The Construct of the Program Control with Probability is Equaled to 1 for the Some Class of Stochastic Systems

Elena Chalykh

Abstract—The definition of the program control is introduced on the theory of the basis of the first integrals SDE system. That definition allows constructing the program control gives opportunity to stochastic system to remain on the given dynamic variety. The program control is considered in terms of dynamically invariant for stochastic process.

Index Terms—Program control, stochastic process, dynamic integrated variety.

1 Introduction

USUALLY the stochastic system program control definition is constructed for the integrated varieties, which have the equations, which do not contain dependence on time. The specificity of our approach is that to take in consider at dynamics, the variability in time of invariance surface. The first integrals system SDE theory allows a variation of a surface in time.

2 The Theoretical Basis

Let $\mathbf{x}(t) \in \mathbb{R}^n$ be a solution of Cauchy problem for system of some linear stochastic differential equations (SDE).

$$dx_i(t) = a_i(t;\mathbf{x}(t)) \cdot dt + \sum_{k=1}^{m} b_{ik}(t;\mathbf{x}(t)) \cdot dw_k(t), \quad \mathbf{x}(t;\mathbf{x}_o)\Big|_{t=0} = \mathbf{x}_o, \tag{1}$$

$\mathbf{w}(t) \in \mathbb{R}^m$ is Wiener process.

The functions $a_i(t;\mathbf{x})$, $b_{ik}(t;\mathbf{x})$ and all their first partial derivatives are bounded and continuous.

If we present a vector $A(t;\mathbf{x}(t)) = \sum_{i=1}^{n} a_i(t;\mathbf{x}(t)) \cdot \vec{e}_i$ of equation (1) as $A(t;\mathbf{x}(t)) = P(t;\mathbf{x}(t)) + Q(t;\mathbf{x}(t)) \cdot \mathbf{s}(t;\mathbf{x}(t))$, then it is possible to study a construction program motion (control) in stochastic system.

Prof. V. Doobko proved that for some class SDE determinate function exists, such as

$$u(t;\mathbf{x}(t;\mathbf{x}_o);\omega) = u(0;\mathbf{x}_o),$$

on all trajectories this class SDE solution. This function has the continuous second partial derivatives with respect to x_i, $i = \overline{1,n}$ and the first continuous derivative with respect to t. The name of this function is the first integral of system SDE (1) [1].

In work [1] the conditions for the SDE system coefficients were determined, in such way that system has first integrals.

According to the prof. V. Doobko theory of the first integrals of the invariant integrated varieties for some classes of the kind (1) equations are determined [1,2]. The construction of class SDE with certain set of first integrals is inverse problem.

Let us introduce the notation:

$$\begin{aligned} \tilde{A}(t;\mathbf{x}) &= \sum_{i=1}^{n} a_i(t;\mathbf{x})\vec{e}_i + \vec{e}_o = \\ &= (0,\, a_1,\, a_2, \cdots, a_n)^* + (1,\, 0,\, 0, \cdots, 0)^* = \\ &= A^o(t;\mathbf{x}) + \vec{e}_o. \end{aligned} \tag{2}$$

- *E. Chalykh is with Department of mathematical modeling and control processes, Pacific National University, Khabarovsk, Russia.*
E-mail: Chalykh@mail.khstu.ru.

The first integrals SDE system theory allows constructing equations with given set of the first integrals. This equation has coefficients from the set of the vector functions

$$\tilde{A}(t;\mathbf{x}) \in \left\{ \begin{array}{l} C^{-1}(t;\mathbf{x}) \cdot \det \begin{bmatrix} \vec{e}_0 & \vec{e}_1 & \dots & \vec{e}_n \\ a_{10} & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \dots & a_{nn} \end{bmatrix} \\ +\frac{1}{2}\left(B_k(t;\mathbf{x}), \nabla_x\right) \cdot B_k(t;\mathbf{x}), \end{array} \right\}$$

$$C(t;\mathbf{x}) = \det \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \neq 0,$$

$$B_k(t;\mathbf{x}) \in \left\{ q_{oo}(t;\mathbf{x}) \cdot \det \begin{bmatrix} \vec{e}_1 & \dots & \vec{e}_n \\ q_{11} & \dots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n-1,1} & \dots & q_{n-1,n} \end{bmatrix} \right\}$$

$$a_{li} = q_{li} = \frac{\partial u^l(t;\mathbf{x})}{\partial x_i}, \quad i = \overline{1,n},$$

$$a_{l0} = \frac{\partial u^l(t;\mathbf{x})}{\partial t}, \quad \forall\ l = \overline{1,N}, \quad N \leq (n-1),$$

In scheme above $B_k(t;\mathbf{x})$ are columns of matrix $B(t;\mathbf{x})$. Then this equation forms the class of the stochastic differential equations, for which each function from the set of linear-independent functions $\{u^l(t;\mathbf{x})\}_{l=1}^{N}$, $N \leq (n-1)$ is the first integral. The other functions and their second partial derivatives are continuous and bounded.

Thus, if the set of first integrals of system (1) exists, then a class of the equations with functions, which forms the integrated invariant, can be constructed with probability is equaled to 1.

3 The New Definition of a Program Control for Dynamic Stochastic System

The problem of the equation construction by its known first integrals adjoins to the problem of construction program motion in controlled systems in a terminology the ordinary differential equations. It is possible under the conditions that we consider do not the whole area of the solution existence and, for example, some neighborhood of a point

, but not at this point only. Because of uniqueness of the solutions and the continuity from the initial data are determined, and then the concept of the first integral can be applied to the program motion.

In the global formulation the program motion can be considered as a motion on the given variety. According to this statement the problem of the program motion adjoins to the theory of the first integrals.

The solution $\mathbf{x}(t) = \mathbf{x}(t;0;\mathbf{x}_o,\mathbf{s})$ of stochastic system

$$d\mathbf{x}(t) = [P(t;\mathbf{x}(t)) + Q(t;\mathbf{x}(t)) \cdot \mathbf{s}(t;\mathbf{x}(t))] \cdot dt + \\ +B(t;\mathbf{x}(t)) \cdot d\mathbf{w}(t) \tag{3}$$

is called a program motion if it allows to stay on the given integrated variety

$$u(t;\mathbf{x}(t;\mathbf{x}_o);\omega) = u(0;\mathbf{x}_o)$$

with probability is equaled to 1 for all time t at some s. This variety defines the first integrals of the equations (3) with the given initial condition $\mathbf{x}(t;\mathbf{x}_o)\Big|_{t=0} = \mathbf{x}_o$. Thus we shall name non-random function $\mathbf{s} = \mathbf{s}(t;\mathbf{x}(t))$ as the program control for dynamic stochastic system.

4 The Construction of a Program Control

Let is consider the following problem. Let $\mathbf{x}(t)$, $\mathbf{y}(t)$ – are stochastic processes, $\mathbf{w}(t)$ – is Wiener process, $t \in [0;+\infty)$ and $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^p$, $\mathbf{w} \in \mathbb{R}^m$, $\mathbf{s}(t;\mathbf{x},\mathbf{y}) \in \mathbb{R}^v$. It's necessary to find the control $\mathbf{s} = \mathbf{s}(t;\mathbf{x}(t),\mathbf{y}(t))$ and the reaction $\Sigma = \Sigma(t;\mathbf{x}(t),\mathbf{y}(t))$ on random effect $\mathbf{w}(t)$ so that the surface

$$\left\{ \begin{array}{l} u_1(t;\mathbf{x}(t),\mathbf{y}(t)) = 0 \\ u_2(t;\mathbf{x}(t),\mathbf{y}(t)) = 0 \end{array} \right. \tag{4}$$

is integrated variety for system

$$\left\{ \begin{array}{l} d\mathbf{x}(t) = F(t;\mathbf{x}(t),\mathbf{y}(t)) \cdot dt \\ d\mathbf{y}(t) = [R(t;\mathbf{x}(t),\mathbf{y}(t)) + \\ \quad +D(t;\mathbf{x}(t),\mathbf{y}(t)) \cdot \mathbf{s}(t;\mathbf{x}(t),\mathbf{y}(t))] \cdot dt + \\ \quad\quad +\sigma(t;\mathbf{x}(t),\mathbf{y}(t)) \cdot d\mathbf{w}(t) \end{array} \right. \tag{5}$$

The integrated variety for system is a set of conditions for its work.

Actually the similar problems exist in the dynamic system, the solution of which must remain on the some fixed surfaces constantly independently of the presence of the random effect.

The given problem can be shown as a problem of a kind (3). The fulfillment of condition:

$$\dim \mathbf{x}(t) + \dim \mathbf{y}(t) \geq \mu + 2, \qquad (6)$$

is necessary, where μ is a quantity of functions $u_j = u_j(\mathbf{x}(t), \mathbf{y}(t), t)$, that determine the given variety. If this condition is not fulfilled, then any of variables can be changed into new variable necessary dimension. We shall increase dimension initial variable through addition a zero component.

Let a condition (6) for variables $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be satisfied.

Let's consider a new vector $\mathbf{z}(t) = (\mathbf{x}^*, \mathbf{y}^*)^*$, $\dim \mathbf{z} = \dim \mathbf{x} + \dim \mathbf{y} = n + p$.

We can rewrite the system (5) as the matrix equation:

$$d\mathbf{z}(t) = \left[\begin{pmatrix} F \\ R \end{pmatrix} + \begin{pmatrix} O_1 \\ D \cdot \mathrm{s} \end{pmatrix}\right] \cdot dt + \begin{pmatrix} O_2 \\ \sigma \end{pmatrix} \cdot d\mathbf{w}(t), \qquad (7)$$

where O_1 and O_2 are zero matrixes with appropriating dimensions. Let $A(t, \mathbf{z}(t)) = \left[\begin{pmatrix} F \\ R \end{pmatrix} + \begin{pmatrix} O_1 \\ D \cdot \mathrm{s} \end{pmatrix}\right]$, $B(t, \mathbf{z}(t)) = \begin{pmatrix} O_2 \\ \sigma \end{pmatrix}$. Then the system (5) is equivalent to

$$d\mathbf{z}(t) = A(t; \mathbf{z}(t)) \cdot dt + B(t; \mathbf{z}(t)) \cdot d\mathbf{w}(t) \qquad (8)$$

Thus we need to find the solution of the equation (8), which allows it to remain on the integrated variety

$$\begin{cases} u_1(t; \mathbf{z}(t)) = 0 \\ u_2(t; \mathbf{z}(t)) = 0 \end{cases} \qquad (9)$$

with probability is equaled to 1. The partial derivatives of all variables for these functions at the given initial conditions are exist.

The control $\mathrm{s}(t; z(t)) = (s_1, \ldots, s_\nu)^*$ and reaction on random effect $\sigma(t; \mathbf{z}(t))$ are defined as the solutions of the systems of the linear equations:

$$\begin{cases} \sum\limits_{j=1}^{\nu} d_{1j} s_j = a'_{n+1} + r_1 \\ \ldots \\ \sum\limits_{j=1}^{\nu} d_{pj} s_j = a'_{n+p} + r_p. \end{cases} \quad \text{and} \quad b_{ik} = 0, \ \forall i = \overline{1, n}$$

respectively. Here $R = (r_j)$, $D = (d_{ij})$, $B = (b_{ij})$, $A = (a'_i)$.

Let is consider this result in details.

4.1 The construction of $\sigma(t; \mathbf{z}(t))$.

Let's consider components of a vector $\mathbf{z} = (\mathbf{x}^*, \mathbf{y}^*)^* = (x_1, \ldots, x_n; y_1, \ldots, y_p)^* = (z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+p})^*$. So as the functions at (9) are the first integrals of the equation (8) we shall define components b_{ik} of matrix $B = B(t; \mathbf{z}(t))$. These components depend on the arbitrary functions

$$g^k_{ll'}(t; \mathbf{z}(t)), \quad k = \overline{1, m}, \quad l = \overline{1, n + p - \mu}, \quad l' = \overline{1, n + p}.$$

For example, if $n = p = 2$, then

$$B_k(t; \mathbf{z}) = q_{oo} \cdot \det \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \\ \frac{\partial u_1}{\partial z_1} & \frac{\partial u_1}{\partial z_2} & \frac{\partial u_1}{\partial z_3} & \frac{\partial u_1}{\partial z_4} \\ \frac{\partial u_2}{\partial z_1} & \frac{\partial u_2}{\partial z_2} & \frac{\partial u_2}{\partial z_3} & \frac{\partial u_2}{\partial z_4} \\ g^k_{11} & g^k_{12} & g^k_{13} & g^k_{14} \end{pmatrix} =$$

$$= b_{1k}\vec{e}_1 + b_{2k}\vec{e}_2 + b_{3k}\vec{e}_3 + b_{4k}\vec{e}_4 = (b_{1k}, b_{2k}, b_{3k}, b_{4k})^*.$$

The coordinates of this vector will depend on the arbitrary functions $g^k_{ll'} = g^k_{ll'}(t; \mathbf{z})$. These functions and all their partial derivatives must be continuous and bounded. Since the presentation (7) is true, then matrix $B = B(t; \mathbf{z}(t))$ looks as $B = B(t; \mathbf{z}(t)) = \begin{pmatrix} O_2 \\ \sigma \end{pmatrix}$. So, conditions $b_{ik} = 0, \ \forall i = \overline{1, n}$ must be fulfilled for elements of a matrix. Then these conditions allow determining some functions $g^k_{ll'}$. Hence, the reaction $\sigma(t; \mathbf{z}(t))$ on random effect $\mathbf{w}(t)$ was constructed.

4.2 The construction of $\mathrm{s} = \mathrm{s}(t; \mathbf{z}(t))$.

According to the first integrals SDE system theory, we shall define vector $A(t; \mathbf{z}(t))$ at (8). For example, if $\tilde{A}(t; \mathbf{z}(t)) = \vec{e}_o + a'_1\vec{e}_1 + \ldots + a'_4\vec{e}_4$, then $A(t; \mathbf{z}(t)) = (a'_1, \ldots, a'_4)^*$. From (2) and (7) we get the following conditions to generalize case:

$$\begin{cases} a'_1 = f_1 \\ \ldots \\ a'_n = f_n \\ a'_{n+1} = r_1 + \sum\limits_{j=1}^{\nu} d_{1j} s_j \\ \ldots \\ a'_{n+p} = r_p + \sum\limits_{j=1}^{\nu} d_{pj} s_j \end{cases}.$$

There are $F = (f_j)$, $R = (r_j)$, $D = (d_{ij})$, $B = (b_{ij})$, $A = (a_i')$. Then we define some of unknown functions from the first n equations. Let's solve system of the rest equations, where s_j are unknown:

$$\begin{cases} \sum\limits_{j=1}^{\nu} d_{1j}s_j = a_{n+1}' + r_1 \\ \dots \\ \sum\limits_{j=1}^{\nu} d_{pj}s_j = a_{n+p}' + r_p. \end{cases} \quad \text{and} \quad b_{ik} = 0, \ \forall i = \overline{1,n}.$$

Hence, the set of stochastic system program control was constructed. We choose the arbitrary functions included in the decision in such way that the control could be realized.

5 The control at the pattern recognition

In practice patterning the recognition we use the classification more often. In this sense the problem of the invariant set searching (or the dynamic stochastic variety) can be considered as a stage at pattern recognition.

A training sample, a set of individual pattern etc. (classes of identification) we shall consider as invariant variety. Let the incoming coded signals received in the presence of strong noises also. Then we shall consider the problem of the pattern recognition as dynamic stochastic system. So it is possible to speak about a construction of control allowing for solution of the given system, i.e. pure signal, with probability is equaled to 1 lies on this variety.

Besides it is possible to consider the surface of invariance depending on time:

$$u_j(t; \mathbf{x}, \mathbf{x}_0^j) = u_j(0; \mathbf{x}_0^j) = C_t,$$

where $C_t = C_j = const$, $t \in (T_j, T_{j+1})$, $j = \overline{1,l}$.

After falling into the point $|\mathbf{x}|_\theta$ at some time $\tau \geq \theta$, the random process will be staying on the surface

$$|\mathbf{x}(t)|^2 = |\mathbf{x}|_\theta^2, \qquad \forall t \geq \tau \geq \theta$$

during some time (may be, this period will be infinitely long).

The equations for definition of the first integrals (attracting varieties of system of the stochastic differential equations) do not depend on the investigating processes in a direct time or a returning time. They can be represented in the following way [by V. Doobko]

$$\frac{\partial u(t;\mathbf{x})}{\partial t} + \sum_{i=1}^{n}\Bigg(a_i(t;\mathbf{x}) - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{m} b_{jk}(t;\mathbf{x})\frac{\partial b_{ik}(t;\mathbf{x})}{\partial x_j}\frac{\partial u(t;\mathbf{x})}{\partial x_i}\Bigg) = 0,$$

$$\sum_{i=1}^{n} b_{jk}(t;\mathbf{x})\frac{\partial u(t;\mathbf{x})}{\partial x_i} = 0, \qquad \forall k = \overline{1,m}.$$

6 Conclusion

So the first integrals SDE system theory makes it possible to construct the program control with probability is equaled to 1 for the some class of stochastic systems. In the article the construct of the control for any initial conditions is shown. We consider the surfaces where their right parts of equations are equaled to 0. But without loss of generality we may change the right parts to nonzero constants. The program control with probability is equaled to 1 can be applied to dynamic stochastic systems which solutions remain on data surfaces constantly.

Acknowledgment

The author is grateful to professor V. A. Doobko for constant attention to this work.

References

[1] V. A. Doobko, *Averaging for a class of stochastic differential equations* // Ukrainian Mathematical Journal, 1978, Volume 30, Number 4, pp.399-403.

[2] V. A. Doobko, *Integration with respect to initial data, the Poincare integral invariant, and "Hamilton's" equations for diffusion processes* // Ukrainian Mathematical Journal, 1981, Volume 33, Number 6, pp. 605-607.

Elena Chalykh is an Associate Professor of the Mathematical Modeling and Control Processes Department, Pacific National University, Khabarovsk, Russia. She received her Ph.D degree on physics in the Mathematic Modeling, Tomsk University, Russia in 1998. Her research interests include the strong solutions of SDE finding, the modeling of dynamic polymeric chain, the integrated invariants for the stochastic dynamic systems with probability is equaled to 1.