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SOME REMARKS ON CENTERED-LINDELÖF SPACES

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Abstract. In this paper, we prove the following two statements: (1) There exists a Hausdorff locally Lindelöf centered-Lindelöf space that is not star-Lindelöf. (2) There exists a T_1 locally compact centered-Lindelöf space that is not star-Lindelöf. The two statements give a partial answer to Bonanzinga and Matveev [2, Question 1].

1. Introduction

By a space, we mean a topological space. A space X is *star-Lindelöf* (see [1], [5], [7]-under different names) if for every open cover $\mathcal{U}$ of X, there exists a countable subset B of X such that $St(B,\mathcal{U}) = X$, where $St(B,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap B \neq \emptyset\}$. It is clear that every separable space is star-Lindelöf as well as every space of countable extent (in particular, every countably compact space or every Lindelöf space).

A space X is *centered-Lindelöf* (*linked-Lindelöf, CCC-Lindelöf*) (see [1], [2], [3], [4], [8]) if every open cover $\mathcal{U}$ of X has a σ-centered (respectively, σ-linked, CCC) subcover. A family of sets is *centeed* (*linked*) if every finite subfamily (resp. every two elements) has a non-empty intersection and a family *σ-centered* (*σ-linked*) if it can be represented as the union of countably many centered-subfamilies (resp. linked-subfamilies). A family of non-empty sets is *a CCC-family* if there is no uncountable pairwise disjoint subfamily.

From the above definitions, it is clear that every star-Lindelöf space is centered-Lindelöf, every centered-Lindelöf space is linked-Lindelöf and every linked-Lindelöf space is CCC-Lindelöf.

A space X is *locally Lindelöf* (*locally compact*) if every point $x \in X$ has an open neighborhood U such that the closure of U is Lindelöf (resp. compact).

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Bonanzinga and Matveev [2] proved that every locally separable centered-Lindelöf space is star-Lindelöf, every Hausdorff locally compact centered-Lindelöf space is star-Lindelöf and asked if every Hausdorff locally compact linked-Lindelöf (or CCC-Lindelöf) space is star-Lindelöf.

The purpose of this paper is to construct two examples stated in the abstract which give a partial answer to the above question.

The cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal and $\mathfrak{c}$ the cardinality of the set of all real numbers. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. For each ordinals α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$ and $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols follow [6].

2. Main results

In this section, we construct two examples stated in the abstract.

Example 2.1. There exists a Hausdorff locally Lindelöf centered-Lindelöf space that is not star-Lindelöf.

Proof. Let $\mathcal{R}$ be a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. Define

$$X = \mathcal{R} \cup ((\mathfrak{c} + 1) \times \omega).$$

We topologize X as follows: $(\mathfrak{c} + 1) \times \omega$ has the usual product topology and is an open subspace of X. On the other hand, a basic neighborhood of $r \in \mathcal{R}$ takes the form

$$G_{\beta,K}(r) = (\{\alpha : \beta < \alpha < \mathfrak{c}\} \times (r \setminus K)) \cup \{r\}$$

for $\beta < \mathfrak{c}$ and a finite subset K of ω. Then, X is locally Lindelöf and Hausdorff, but is not regular, since each $r \in \mathcal{R}$ can not be separated from the closed subset $\{\langle \mathfrak{c}, n \rangle : n \in \omega\}$ by disjoint open subsets of X.

To show that X is centered-Lindelöf. Let $\mathcal{U}$ be an open cover of X. Since $(\mathfrak{c} + 1) \times \{n\}$ is compact for each $n \in \omega$, hence centered-Lindelöf, there exists a finite centered-subfamily $\mathcal{V}_n$ of covering $(\mathfrak{c} + 1) \times \{n\}$. Therefore, $(\mathfrak{c} + 1) \times \omega$ is covered by countably many centered subfamilies $\{\mathcal{V}_n : n \in \omega\}$ of $\mathcal{U}$. On the other hand, for each $r \in \mathcal{R}$, there exists $U_r \in \mathcal{U}$ such that $r \in U_r$. Then, there exist $n_r \in \omega$ and $\alpha_r < \mathfrak{c}$ such that

$$(\alpha_r, \mathfrak{c}) \times \{n_r\} \subseteq U_r.$$

For each $n \in \omega$, let $\mathcal{W}_n = \{U_r : n_r = n\}$. Then, $\mathcal{W}_n$ is a centered subfamily of $\mathcal{U}$ and $\mathcal{R} \subseteq \cup \bigcup \{\mathcal{W}_n : n \in \omega\}$. Hence, $\{\mathcal{V}_n : n \in \omega\} \cup \{\mathcal{W}_n : n \in \omega\}$ is σ-centered subcover of $\mathcal{U}$, which shows that X is centered-Lindelöf.

Next, we show that X is not star-Lindelöf. Since $|\mathcal{R}| = \mathfrak{c}$, we can enumerate $\mathcal{R}$ as $\{r_\alpha : \alpha < \mathfrak{c}\}$. For each $\alpha < \mathfrak{c}$, let

$$U_\alpha = \{r_\alpha\} \cup ((\alpha, \mathfrak{c}) \times r_\alpha).$$

Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{(\mathfrak{c}+1) \times \{n\} : n \in \omega\}$$

of X and let F be a countable subset of X. Since $|\mathcal{R}| = \mathfrak{c}$ and $F \cap \mathcal{R}$ is countable. Hence, there exists $\beta' < \mathfrak{c}$ such that

$$F \cap \{r_\alpha : \alpha > \beta'\} = \emptyset. \tag{1}$$

On the other hand, $F \cap (\mathfrak{c} \times \{n\})$ is bounded in $\mathfrak{c} \times \{n\}$ for each $n < \omega$. Thus, there exists $\beta_n < \mathfrak{c}$ such that

$$\beta_n > \sup\{\alpha < \mathfrak{c} : \langle \alpha, n \rangle \in F\}.$$

Pick $\beta'' < \mathfrak{c}$ such that $\beta'' > \beta_n$ for each $n \in \omega$. Then,

$$((\beta'', \mathfrak{c}) \times \omega) \cap F = \emptyset. \tag{2}$$

Choose $\gamma < \mathfrak{c}$ such that $\gamma > \max\{\beta', \beta''\}$. Then, U_γ is the only element of $\mathcal{U}$ containing the point r_γ and $U_\gamma \cap F = \emptyset$ by (1) and (2). It follows that $r_\gamma \notin St(F, \mathcal{U})$, which shows that X is not star-Lindelöf. □

Example 2.2. There exists a T_1-locally compact centered-Lindelöf space that is not star-Lindelöf.

Proof. Let $X = (\omega_1 + 1) \cup A$, where $A = \{a_\alpha\ \alpha \in \omega_1\}$ is a set of cardinality ω_1. We topologize X as follows: $\omega_1 + 1$ has the usual order topology and is an open subspace of X; a basic neighborhood of a point $a_\alpha \in A$ takes the form

$$O_\beta(a_\alpha) = \{a_\alpha\} \cup (\beta, \omega_1), \quad \text{where } \beta < \omega_1.$$

Then, X is a T_1-space.

First, we show that X is centered-Lindelöf, let $\mathcal{U}$ be an open cover of X. Without loss of generality, we can assume that $\mathcal{U}$ consists of basic open subsets of X. Since $\omega_1 + 1$ is compact, there is a countable subset F_1 of $\mathcal{U}$ such that $\omega_1 + 1 \subseteq St(F_1, \mathcal{U})$; On the other hand, for each $\alpha < \omega_1$, there is $U_\alpha \in \mathcal{U}$ such that $a_\alpha \in U_\alpha$, therefore $\{U_\alpha : \alpha < \omega_1\}$ is centered by the definition of the topology of X.

If we put

$$\bigcup_{x \in F_1} \{U \in \mathcal{U} : x \in U\} \cup \{U_\alpha : \alpha < \omega_1\}.$$

Then it is σ-centered subcover of $\mathcal{U}$, which shows that X is centered-Lindelöf.

Next, we show that X is not star-Lindelöf. Let us consider the open cover

$$\mathcal{V} = \{\omega_1 + 1\} \cup \{O_\alpha(a_\alpha) : \alpha < \omega_1\}.$$

Let F be a countable subset of X. Since $|A| = \omega_1$ and $F \cap A$ is countable, then there exists an $\alpha_1 < \omega_1$ such that $F \cap \{a_\alpha : \alpha > \alpha_1\} = \emptyset$; on the other hand, there exists $\alpha_2 < \omega_1$ such that $F \cap (\alpha_2, \omega_1) = \emptyset$, since $F \cap \omega_1$ is countable in ω_1. Choose $\beta > \max\{\alpha_1, \alpha_2\}$. Then $a_\beta \notin St(F, \mathcal{V})$, since $O_\beta(a_\beta)$ is only element of $\mathcal{V}$ containing a_β and $O_\beta \cap F = \emptyset$. This shows that X is not star-Lindelöf. □

Remark. Since every centered-Lindelöf space is linked-Lindelöf and every linked-Lindelöf space is CCC-Lindelöf, Examples 2.1 and 2.2 give a partial answer to Bonanzinga and Matveev [2, Question 1].

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