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SOME INEQUALITIES FOR BIVARIATE MEANS

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Abstract. In the paper, some new inequalities for certain bivariate means are obtained, which extend some known results.

1. Introduction

The logarithmic and identric means of two positive numbers a and b are defined by

$$L \equiv L(a,b) = \begin{cases} \frac{b-a}{\log b - \log a} & a \neq b, \\ a & a = b \end{cases}$$

and

$$I \equiv I(a,b) = \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} & a \neq b, \\ a & a = b \end{cases}$$

respectively. Let

$$A_k \equiv A_k(a,b) = \left(\frac{a^k + b^k}{2}\right)^{1/k}$$

denote the power mean of order $k \neq 0$ of a and b. In particular, the arithmetic and geometric mean of a and b are

$$A \equiv A_1(a,b) = \frac{a+b}{2}, \quad G \equiv \lim_{k\to 0} A_k(a,b) = \sqrt{ab}.$$

There are many remarkable inequalities and identities for all means defined above have been established and studied extensively by many researchers (see [1]-[15]).

For instance, Stolarsky [14] proved that for all $a \neq b$ one has

$$A_{2/3} < I,$$

and that the order 2/3 of the power mean is the best one. Bullen [2] obtained

$$L < I < A,$$

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Sándor gave

$$A < \frac{e}{2} I,$$

Neuman-Sándor [7] got

$$I < \frac{2}{e}(A+G)$$

and recently, Neuman-Sándor [7] proved the following companion inequalities,

$$I < A < \frac{e}{2} I, \quad A_{2/3} < I < \frac{2\sqrt{2}}{e} A_{2/3}$$

where the constants above are best possible.

Next, we introduce the weighted geometric mean S (see, e.g., [8, 9, 12]) of a and b with weights $a/(a+b)$ and $b/(a+b)$:

$$S \equiv S(a,b) = a^{a/(a+b)} b^{b/(a+b)}.$$

The Heronian mean denoted by He (see [2]) and defined as follows

$$He \equiv He(a,b) = \frac{a + \sqrt{ab} + b}{3} = \frac{2A+G}{3}.$$

Neuman-Sándor [7] obtained two companion inequalities

$$He < I < \frac{3}{e} He, \quad \text{and} \quad A_2 < S < \sqrt{2} A_2.$$

In the present paper, we will establish some new inequalities for bivariate means which extend some known results.

2. Main results

In what follows, without loss of generality, we will assume that $b > a > 0$.

Theorem 2.1. *Let $0 < k \le 1$, then we have*

$$A_k(a,b) > a^{1-k} I(a^k, b^k)$$

where the constant 1 *is best possible.*

Proof. Let $x = b/a$ and

$$f(x) = \frac{A_k(x,1)}{I(x^k,1)},$$

then we have

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{A_k'(x,1)}{A_k(x,1)} - \frac{I'(x^k,1)}{I(x^k,1)} \\ &= \frac{x^{k-1}}{x^k+1} + \frac{kx^{k-1}}{1-x^k} + \frac{k^2 x^{k-1} \log x}{(1-x^k)^2} \\ &= \frac{x^{k-1}}{(1-x^k)^2(1+x^k)} \left[1 + k + (1-k)x^{2k} - 2x^k + k^2(x^k+1)\log x\right]. \end{aligned}$$

Let

$$g(x) = 1 + k + (1-k)x^{2k} - 2x^k + k^2(x^k+1)\log x$$

then

$$g'(x) = \frac{k}{x}\left[2(1-k)x^{2k} - (2-k)x^k + k^2x^k\log x + k\right].$$

Let

$$h(x) = 2(1-k)x^{2k} - (2-k)x^k + k^2x^k\log x + k,$$

then

$$h'(x) = kx^{k-1}\left[4(1-k)x^k - 2(1-k) + k^2\log x\right].$$

For $0 < k \le 1$, $x > 1$, it is easy to check that

$$h'(x) > 0, \quad \lim_{x\to 1} h(x) = 0$$

which implies that

$$g'(x) > 0, \quad \forall \ x > 1.$$

By $g(1) = 0$, we have $g(x) > 0$. Then it follows that

$$f'(x) > 0, \text{ for } x > 1, \ 0 < k \le 1.$$

Thus $f(x)$ is strictly increasing and

$$f(x) > \lim_{x\to 1} f(x) = 1.$$

Moreover, as $x \to \infty$, we have

$$A_k(x,1) = \frac{x}{2^{1/k}} + o\,(x)\,, \quad I(x^k,1) = \frac{1}{e}x^k + o(x^k), \tag{2.1}$$

then

$$f(x) < \lim_{x\to\infty} f(x) = +\infty.$$

Since $f(x)$ is continuous for $x > 1$, it follows that the constant 1 is best possible. Furthermore, by noting

$$A_k(x,1) = \left(\frac{\left(\frac{b}{a}\right)^k + 1}{2}\right)^{\frac{1}{k}} = \frac{1}{a}\left(\frac{b^k + a^k}{2}\right)^{\frac{1}{k}} = \frac{1}{a}A_k(a,b) \tag{2.2}$$

and

$$I(x^k,1) = \frac{1}{e}\left(\frac{1}{\left(\frac{b}{a}\right)^{k\left(\frac{b}{a}\right)^k}}\right)^{\frac{1}{1-\left(\frac{b}{a}\right)^k}} = \frac{1}{e}\left(\frac{b^{kb^k}}{a^{ka^k}}\right)^{\frac{1}{b^k-a^k}}\frac{1}{a^k} = \frac{1}{a^k}I(a^k,b^k), \tag{2.3}$$

the desired result can be obtained. □

Remark 2.1. If taking $k = 1$, then we have the following well-known inequality (e.g., [2])

$$I(a,b) < A(a,b).$$

Furthermore, from (2.1), we can obtain the following companion inequality (e.g., [7])

$$I(a,b) < A(a,b) < \frac{e}{2}I(a,b).$$

Theorem 2.2. *Let $0 \le k \le 1/2$, we have*

$$A_k(a,b) < I(a,b),$$

where the constant 1 *is best possible.*

Proof. Let $x = b/a$ and

$$f(x) = \frac{A_k(x,1)}{I(x,1)},$$

then we have

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{A_k'(x,1)}{A_k(x,1)} - \frac{I'(x,1)}{I(x,1)}\\ &= \frac{x^{k-1}}{x^k+1} + \frac{1}{1-x} + \frac{\log x}{(1-x)^2}\\ &= \frac{x^{k-1} - x^k + 1 - x + \log x + x^k \log x}{(1-x)^2(1+x^k)}.\end{aligned}$$

Let $f_1(x) = x^{k-1} - x^k + 1 - x + \log x + x^k \log x$, then

$$\begin{aligned}f_1'(x) &= (k-1)x^{k-2} + x^{-1} + x^{k-1} + kx^{k-1}\log x - 1 - kx^{k-1}\\ &= \frac{1}{x^{1-k}}\left\{1 + k\log x + x^{-k} - k - (1-k)x^{-1} - x^{1-k}\right\}.\end{aligned}$$

In addition, let $f_2(x) = 1 + k\log x + x^{-k} - k - (1-k)x^{-1} - x^{1-k}$, then

$$\begin{aligned}f_2'(x) &= \frac{k}{x} - kx^{-k-1} + (1-k)x^{-2} - (1-k)x^{-k}\\ &= k\left(\frac{1}{x} - \frac{1}{x^{k+1}}\right) - (1-k)\left(\frac{1}{x^k} - \frac{1}{x^2}\right)\\ &\le (2k-1)\left(\frac{1}{x} - \frac{1}{x^{k+1}}\right) \le 0.\end{aligned}$$

Since $f_2(1) = 0$, then $f_2(x) \le 0$, which implies ${f_1}'(x) \le 0$ and $f'(x) \le 0$. Thus $f(x)$ is decreasing, and it follows that $f(x) \le 1$ by $\lim_{x\to 1} f(x) = 1$. The desired result can be obtained. □

Theorem 2.3. *Let $\beta \ge 2/3$, then for any $k > 0$, we have*

$$He(a^k,b^k) < A_\beta(a^k,b^k) < \frac{3}{2^{1/\beta}} He(a^k,b^k),$$

where the constants 1 *and* $\frac{3}{2^{1/\beta}}$ *are best possible.*

Proof. Let

$$f(x) = \frac{He(x^k,1)}{A_\beta(x^k,1)},$$

then we have

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{He'(x^k,1)}{He(x^k,1)} - \frac{A'_\beta(x^k,1)}{A_\beta(x^k,1)} \\ &= \frac{kx^{k-1}+\frac{k}{2}x^{\frac{k}{2}-1}}{x^k+x^{\frac{k}{2}}+1} - \frac{kx^{k\beta-1}}{1+x^{k\beta}} \\ &= \frac{k}{2}\left(\frac{2x^{k-1}+x^{\frac{k}{2}-1}-x^{k(\beta+1/2)-1}-2x^{k\beta-1}}{(x^k+x^{\frac{k}{2}}+1)(1+x^{k\beta})}\right) \\ &= \frac{k}{2}\frac{x^{k-1}}{x^{k/2}}\left(\frac{2x^{k/2}+1-x^{k\beta}-2x^{k\beta-\frac{k}{2}}}{(x^k+x^{\frac{k}{2}}+1)(1+x^{k\beta})}\right).\end{aligned}$$

Let

$$g(x) = 2x^{k/2}+1-x^{k\beta}-2x^{k\beta-\frac{k}{2}}$$

then it follows that

$$\begin{aligned}g'(x) &= k\left[x^{\frac{k}{2}-1}-\beta x^{k\beta-1}-2\left(\beta-\frac{1}{2}\right)x^{k\beta-\frac{k}{2}-1}\right] \\ &= kx^{\frac{k}{2}-1}\left[1-\beta x^{k\beta-\frac{k}{2}}-2\left(\beta-\frac{1}{2}\right)x^{k\beta-k}\right].\end{aligned}$$

Let

$$h(x) = 1-\beta x^{k\beta-\frac{k}{2}}-2\left(\beta-\frac{1}{2}\right)x^{k\beta-k}$$

then

$$\begin{aligned}h'(x) &= -k\left(\beta-\frac{1}{2}\right)x^{k\beta-k-1}\left(\beta x^{\frac{k}{2}}+2(\beta-1)\right) \\ &\le -k\left(\beta-\frac{1}{2}\right)x^{k\beta-k-1}\left(\beta+2(\beta-1)\right).\end{aligned}$$

If $\beta > 2/3$, then $h'(x) < 0$. From above discussions, we have the following claims

$$h(x) < h(1) = 2-3\beta < 0, \quad g'(x) < 0, \quad g(x) < g(1) = 0.$$

Hence, $f'(x) < 0$, and $f(x)$ is strictly decreasing for all $x > 1$. If $\beta = 2/3$, by similar discussion, $f(x)$ is strictly decreasing for all $x > 1$. Furthermore, it is easy to check that

$$\lim_{x\to 1} f(x) = 1, \quad \text{and} \quad \lim_{x\to\infty} f(x) = \frac{2^{1/\beta}}{3},$$

which implies our result. □

Remark 2.2. Here if taking $k = 1$ and $\beta = 2/3$, then we have the following known inequality [7],

$$He(a,b) < A_{2/3}(a,b) < \frac{3}{2\sqrt{2}}He(a,b).$$

Theorem 2.4. *Let $1 \le k \le 2$, then we have*

$$A_k(a,b) < S < 2^{1/k} A_k(a,b),$$

where the constants 1 *and* $2^{1/k}$ *are best possible.*

Proof. Let

$$f(x) = \frac{A_k(x,1)}{S(x,1)},$$

then

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{A_k'(x,1)}{A_k(x,1)} - \frac{S'(x,1)}{S(x,1)} \\ &= \frac{x^{k-1}}{x^k+1} - \frac{1}{1+x} - \frac{\log x}{(1+x)^2} \\ &= \frac{x^k + x^{k-1} - x - 1 - (1+x^k)\log x}{(x^k+1)(1+x)^2} \\ &=: \ \frac{g(x)}{(x^k+1)(1+x)^2},\end{aligned}$$

$$g'(x) = kx^{k-1} + (k-1)x^{k-2} - 1 - kx^{k-1}\log x - \frac{1+x^k}{x}$$

and

$$\begin{aligned}g''(x) &= (k-1)(k-2)x^{k-3} + x^{k-2}\left[k^2 - 3k + 1 - k(k-1)\log x\right] + \frac{1}{x^2} \\ &= x^{k-2}\left\{(k-1)(k-2)x^{-1} + \left[k^2 - 3k + 1 - k(k-1)\log x\right] + x^{-k}\right\}.\end{aligned}$$

Let

$$h(x) = (k-1)(k-2)x^{-1} + \left[k^2 - 3k + 1 - k(k-1)\log x\right] + x^{-k}$$

then we have

$$\begin{aligned}h'(x) &= -(k-1)(k-2)x^{-2} - k(k-1)x^{-1} - kx^{-k-1} \\ &= x^{-2}\left[-(k-1)(k-2) - k(k-1)x - kx^{-k+1}\right] =: x^{-2} l(x)\end{aligned}$$

and

$$l'(x) = k(k-1)(x^{-k} - 1).$$

By $1 \le k \le 2$ and $x > 1$, we can get the following claims

$$l'(x) < 0, \quad l(1) = -(2k^2 - 3k + 2) < 0, \quad h'(x) < 0,$$

$$h(x) < h(1) = 2(k^2 - 3k + 2) \le 0, \quad g''(x) < 0, \quad g'(x) < g'(1) \le 0.$$

Thus, $g(x)$ is strictly decreasing and, by $g(1) = 0$, we have $f'(x) < 0$ for all $x > 1$, that is to say, $f(x)$ is strictly decreasing. Furthermore,

$$\frac{1}{2^{1/k}} = \lim_{x\to\infty} f(x) < f(x) < \lim_{x\to 1} f(x) = 1,$$

which implies our result. □

Remark 2.3. By taking $k = 2$, we get the following know inequality [7],

$$A_2(a,b) < S < \sqrt{2}A(a,b).$$

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