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CONGRUENCE PROPERTIES OF A DRINFELD MODULAR FUNCTION μ

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Abstract. The Drinfeld modular function μ is a generator of the function field of the Drinfeld modular curve $X_0(T)$ and has an t-expansion with the integral coefficients at infinity. In this paper, we show that the coefficients of μ has congruence properties modulo powers of T.

1. Introduction

Vincent Bosser [1] showed that the coefficients of the Drinfeld modular invariant j has congruence properties modulo powers of polynomials of degree 1 in $\mathbb{F}_q[T]$. It can be applied for a generator μ of the function field of the Drinfeld modular curve $X_0(T)$. The generator μ plays an important role in the study of $X_0(T)$ and the construction of class fields over function fields. Jeon and Kim [2] show that μ gives a plane model for $X_0(T)$ and the singular values of μ generate class fields over imaginary quadratic function fields.

In this paper, by using tools of Bosser we show that the coefficients of μ has congruence properties modulo powers of T.

2. Preliminaries

Let K be the rational function field $\mathbb{F}_q(T)$ over the finite field $\mathbb{F}_q$ of characteristic p and $A = \mathbb{F}_q[T]$. Let K_∞ be the completion of K at $\infty = (1/T)$ and C be the completion of an algebraic closure of K_∞. On K, we consider the degree valuation — deg associated with the infinite place ∞ of K, where $\deg : K \to \mathbb{Z} \cup \{-\infty\}$, $x \mapsto \deg x$. The corresponding absolute value $|\cdot|$ is normalized by $|T| = q$. There is a unique extension of $|\cdot|$ to C, labelled by the same symbol.

Let $\Omega = C - K_\infty$. Then the group $GL_2(A)$ acts on Ω in the following way: if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(A)$ and $z \in \Omega$, then

$$\gamma z = \frac{az+b}{cz+d}.$$

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Let Q be a monic polynomial of A. Consider the following Hecke congruence subgroup of $GL_2(A)$:

$$\Gamma_0(Q) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in GL_2(A) \mid c \equiv 0 \mod Q \}.$$

For each group $\Gamma_0(Q)$, the rigid analytic space $\Gamma_0(Q)\backslash\Omega$ is endowed with a unique structure of a smooth affine algebraic curve over C. We let $\overline{\Gamma_0(Q)\backslash\Omega}$ be its smooth projective model.

A cusp of $\overline{\Gamma_0(Q)\backslash\Omega}$ is a point of $\overline{\Gamma_0(Q)\backslash\Omega} - \Gamma_0(Q)\backslash\Omega$. Set-theoretically, we have $\overline{\Gamma_0(Q)\backslash\Omega} = \Gamma_0(Q)\backslash(\Omega \cup \mathbb{P}^1(K))$.

Let $L = \widetilde{\pi} A$ be the rank 1 A-lattice in C corresponding to the Carlitz module,

$$\rho_T = TX + X^q.$$

Let e_L be the exponential function associated to L, i.e.,

$$e_L : C \to C, \qquad e_L(z) := z \prod_{\lambda \in L - \{0\}} \left(1 - \frac{z}{\lambda}\right).$$

We define

$$t = t(z) := 1/e_L(\widetilde{\pi} z)$$

and

$$s = t^{q-1}.$$

A Drinfeld modular function for $\Gamma_0(Q)$ is a meromorphic function $f : \Omega \to C$ that satisfies:

(i) $f(\gamma z) = f(z)$ for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(Q)$,
(ii) f is meromorphic at the cusps of $\Gamma_0(Q)$.

We briefly explain the last condition. Let α be a cusp of $\Gamma_0(Q)$ and $v \in GL_2(K)$ with $v(\infty) = \alpha$. Now (ii) means that $f(vz)$ has a convergent series expansion with respect to a local parameter at α for $\overline{\Gamma_0(Q)\backslash\Omega}$.

Now we define a Drinfeld modular function μ for $\Gamma_0(T)$. Define the ath inverse cyclotomic polynomial $f_a(X) \in A[X]$ for $a \in A$ by

$$f_a(X) = \rho_a(X^{-1}) X^{|a|}.$$

Then we have $t(az) = t^{|a|}/f_a(t)$.

Now we define

$$\mu(z) = \frac{\eta^{q+1}(z)}{\eta^{q+1}(Tz)},$$

where

$$\eta = \widetilde{\pi} t^{\frac{1}{q+1}} \prod_{a \in A, a:\ \text{monic}} f_a^{q-1}(t).$$

Then we have the following property.

Proposition 2.1. (1) *$\mu(z)$ generates the function field $C(X_0(T))$.*

(2) *$\mu(z)$ has an s-expansion with integral coefficients at infinity as follows*:

$$\mu(z) = \frac{1}{s} + \sum_{n\geq 0} c_n s^n \ (c_n \in A).$$

(3) *$\mu(z)$ is non-vanishing on Ω.*

(4) *$\mu(1/(Tz)) = T^{q+1}/\mu(z)$.*

Proof. See [2, p. 277] □

3. Congruence properties of μ modulo power of T

We define the meromorphic function $U_T\mu$ on Ω by

$$U_T\mu(z) = \frac{1}{T}\sum_{\lambda\in\mathbb{F}_q} \mu\left(\frac{z+\lambda}{T}\right).$$

For $i = (i_0, i_1) \in \mathbb{N}^2$ and $n \in \mathbb{N}$, we denote by $\binom{n}{i}$ the multinomial coefficient $n!/(i_0!i_1!)$. Then we have the following lemma.

Lemma 3.1. (1)

$$U_T\mu\left(\frac{1}{z}\right) = U_T\mu(z) + \frac{1}{T}\mu\left(\frac{1}{Tz}\right) - \frac{1}{T}\mu\left(\frac{z}{T}\right).$$

(2) *$U_T\mu(z)$ is invariant under the action of $\Gamma_0(T)$.*

(3) *Write $\mu(z) = \sum_{n\geq 1-q} b_n t^n$ $(b_n \in A)$. Then $U_T\mu(z)$ is holomorphic at infinity with the following expansion*

$$U_T\mu(z) = 1 + \sum_{j\geq 1} a_j t^j,$$

where $a_j = \sum_{j\leq n\leq 1+(j-1)q} \sum_{i\in\mathbb{N}^2, i_0+i_1=j-1, i_0+qi_1=n-1} \binom{j-1}{i} b_n T^{i_0}$ if $j \geq 1$.

Proof. see [1, Corollary 2.8, 2.10 and Lemma 2.12] □

Proposition 3.2. (1) *$U_T\mu(z)$ is holomorphic in Ω.*

(2) *$U_T\mu(z)$ is holomorphic at infinity and has the following expansion for $|z|_i \gg 0$*

$$U_T\mu(z) = 1 + \sum_{n\geq 1} a_n s^n,$$

where $a_n = \sum_{0\leq i\leq n(q-1)-1} \binom{n(q-1)-1}{i} T^i c_{nq-i-1}$ $(n \geq 1)$.

(3) *$U_T\mu(z)$ has a simple pole at the cusp 0.*

(4) *$U_T\mu(z)$ generates the function field $C(X_0(T))$.*

Proof. From Lemma 3.1(3), we have $a_j = 0$ for any $j \not\equiv 0 \pmod{q-1}$ because $n - j = i_1(q-1) \equiv 0 \pmod{q-1}$ implies $n \equiv j \pmod{q-1}$. Hence we obtain the s-expansion of $U_T\mu(z)$ at infinity as follows:

$$U_T\mu(z) = 1 + \sum_{n\geq 1} a_n s^n,$$

where $a_n = \sum_{0\le i\le n(q-1)-1} \binom{n(q-1)-1}{i} T^i c_{nq-i-1}$ $(n \ge 1)$. Now observe the behavior of $U_T\mu(z)$ at the other cusp 0 of $\Gamma_0(T)$. By Lemma 3.1(1), we have

$$\begin{aligned} TU_T\mu\left(\frac{1}{Tz}\right) &= TU_T\mu(Tz) + \mu\left(\frac{1}{T^2z}\right) - \mu(z) \\ &= TU_T\mu(Tz) + \frac{T^{q+1}}{\mu(T^2z)} - \mu(z) \\ &= -\frac{1}{s} + h(s) \ (h(s) \in C[[s]]). \end{aligned}$$

Therefore $U_T\mu(z)$ has a simple pole at 0. Consequently, $U_T\mu(z)$ generates the function field $C(X_0(T))$ by Lemma 3.1(2). □

Theorem 3.3. *The Drinfeld modular function $\mu(z)$ has an s-expansion with integral coefficients at infinity as follows*:

$$\mu(z) = \frac{1}{s} + \sum_{n\ge 0} c_n s^n \ (c_n \in A).$$

Then we obtain that

$$\begin{aligned} \sum_{0\le i\le q-1} \binom{n(q-1)-1}{i} T^i c_{nq-i-1} = \sum_{0\le i\le q-1} (-1)^i \binom{i+n}{n} \\ T^i c_{nq-i-1} \equiv 0 \pmod{T^q} \ (n \ge 1). \end{aligned}$$

Here $\binom{k}{j}$ *denote binomial coefficients.*

Proof. Note that

$$U_T\mu(z) = 1 + \sum_{n\ge 1} a_n s^n,$$

where a_n are in Proposition 3.2. Since $TU_T\mu(z)+T^{q+1}/\mu(z)$ is holomorphic on $\Omega \cup \mathbb{P}^1(K)$, we can conclude that $TU_T\mu(z)+T^{q+1}/\mu(z) = c$ for some constant $c \in C$. This means that $(T + \sum_{n\ge 1} Ta_n s^n)(1/s + \sum_{n\ge 0} c_n s^n) + T^{q+1} = c(1/s + \sum_{n\ge 0} c_n s^n)$ which implies

$$c = T \quad \text{and} \quad (1/s + \sum_{n\ge 0} c_n s^n)(\sum_{n\ge 1} a_n s^n) = -T^q.$$

From this equation we obtain that $a_1 = -T^q$ and $a_n = -c_0 a_{n-1} - c_1 a_{n-2} - \cdots - c_{n-2} a_1$ for $n \ge 2$. Assume that $a_k \equiv 0 \pmod{T^q}$ for $1 \le k \le n-1$. Then $a_n = -c_0 a_{n-1} - c_1 a_{n-2} - \cdots - c_{n-2} a_1 \equiv 0 \pmod{T^q}$ because $c_n \in A$. By mathematical induction, $a_n \equiv 0 \pmod{T^q}$ for all $n \ge 1$. Consequently, the assertion is true because $\binom{n(q-1)-1}{i} = (-1)^i \binom{i+n}{n}$ in $\mathbb{F}_q$. □

Corollary 3.4. *For all $n \ge 1$, we have*

$$c_{nq-1} \equiv (n+1)Tc_{nq-2} \pmod{T^2}.$$

Proof. It follows from Theorem 3.3. □

Theorem 3.5. *Let $\mu(z)$ have an s-expansion with integral coefficients at infinity as follows*:

$$\mu(z) = \frac{1}{s} + \sum_{n \geq 0} c_n s^n \ (c_n \in A).$$

Define $r \in \mathbb{N}$ by $0 \leq r \leq q-1$ and $n \equiv r \pmod q$. Then we have

$$\sum_{0 \leq i \leq q-1-r} \binom{q-1-r}{i} T^i c_{nq-i-1} \equiv 0 \pmod{T^q}.$$

Proof. Let $0 \leq i \leq q-1-r$ and $0 \leq r \leq q-1-r$ such that $n \equiv r \pmod q$. We have

$$\binom{n(q-1)-1}{i} = (-1)^i \binom{i+n}{n} = \binom{q-1-r}{i},$$

and this is 0 if $i \geq q-r$. □

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