Commun. Korean Math. Soc. **24** (2009), No. 4, pp. 581–594
DOI 10.4134/CKMS.2009.24.4.581

COMMON FIXED POINT OF COMPATIBLE MAPS OF TYPE (γ) ON COMPLETE FUZZY METRIC SPACES

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Abstract. In this paper, we establish a common fixed point theorem in complete fuzzy metric spaces which generalizes some results in [9].

1. Introduction and preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [16] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [5] and Kramosil and Michalek [8] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and $\epsilon^{(\infty)}$ theory which were given and studied by El Naschie [1, 2, 3, 4, 15]. Many authors [6, 10, 13] have proved fixed point theorem in fuzzy (probabilistic) metric spaces.

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if it satisfies the following conditions

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $a*b = ab$ and $a*b = \min(a, b)$.

Definition 1.2. A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

Received September 11, 2006.
2000 *Mathematics Subject Classification.* 54E40, 54E35, 54H25.
Key words and phrases. fuzzy contractive mapping, complete fuzzy metric space.

(5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Let $(X, M, *)$ be a fuzzy metric space. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the fuzzy metric M). This topology is Hausdorff and first countable. A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exits $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$. The fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Example 1.3. Let $X = \mathbb{R}$. Denote $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

for all $x, y \in X$.

Lemma 1.4. *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, t)$ is non-decreasing with respect to t, for all x, y in X.*

Definition 1.5. Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t).$$

Whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$, i.e.,

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 1.6. *Let $(X, M, *)$ be a fuzzy metric space. Then M is continuous function on $X^2 \times (0, \infty)$.*

Proof. See Proposition 1 of [11]. □

Definition 1.7. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

Definition 1.8. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be compatible if

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1, \forall t > 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X.$$

Proposition 1.9 ([14])**.** *Self-mappings A and S of a fuzzy metric space $(X, M, *)$ are compatible, then they are weak compatible.*

The converse is not true as seen in following example.

Example 1.10. Let $(X, M, *)$ be a fuzzy metric space, where $X = [0, 2]$, with t-norm defined $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $t > 0$ and $x, y \in X$. Define self-maps A and S on X as follows:

$$Ax = \begin{cases} 2 & \text{if } 0 \le x \le 1, \\ \frac{x}{2} & \text{if } 1 < x \le 2, \end{cases} \qquad Sx = \begin{cases} 2 & \text{if } x = 1, \\ \frac{x+3}{5} & \text{otherwise}, \end{cases}$$

and $x_n = 2 - \frac{1}{2n}$. Then we have $S1 = A1{=}2$ and $S2 = A2 = 1$. Also $SA1 = AS1 = 1$ and $SA2 = AS2 = 2$. Thus (A, S) is weak compatible. Again,

$$Ax_n = 1 - \frac{1}{4n}, \qquad Sx_n = 1 - \frac{1}{10n}.$$

Thus,

$$Ax_n \to 1, \qquad Sx_n \to 1.$$

Further,

$$SAx_n = \frac{4}{5} - \frac{1}{20n}, \qquad ASx_n = 2.$$

Now,

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = \lim_{n\to\infty} M(2, \frac{4}{5} - \frac{1}{20n}, t) = \frac{t}{t + \frac{6}{5}} < 1,\ \forall\ t > 0.$$

Hence (A, S) is not compatible.

2. Compatible maps of type (γ)

In this section, we give the concept of compatible maps of type (γ) in fuzzy metric spaces and some properties of these maps.

Definition 2.1. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be compatible maps of type (γ) if satisfying:

(i) A and S are compatible, that is

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1, \forall t > 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X,$$

(ii) they are continuous at x.
On the other hand we have,

$$A(x) = A(\lim_{n\to\infty} Ax_n) = A(\lim_{n\to\infty} Sx_n) = \lim_{n\to\infty} ASx_n$$
$$= \lim_{n\to\infty} SAx_n = S(\lim_{n\to\infty} Ax_n) = S(x).$$

Definition 2.2. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. The maps A and S are said to be weak compatible of type (γ) if

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$$

for some $x \in X$ implies that $Ax = Sx$.

Clearly if self-mappings A and S of a fuzzy metric space $(X, M, *)$ are compatible of type (γ), then they are weak compatible of type (γ). But the converse is not true as seen in following example.

Example 2.3. Let $(X, M, *)$ be a fuzzy metric space, where $X = [0, 2]$, with t-norm defined $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $t > 0$ and $x, y \in X$. Define self-maps A and S on X as follows:

$$Ax = \begin{cases} 1 & \text{if } x \in Q, \\ \frac{1}{2} & \text{otherwise,} \end{cases} \qquad Sx = \begin{cases} 1 & \text{if } x \in Q, \\ 0 & \text{otherwise,} \end{cases}$$

and $x_n = 2 - \frac{1}{2n}$. Then we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1,$$

and $S1 = A1 = 1$. That is (A, S) is weak compatible of type (γ) also (A, S) is compatible, for $Ax_n = A(2 - \frac{1}{2n}) = 1$ and $Sx_n = S(2 - \frac{1}{2n}) = 1$ hence

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} SAx_n = 1,$$

but A, S are not continuous at 1 and hence (A, S) is not compatible of type (γ).

Lemma 2.4. *Let $(X, M, *)$ be a fuzzy metric space.*
(i) *If we define $E_{\lambda,M} : X^2 \to^+ \cup\{0\}$ by*

$$E_{\lambda,M}(x, y) = \inf\{t > 0 \ : \ M(x, y, t) > 1 - \lambda\}$$

for each $\mu \in (0, 1)$ there exists $\lambda \in (0, 1)$ such that

$$E_{\mu,M}(x_1, x_n) \le E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n)$$

for any $x_1, x_2, \ldots, x_n \in X$.
(ii) *The sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent in fuzzy metric space $(X, M, *)$ if and only if $E_{\lambda,M}(x_n, x) \to 0$. Also the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy sequence if and only if it is Cauchy with $E_{\lambda,M}$.*

Proof. (i) For every $\mu \in (0,1)$, we can find a $\lambda \in (0,1)$ such that

$$\overbrace{(1-\lambda) * (1-\lambda) * \cdots * (1-\lambda)}^{n} \geq 1-\mu$$

by triangular inequality we have

$$\begin{aligned} & M(x_1, x_n, E_{\lambda,M}(x_1,x_2) + E_{\lambda,M}(x_2,x_3) + \cdots + E_{\lambda,M}(x_{n-1},x_n)) + n\delta) \\ \geq\ & M(x_1, x_2, E_{\lambda,M}(x_1,x_2) + \delta) * \cdots * M(x_{n-1}, x_n, E_{\lambda,M}(x_{n-1},x_n) + \delta) \\ \geq\ & \overbrace{(1-\lambda) * (1-\lambda) * \cdots * (1-\lambda)}^{n} \geq 1-\mu \end{aligned}$$

for very $\delta > 0$, which implies that

$$E_{\mu,M}(x_1,x_n) \leq E_{\lambda,M}(x_1,x_2) + E_{\lambda,M}(x_2,x_3) + \cdots + E_{\lambda,M}(x_{n-1},x_n) + n\delta.$$

Since $\delta > 0$ is arbitrary, we have

$$E_{\mu,M}(x_1,x_n) \leq E_{\lambda,M}(x_1,x_2) + E_{\lambda,M}(x_2,x_3) + \cdots + E_{\lambda,M}(x_{n-1},x_n).$$

(ii) Note that since M is continuous in its third place and

$$E_{\lambda,M}(x,y) = \inf\{t > 0 \ :\ M(x,y,t) > 1-\lambda\}.$$

Hence, we have

$$M(x_n, x, \eta) > 1-\lambda \Longleftrightarrow E_{\lambda,M}(x_n,x) < \eta$$

for every $\eta > 0$. □

Lemma 2.5. *Let* $(X,M,*)$ *be a fuzzy metric space. If*

$$M(x_n, x_{n+1}, t) \geq M(x_0, x_1, k^n t)$$

for some $k > 1$ *and for every* $n \in \mathbb{N}$. *Then sequence* $\{x_n\}$ *is a Cauchy sequence.*

Proof. For every $\lambda \in (0,1)$ and $x_n, x_{n+1} \in X$, we have

$$\begin{aligned} E_{\lambda,M}(x_{n+1},x_n) &= \inf\{t > 0 \ :\ M(x_{n+1},x_n,t) > 1-\lambda\} \\ &\leq \inf\{t > 0 \ :\ M(x_0,x_1,k^n t) > 1-\lambda\} \\ &= \inf\{\frac{t}{k^n} \ :\ M(x_0,x_1,t) > 1-\lambda\} \\ &= \frac{1}{k^n} \inf\{t > 0 \ :\ M(x_0,x_1,t) > 1-\lambda\} \\ &= \frac{1}{k^n} E_{\lambda,M}(x_0,x_1). \end{aligned}$$

By Lemma 2.4, for every $\mu \in (0,1)$ there exists $\lambda \in (0,1)$ such that

$$\begin{aligned} E_{\mu,M}(x_n, x_m) &\leq E_{\lambda,M}(x_n, x_{n+1}) + E_{\lambda,M}(x_{n+1}, x_{n+2}) \\ &\quad + \cdots + E_{\lambda,M}(x_{m-1}, x_m) \\ &\leq \frac{1}{k^n} E_{\lambda,M}(x_0, x_1) + \frac{1}{k^{n+1}} E_{\lambda,M}(x_0, x_1) \\ &\quad + \cdots + \frac{1}{k^{m-1}} E_{\lambda,M}(x_0, x_1) \\ &= E_{\lambda,M}(x_0, x_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \longrightarrow 0. \end{aligned}$$

Hence sequence $\{x_n\}$ is Cauchy sequence. □

3. The main results

Lemma 3.1. *Let P and Q be self-mappings of a complete fuzzy metric space $(X, M, *)$ satisfying*:

(i) *there exists a constant $k \in (0,1)$ such that*

$$\begin{aligned} &M^2(Px, Qy, kt) * [M(x, Px, kt)M(y, Qy, kt)] \\ &\quad * M^2(y, Qy, kt) + aM(y, Qy, kt)M(x, Qy, 2kt) \\ &\geq [pM(x, Px, t) + qM(x, y, t)]M(x, Qy, 2kt) \end{aligned}$$

for every x, y in X and $t > 0$, where $0 < p, q < 1$, $0 \leq a < 1$ such that $p + q - a = 1$. Then P and Q have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point, there exist $x_1 \in X$ such that $Px_0 = x_1$, $Qx_1 = x_2$. Inductively, construct sequence $\{x_n\}$ in X such that $x_{2n+1} = Px_{2n}$, $x_{2n+2} = Qx_{2n+1}$ for $n = 0, 1, 2, \ldots$.

Now, we prove $\{x_n\}$ is a Cauchy sequence. For $x = x_{2n}$ and $y = x_{2n+1}$ by (i) we have

$$\begin{aligned} &M^2(Px_{2n}, Qx_{2n+1}, kt) * [M(x_{2n}, Px_{2n}, kt)M(x_{2n+1}, Qx_{2n+1}, kt)] \\ &\quad * M^2(x_{2n+1}, Qx_{2n+1}, kt) + aM(x_{2n+1}, Qx_{2n+1}, kt)M(x_{2n}, Qx_{2n+1}, 2kt) \\ &\geq [pM(x_{2n}, Px_{2n}, t) + qM(x_{2n}, x_{2n+1}, t)]M(x_{2n}, Qx_{2n+1}, 2kt) \end{aligned}$$

and

$$\begin{aligned} &M^2(x_{2n+1}, x_{2n+2}, kt) * [M(x_{2n}, x_{2n+1}, kt)M(x_{2n+1}, x_{2n+2}, kt)] \\ &\quad * M^2(x_{2n+1}, x_{2n+2}, kt) + aM(x_{2n+1}, x_{2n+2}, kt)M(x_{2n}, x_{2n+2}, 2kt) \\ &\geq [pM(x_{2n}, x_{2n+1}, t) + qM(x_{2n}, x_{2n+1}, t)]M(x_{2n}, x_{2n+2}, 2kt) \end{aligned}$$

then

$$M^2(x_{2n+1}, x_{2n+2}, kt) * [M(x_{2n}, x_{2n+1}, kt)M(x_{2n+1}, x_{2n+2}, kt)] \\ + aM(x_{2n+1}, x_{2n+2}, kt)M(x_{2n}, x_{2n+2}, 2kt) \\ \geq (p+q)M(x_{2n}, x_{2n+1}, t)M(x_{2n}, x_{2n+2}, 2kt).$$

Hence we have

$$M(x_{2n+1}, x_{2n+2}, kt)M(x_{2n}, x_{2n+2}, 2kt) \\ + aM(x_{2n+1}, x_{2n+2}, kt)M(x_{2n}, x_{2n+2}, 2kt) \\ \geq (p+q)M(x_{2n}, x_{2n+1}, t)M(x_{2n}, x_{2n+2}, 2kt)$$

so

$$M(x_{2n+1}, x_{2n+2}, kt) + aM(x_{2n+1}, x_{2n+2}, kt) \geq (p+q)M(x_{2n}, x_{2n+1}, t)$$

and

$$M(x_{2n+1}, x_{2n+2}, kt) \geq M(x_{2n}, x_{2n+1}, t) \cdot \frac{p+q}{1+a} = M(x_{2n}, x_{2n+1}, t).$$

Hence we have

$$M(x_{2n+1}, x_{2n+2}, kt) \geq M(x_{2n}, x_{2n+1}, t).$$

Similarly, we also have

$$M(x_{2n+2}, x_{2n+3}, kt) \geq M(x_{2n+1}, x_{2n+2}, t).$$

For $k \in (0,1)$ if set $k_1 = \frac{1}{k} > 1$ and set $t = k_1 t_1$, then we have

$$M(x_n, x_{n+1}, t_1) \geq M(x_{n-1}, x_n, k_1 t_1) \geq \cdots \geq M(x_0, x_1, k_1^n t_1).$$

Hence by Lemma 2.5 $\{x_n\}$ is Cauchy and the completeness of X, $\{x_n\}$ converges to z in X. That is, $\lim_{n\to\infty} x_n = z$. Hence

$$\lim_{n\to\infty} Px_{2n} = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} Qx_{2n+1} = z.$$

Now, taking $x = z$ and $y = x_{2n+1}$ in (i), we have

$$M^2(Pz, Qx_{2n+1}, kt) * [M(z, Pz, kt)M(x_{2n+1}, Qx_{2n+1}, kt)] \\ * M^2(x_{2n+1}, Qx_{2n+1}, kt) + aM(x_{2n+1}, Qx_{2n+1}, kt)M(z, Qx_{2n+1}, 2kt) \\ \geq [pM(z, Pz, t) + qM(z, x_{2n+1}, t)]M(z, Qx_{2n+1}, 2kt)$$

as $n \longrightarrow \infty$

$$M^2(Pz, z, kt) * [M(z, Pz, kt)M(z, z, kt)] \\ * M^2(z, z, kt) + aM(z, z, kt)M(z, z, 2kt) \\ \geq [pM(z, Pz, t) + qM(z, z, t)]M(z, z, 2kt)$$

then, it follows that

$$M(Pz, z, t) + a \geq pM(Pz, z, t) + q$$

thus

$$M(Pz,z,t) \geq \frac{q-a}{1-p} = 1$$

for all $t > 0$, so $Pz = z$. Taking $x = x_{2n}$ and $y = z$ in (i), we have

$$\begin{aligned} M^2(Px_{2n}, Qz, kt) * [M(x_{2n}, Px_{2n}, kt)M(z, Qz, kt)] \\ * M^2(z, Qz, kt) + aM(z, Qz, kt)M(x_{2n}, Qz, 2kt) \\ \geq [pM(x_{2n}, Px_{2n}, t) + qM(x_{2n}, z, t)]M(x_{2n}, Qz, 2kt) \end{aligned}$$

as $n \longrightarrow \infty$

$$\begin{aligned} M^2(z, Qz, kt) * [M(z, z, kt)M(z, Qz, kt)] \\ * M^2(z, Qz, kt) + aM(z, Qz, kt)M(z, Qz, 2kt) \\ \geq [pM(z, z, t) + qM(z, z, t)]M(z, Qz, 2kt) \end{aligned}$$

then

$$M(z, Qz, kt)M(z, Qz, 2kt) + aM(z, Qz, 2kt) \geq [p+q]M(z, Qz, 2kt).$$

$$M(z, Qz, t) + a \geq p + q$$

and

$$M(z, Qz, t) \geq p + q - a = 1$$

for all $t > 0$, so $Qz = z$.

Therefore, z is a common fixed of P and Q.

Uniqueness, let v be second common fixed point of P and Q. Then using inequality (i), we have

$$\begin{aligned} M^2(Pz, Qv, kt) * [M(z, Pz, kt)M(v, Qv, kt)] \\ * M^2(v, Qv, kt) + aM(v, Qv, kt)M(z, Qv, 2kt) \\ \geq [pM(z, Pz, t) + qM(z, v, t)]M(z, Qv, 2kt) \end{aligned}$$

so

$$M^2(z, v, kt) + aM(z, v, 2kt) \geq [p + qM(z, v, t)]M(z, v, 2kt)$$

and

$$M(z, v, t)M(z, v, 2kt) + aM(z, v, 2kt) \geq [p + qM(z, v, t)]M(z, v, 2kt).$$

Thus, it follows that

$$M(z, v, t) \geq \frac{p-a}{1-q} = 1$$

for all $t > 0$, so $z = v$. Hence P and Q have unique common fixed point. □

Theorem 3.2. *Let P, S, T and Q be self-mappings of a complete fuzzy metric space $(X, M, *)$ satisfying*:

(i) $P(X) \subseteq T(X),\ Q(X) \subseteq S(X)$,

(ii) *there exists a constant $k \in (0, 1)$ such that*

$$\begin{aligned}&M^2(Px, Qy, kt) * [M(Sx, Px, kt)M(Ty, Qy, kt)]\\&* M^2(Ty, Qy, kt) + aM(Ty, Qy, kt)M(Sx, Qy, 2kt)\\&\geq [pM(Sx, Px, t) + qM(Sx, Ty, t)]M(Sx, Qy, 2kt)\end{aligned}$$

for every x, y in X and $t > 0$, where $0 < p, q < 1$, $0 \leq a < 1$ such that $p + q - a = 1$,

(iii) *the pairs (P, S) and (Q, T) are weak compatible of type (γ).*

Then P, S, Q and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point as $P(X) \subseteq T(X), Q(X) \subseteq S(X)$, there exist $x_1, x_2 \in X$ such that $Px_0 = Tx_1 = y_1$, $Qx_1 = Sx_2 = y_2$. Inductively, construct sequence $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n+1} = Px_{2n} = Tx_{2n+1}$, $y_{2n+2} = Qx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, \ldots$.

Now, we prove $\{y_n\}$ is a Cauchy sequence. For $x = x_{2n}$ and $y = x_{2n+1}$ by (ii) we have

$$\begin{aligned}&M^2(Px_{2n}, Qx_{2n+1}, kt) * [M(Sx_{2n}, Px_{2n}, kt)M(Tx_{2n+1}, Qx_{2n+1}, kt)]\\&*M^2(Tx_{2n+1}, Qx_{2n+1}, kt) + aM(Tx_{2n+1}, Qx_{2n+1}, kt)M(Sx_{2n}, Qx_{2n+1}, 2kt)\\&\geq [pM(Sx_{2n}, Px_{2n}, t) + qM(Sx_{2n}, Tx_{2n+1}, t)]M(Sx_{2n}, Qx_{2n+1}, 2kt)\end{aligned}$$

and

$$\begin{aligned}&M^2(y_{2n+1}, y_{2n+2}, kt) * [M(y_{2n}, y_{2n+1}, kt)M(y_{2n+1}, y_{2n+2}, kt)]\\&* M^2(y_{2n+1}, y_{2n+2}, kt) + aM(y_{2n+1}, y_{2n+2}, kt)M(y_{2n}, y_{2n+2}, 2kt)\\&\geq [pM(y_{2n}, y_{2n+1}, t) + qM(y_{2n}, y_{2n+1}, t)]M(y_{2n}, y_{2n+2}, 2kt)\end{aligned}$$

then

$$\begin{aligned}&M^2(y_{2n+1}, y_{2n+2}, kt) * [M(y_{2n}, y_{2n+1}, kt)M(y_{2n+1}, y_{2n+2}, kt)]\\&+ aM(y_{2n+1}, y_{2n+2}, kt)M(y_{2n}, y_{2n+2}, 2kt)\\&\geq (p + q)M(y_{2n}, y_{2n+1}, t)M(y_{2n}, y_{2n+2}, 2kt).\end{aligned}$$

Hence we have

$$\begin{aligned}&M(y_{2n+1}, y_{2n+2}, kt)M(y_{2n}, y_{2n+2}, 2kt)\\&+ aM(y_{2n+1}, y_{2n+2}, kt)M(y_{2n}, y_{2n+2}, 2kt)\\&\geq (p + q)M(y_{2n}, y_{2n+1}, t)M(y_{2n}, y_{2n+2}, 2kt)\end{aligned}$$

so

$$M(y_{2n+1}, y_{2n+2}, kt) + aM(y_{2n+1}, y_{2n+2}, kt) \geq (p + q)M(y_{2n}, y_{2n+1}, t)$$

and

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) \cdot \frac{p+q}{1+a} = M(y_{2n}, y_{2n+1}, t).$$

Hence we have

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t).$$

Similarly, we also have

$$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t).$$

For $k \in (0,1)$ if set $k_1 = \frac{1}{k} > 1$ and set $t = k_1 t_1$, then we have

$$M(y_n, y_{n+1}, t_1) \geq M(y_{n-1}, y_n, k_1 t_1) \geq \cdots \geq M(y_0, y_1, k_1^n t_1).$$

Hence by Lemma 2.5 $\{y_n\}$ is Cauchy and the completeness of X, $\{y_n\}$ converges to z in X. That is, $\lim_{n\to\infty} y_n = z$. Hence

$$\begin{aligned} \lim_{n\to\infty} Px_{2n} &= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} \\ &= \lim_{n\to\infty} y_{2n+2} = \lim_{n\to\infty} Qx_{2n+1} \\ &= \lim_{n\to\infty} Sx_{2n+2} = \lim_{n\to\infty} Sx_{2n} = z. \end{aligned}$$

Since P, S are weak compatible of type (γ) we get $Pz = Sz$. Now, taking $x = z$ and $y = x_{2n+1}$ in (ii), we have

$$\begin{aligned} &M^2(Pz, Qx_{2n+1}, kt) * [M(Sz, Pz, kt) M(Tx_{2n+1}, Qx_{2n+1}, kt)] \\ &* M^2(Tx_{2n+1}, Qx_{2n+1}, kt) + aM(Tx_{2n+1}, Qx_{2n+1}, kt) M(Sz, Qx_{2n+1}, 2kt) \\ &\geq [pM(Sz, Pz, t) + qM(Sz, Tx_{2n+1}, t)] M(Sz, Qx_{2n+1}, 2kt) \end{aligned}$$

as $n \longrightarrow \infty$

$$\begin{aligned} &M^2(Pz, z, kt) * [M(Sz, Pz, kt) M(z, z, kt)] \\ &* M^2(z, z, kt) + aM(z, z, kt) M(Sz, z, 2kt) \\ &\geq [pM(Sz, Pz, t) + qM(Sz, z, t)] M(Sz, z, 2kt) \end{aligned}$$

then, it follows that

$$M^2(Pz, z, kt) + aM(Pz, z, 2kt) \geq [p + qM(Pz, z, t)] M(Pz, z, 2kt)$$

and since $M(x, y, \cdot)$ is non-decreasing for all x, y in X, we have

$$M(Pz, z, 2kt) M(Pz, z, t) + aM(Pz, z, 2kt) \geq [p + qM(Pz, z, t)] M(Pz, z, 2kt)$$

thus

$$M(Pz, z, t) + a \geq p + qM(Pz, z, t)$$

and

$$M(Pz, z, t) \geq \frac{p-a}{1-q} = 1$$

for all $t > 0$ so $Pz = z$. Therefore $Pz = Sz = z$.

Similarly since pair Q, T are weak compatible of type (γ) hence we get $Qz = Tz$. Now, we show that $Qz = z$. For this taking $x = x_{2n}$ and $y = z$ in (ii), we have

$$\begin{aligned} &M^2(Px_{2n}, Qz, kt) * [M(Sx_{2n}, Px_{2n}, kt)M(Tz, Qz, kt)] \\ &* M^2(Tz, Qz, kt) + aM(Tz, Qz, kt)M(Sx_{2n}, Qz, 2kt) \\ &\geq [pM(Sx_{2n}, Px_{2n}, t) + qM(Sx_{2n}, Tz, t)]M(Sx_{2n}, Qz, 2kt) \end{aligned}$$

as $n \longrightarrow \infty$

$$\begin{aligned} &M^2(z, Qz, kt) * [M(z, z, kt)M(Tz, Qz, kt)] \\ &* M^2(Tz, Qz, kt) + aM(Tz, Qz, kt)M(z, Qz, 2kt) \\ &\geq [pM(z, z, t) + qM(z, Tz, t)]M(z, Qz, 2kt) \end{aligned}$$

then

$$M^2(z, Qz, kt) + aM(z, Qz, 2kt) \geq [p + qM(z, Tz, t)]M(z, Qz, 2kt).$$

Since $M(x, y, \cdot)$ is non-decreasing for all x, y in X, we have

$$M(z, Qz, kt)M(z, Qz, 2kt) + aM(z, Qz, 2kt) \geq [p + qM(z, Tz, t)]M(z, Qz, 2kt).$$

Thus it follows that

$$M(z, Qz, t) + a \geq p + qM(z, Tz, t)$$

and

$$M(z, Qz, t) \geq \frac{p - a}{1 - q} = 1$$

for all $t > 0$ so $Qz = z$. Hence, we have $Qz = Tz = z$.

Therefore, z is a common fixed of P, Q, S and T.

Uniqueness, let v be second common fixed point of P, Q, S and T. Then using inequality (ii), we have

$$\begin{aligned} &M^2(Pz, Qv, kt) * [M(Sz, Pz, kt)M(Tv, Qv, kt)] \\ &* M^2(Tv, Qv, kt) + aM(Tv, Qv, kt)M(Sz, Qv, 2kt) \\ &\geq [pM(Sz, Pz, t) + qM(Sz, Tv, t)]M(Sz, Qv, 2kt) \end{aligned}$$

so

$$M^2(z, v, kt) + aM(z, v, 2kt) \geq [p + qM(z, v, t)]M(z, v, 2kt)$$

and

$$M(z, v, t)M(z, v, 2kt) + aM(z, v, 2kt) \geq [p + qM(z, v, t)]M(z, v, 2kt).$$

Thus, it follows that

$$M(z, v, t) \geq \frac{p - a}{1 - q} = 1$$

for all $t > 0$ so $z = v$. Hence P, Q, S and T have unique common fixed point. □

A class of implicit relation

Let $\{S_\alpha\}_{\alpha\in A}$ and $\{T_\beta\}_{\beta\in B}$ be the set of all self-mappings of a complete fuzzy metric space $(X, M, *)$.

Theorem 3.3. *Let T, S and $\{P_\alpha\}_{\alpha\in A}, \{Q_\beta\}_{\beta\in B}$ be self-mappings of a complete fuzzy metric space $(X, M, *)$ satisfying*:

(i) *there exist $\alpha_0 \in A$ and $\beta_0 \in B$ such that $P_{\alpha_0}(X) \subseteq T(X)$, $Q_{\beta_0}(X) \subseteq S(X)$,*

(ii) *there exists a constant $k \in (0,1)$ such that*

$$\begin{aligned}&M^2(P_\alpha x, Q_\beta y, kt) * [M(Sx, P_\alpha x, kt)M(Ty, Q_\beta y, kt)]\\&* M^2(Ty, Q_\beta y, kt) + aM(Ty, Q_\beta y, kt)M(Sx, Q_\beta y, 2kt)\\ \geq\ &[pM(Sx, P_\alpha x, t) + qM(Sx, Ty, t)]M(Sx, Q_\beta y, 2kt)\end{aligned}$$

for every x, y in X and every $\alpha \in A, \beta \in B$ and $t > 0$, where $0 < p, q < 1$, $0 \leq a < 1$ such that $p + q - a = 1$.

(iii) *the pairs (P_{α_0}, S) and (Q_{β_0}, T) are weak compatible of type (γ).*
Then P_α, S, Q_β and T have a unique common fixed point in X.

Proof. By Theorem 3.2 S, T, P_{α_0} and Q_{β_0} for some $\alpha_0 \in A, \beta_0 \in B$ have a unique common fixed point in X. That is there exist a unique $a \in X$ such that $T(a) = S(a) = P_{\alpha_0}(a) = Q_{\beta_0}(a) = a$. Let there exists $\lambda \in B$ such that $\lambda \neq \beta_0$ then we have

$$\begin{aligned}&M^2(P_{\alpha_0}a, Q_\lambda a, kt) * [M(Sa, P_{\alpha_0}a, kt)M(Ta, Q_\lambda a, kt)]\\&* M^2(Ta, Q_\lambda a, kt) + aM(Ta, Q_\lambda a, kt)M(Sa, Q_\lambda a, 2kt)\\ \geq\ &[pM(Sa, P_{\alpha_0}a, t) + qM(Sa, Ta, t)]M(Sa, Q_\lambda a, 2kt)\end{aligned}$$

then

$$\begin{aligned}&M^2(a, Q_\lambda a, kt) * [M(a, a, kt)M(a, Q_\lambda a, kt)]\\&* M^2(a, Q_\lambda a, kt) + aM(a, Q_\lambda a, kt)M(a, Q_\lambda a, 2kt)\\ \geq\ &[pM(a, a, t) + qM(a, a, t)]M(a, Q_\lambda a, 2kt)\end{aligned}$$

so

$$\begin{aligned}&M^2(a, Q_\lambda a, kt) * M(a, Q_\lambda a, kt)\\&* M^2(a, Q_\lambda a, kt) + aM(a, Q_\lambda a, kt)M(a, Q_\lambda a, 2kt)\\ \geq\ &[p + q]M(a, Q_\lambda a, 2kt).\end{aligned}$$

Since $M(x, y, \cdot)$ is non-decreasing for all x, y in X, we have

$$\begin{aligned}&M(a, Q_\lambda a, kt)M(a, Q_\lambda a, 2kt)\\&+ aM(a, Q_\lambda a, kt)M(a, Q_\lambda a, 2kt)\\ \geq\ &[p + q]M(a, Q_\lambda a, 2kt).\end{aligned}$$

That is

$$M(a, Q_\lambda a, kt) \geq \frac{p+q}{1+a} = 1.$$

Hence for every $\lambda \in B$ we have $Q_\lambda(a) = a = T(a) = S(a)$. Similarly for every $\gamma \in A$ we get $P_\gamma(a) = a$. Therefore for every $\gamma \in A, \lambda \in B$ we have

$$P_\gamma(a) = Q_\lambda(a) = T(a) = S(a) = a. \qquad \square$$

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