q^2)_{m+1}} \psi_{LCm}(q) + \sum_{r=0}^{n-1} \frac{q^{(r+2)(r+3)/2}}{(q;q^2)_{r+2}} \phi_{LCr}(q).$$

This is the relation between partial mock theta functions of order 10.

$$\psi_{LC}(q)\phi_{LC}(q) = \sum_{m=0}^{\infty} \frac{q^{m(m+1)/2}}{(q;q^2)_{m+1}} \psi_{LCm}(q) + \sum_{r=0}^{\infty} \frac{q^{(r+2)(r+3)/2}}{(q;q^2)_{r+2}} \phi_{LCr}(q). \tag{43}$$

This is the relation between mock theta functions of order 10.

Similarly we can give the relations between other partial mock theta functions and mock theta functions of same order.

4. Proof of main results

As an illustration, we shall prove result (2).

Taking $\delta_m = \frac{q^{m^2}(-q;q^2)_m}{(-q^2;q^2)_m}$ and $\alpha_r = \frac{q^{(r+1)^2}(-q;q^2)_r}{(q;q^2)^2_{r+1}}$ in identity (1) and simplifying further, we obtain the result (2).

Taking the limit $n \to \infty$ in (2), result (3) can be established.

Choosing δ_m and α_r appropriately in identity (1), results (4) to (43) can be obtained.

5. Continued fraction representation for the ratio of two mock theta functions of order 2

Singh [9] has established the result:

$$\frac{{}_2\phi_1(\alpha,\beta;\gamma;q;x)}{{}_2\phi_1(\alpha,\beta q;\gamma;q;x)} = 1 - \frac{\mu_0}{1+}\ \frac{\nu_0}{1-}\ \frac{\mu_1}{1+}\ \frac{\nu_1}{1-}\ \frac{\mu_2}{1+\cdots}, \tag{44}$$

where

$$\mu_i = x\beta q^i(1-\alpha q^i), \quad i = 0,1,2,3,\ldots,$$
$$\nu_i = \tfrac{\gamma}{\beta q}(1-\beta q^{i+1})(1-\tfrac{\alpha\beta x q^{i+1}}{\gamma}), \quad i = 0,1,2,3,\ldots,$$

Now let $q \to q^2$ and $\alpha = q^2$, $\beta = -q$, $\gamma = q^3$ and $x = q$ in identity (44), we get

$$\frac{{}_2\phi_1(q^2,-q;q^3;q^2;q)}{{}_2\phi_1(q^2,-q^2;q^3;q^2;q)} = 1 - \frac{s_0}{1+}\ \frac{t_0}{1-}\ \frac{s_1}{1+}\ \frac{t_1}{1-}\ \frac{s_2}{1+\cdots}, \tag{45}$$

where

$$s_i = -q^{i+2}(1-q^{i+2}) \quad \text{and} \quad t_i = -q(1+q^{i+2})^2, \quad i = 0,1,2,3,\ldots.$$

The basic hypergeometric series structure of second order mock theta functions $A(q)$ and $B(q)$ given by Srivastava [12] is as:

$$A(q) = \frac{q}{1-q}\ {}_2\phi_1(q^2,-q^2;q^3;q^2,q), \tag{46}$$

$$B(q) = \frac{1}{1-q}\ {}_2\phi_1(q^2,-q;q^3;q^2,q). \tag{47}$$

Taking the ratio of (47) by (46), now using the result (45) and after simplification, the continued fraction representation for the ratio of two mock theta functions of order 2 can be obtained.

$$\frac{B(q)}{A(q)} = \frac{1}{q}\left[1 - \frac{s_0}{1+}\ \frac{t_0}{1-}\ \frac{s_1}{1+}\ \frac{t_1}{1-}\ \frac{s_2}{1+\cdots}\right]. \tag{48}$$

Remark. In §3 $A_n, S_{0n}, B_n, S_{1n}, \ldots$ represent partial mock theta functions corresponding to the mock theta functions $A, S_0, B, S_1, \ldots$ defined in §2.

Acknowledgment. We are grateful to the referee for his valuable suggestions.

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