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MARK SEQUENCES IN 3-PARTITE 2-DIGRAPHS

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Abstract. A 3-partite 2-digraph is an orientation of a 3-partite multi-graph that is without loops and contains at most two edges between any pair of vertices from distinct parts. Let D(X, Y, Z) be a 3-partite 2-digraph with $|X| = l$, $|Y| = m$, $|Z| = n$. For any vertex v in D(X, Y, Z), let d_v^+ and d_v^- denote the outdegree and indegree respectively of v. Define $p_x = 2(m+n) + d_x^+ - d_x^-$, $q_y = 2(l+n) + d_y^+ - d_y^-$ and $r_z = 2(l+m) + d_z^+ - d_z^-$ as the marks (or 2-scores) of x in X, y in Y and z in Z respectively. In this paper, we characterize the marks of 3-partite 2-digraphs and give a constructive and existence criterion for sequences of non-negative integers in non-decreasing order to be the mark sequences of some 3-partite 2-digraph.

1. INTRODUCTION

An oriented graph is a digraph with no symmetric pairs of directed arcs and without loops. Suppose V = $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of an oriented graph, and let d_v^+ and d_v^- denote the outdegree and indegree respectively of a vertex v. Avery [1] defined $a_v = n - 1 + d_v^+ - d_v^-$, the score (or 1-score) of v, so $0 \le a_v \le 2n - 2$. Then, the sequence $[a_1, a_2, \ldots, a_n]$ in non-decreasing order is called the score sequence of the oriented graph.

Key Words :

Avery obtained the following criterion for score sequences in oriented graphs.

Theorem 1.1[1]. A non-decreasing sequence of non-negative integers $[a_1, a_2, \ldots, a_n]$ is the score sequence of an oriented graph if and only

$$\sum_{i=1}^{k} a_i \geq k(k-1), \qquad for \;\; 1 \leq k \leq n,$$

with equality when $k = n$.

An r-digraph is an orientation of a multi-graph that is without loops and contains at most r edges between any pair of distinct vertices. Clearly, 1-digraph is an oriented graph and complete 1-digraph is a tournament.

Suppose $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of a 2-digraph, and let d_v^+ and d_v^- denote the outdegree and indegree respectively of a vertex v. Define $p_v = 2n - 2 + d_v^+ - d_v^-$, the mark (or 2-score) of v, so $0 \leq p_v \leq 4n - 4$. Then, the sequence $[p_1, p_2, \ldots, p_n]$ in non-decreasing order is called the mark sequence of the 2-digraph.

The following result is given by Pirzada and Samee [4].

Theorem 1.2[4]. A non-decreasing sequence of non-negative integers $[p_1, p_2, \ldots, p_n]$ is the mark sequence of a 2-digraph if and only if

$$\sum_{i=1}^{k} p_i \geq 2k(k-1), \qquad for \;\; 1 \leq k \leq n,$$

with equality when $k = n$.

Some stronger inequalities for marks in 2-digraphs can be found in Pirzada and Naikoo [3].

The scores of oriented bipartite graphs have been characterized by Pirzada et al. [5] and those of marks in bipartite 2-digraphs by Samee et al. [6].

An oriented 3-partite graph is the result of assigning a direction to each edge of a simple 3-partite graph. Thus, it has no loops or parallel arcs. Suppose $U = \{u_1, u_2, \ldots, u_p\}$, $V = \{v_1, v_2, \ldots, v_q\}$ and $W = \{w_1, w_2, \ldots, w_r\}$ be the parts of an oriented 3-partite graph, and let d_u^+ (d_v^+ and d_w^+) and d_u^- (d_v^- and d_w^-) be the outdegree and indegree respectively of vertex u in U (v in V and w in W). Define $a_u = q + r + d_u^+ - d_u^-$, $b_v = p + r + d_v^+ - d_v^-$ and $c_w = p + q + d_w^+ - d_w^-$, the scores (or 1-scores) of u, v and w respectively. So, $0 \le a_u \le 2(q + r)$, $0 \le b_v \le 2(p + r)$ and $0 \le c_w \le 2(p + q)$. Then, the sequences $[a_1, a_2, \ldots, a_p]$, $[b_1, b_2, \ldots, b_q]$ and $[c_1, c_2, \ldots, c_r]$ in non-decreasing order are called the score sequences of the oriented bipartite graph.

The next result is due to Pirzada and Merajuddin [2].

Theorem 1.3[2]. Let $A = [a_1, a_2, \ldots, a_p]$, $B = [b_1, b_2, \ldots, b_q]$ and $C = [c_1, c_2, \ldots, c_r]$ be the sequences of non-negative integers in non-decreasing order. Then, A, B and C are the score sequences of some oriented 3-partite graph if and only if

$$\sum_{i=1}^{l} a_i + \sum_{j=1}^{m} b_j + \sum_{k=1}^{n} c_k \ge 2(lm + mn + nl), \quad for\ 1 \le l \le p,\ 1 \le m \le q\ and\ 1 \le n \le r,$$

with equality when $l = p$, $m = q$ and $n = r$.

A 3-partite 2-digraph is an orientation of a 3-partite multi-graph that is without loops and contains at most two edges between any pair of vertices from distinct parts. Suppose $X = \{x_1, x_2, \ldots, x_l\}$, $Y = \{y_1, y_2, \ldots, y_m\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$ be the parts of a 3-partite 2-digraph D(X, Y, Z), and let d_x^+ (d_y^+ and d_z^+) and d_x^- (d_y^- and d_z^-) be the outdegree and indegree respectively of vertex x in X (y in Y and z in Z). Define $p_x = 2(m + n) + d_x^+ - d_x^-$, $q_y = 2(l + n) + d_y^+ - d_y^-$ and $r_z = 2(l + m) + d_z^+ - d_z^-$, the marks (or 2-scores) of x, y and z respectively. So, $0 \leq p_x \leq 4(m + n)$, $0 \leq q_y \leq 4(l + n)$ and $0 \leq r_z \leq 4(l + m)$. Then, the sequences $P = [p_1, p_2,\ldots, p_l]$, $Q = [q_1, q_2,\ldots, q_m]$ and $R = [r_1, r_2,\ldots, r_n]$ in non-decreasing order are called the mark sequences of D(X, Y, Z). We can interpret a 3-partite 2-digraph as a result of competition between three teams in which each player of one team plays against everyone on the other two teams at most twice, with ties (draws) being allowed. A player receives two points for each win, and one point for each tie, and with this marking system, player x (y and z) receives a total of p_x (q_y and r_z) points. The sequences P, Q and R of non-negative integers in non-decreasing order are said to be realizable if there exists a 3-partite 2-digraph with mark sequences P, Q and R.

2. Criteria for Realizability

If u and v are two vertices from distinct parts X, Y, Z of a 3-partite 2–digraph D(X, Y, Z), then we have one of the following six possibilities.

(i) Exactly two arcs directed from u to v, and no arc directed from v to u, and this is denoted by u(2–0)v, see Figure 1(a).

(ii) Exactly two arcs directed from v to u, and no arc directed from u to v, and this is denoted by u(0–2)v, see Figure 1(b).

(iii) Exactly one arc directed from u to v, and exactly one arc directed from v to u, and this is denoted by u(1–1)v, and is called a pair of symmetric arcs between u and v, see Figure 1(c).

(iv) Exactly one arc directed from u to v, and no arc directed from v to u, and this is denoted by u(1–0)v, see Figure 1(d).

(v) Exactly one arc directed from v to u, and no arc directed from u to v, and this is denoted by u(0–1)v, see Figure 1(e).

(vi) No arc directed from u to v, and no arc directed from v to u, and this is denoted by u(0–0)v, see Figure 1(f).

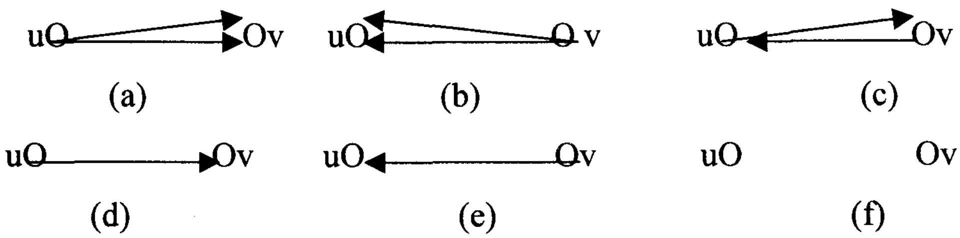

Figure 1

Figure 2 shows a 3-partite 2-digraph with mark sequences [10,11], [8,9], [8,9,9].

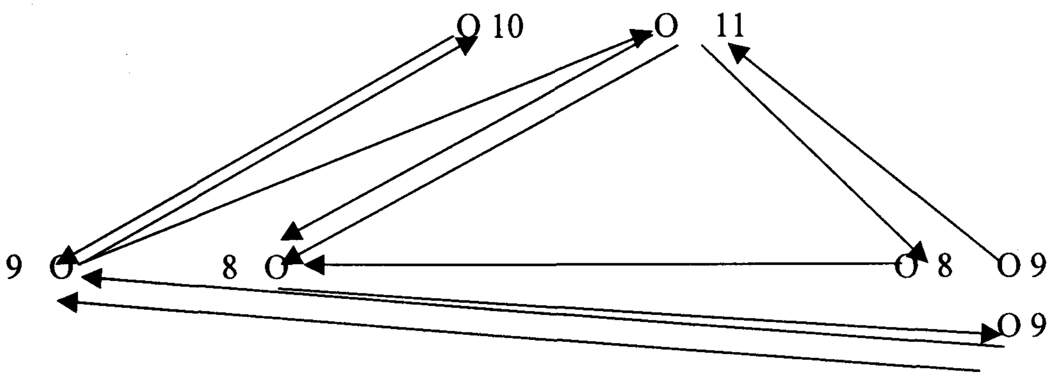

Figure 2

A triple in a 3-partite 2-digraph is an induced 2-subdigraph with one vertex from each part, and is of the form $x(a_1-a_2)y(b_1-b_2)z(c_1-c_2)x$, where for $1 \leq i \leq 2$, $0 \leq a_i$, b_i , $c_i \leq 2$ and $0 \leq \sum_{i=1}^{2} a_i, \sum_{i=1}^{2} b_i, \sum_{i=1}^{2} c_i \leq 2$.

In a 3-partite 2-digraph, an oriented triple is an induced 1-subdigraph with one vertex from each part. An oriented triple is said to be transitive if it is of the form x(1−0)y(1−0)z(0−1)x, or x(1−0)y(0−1)z(0−0)x, or x(1−0)y(0−0)z(0−1)x, or x(1−0)y(0−0)z(0−0)x, or x(0−0)y(0−0)z(0−0)x, otherwise it is intransitive. A 3-partite 2-digraph is said to be transitive if every of its oriented triple is transitive. In particular, a triple C in a 3-partite 2-digraph is transitive if every oriented triple of C is transitive.

First, we have the following observation.

Theorem 2.1. Let D and D' be two 3-partite 2-digraphs with the same mark sequences. Then, D can be transformed to D' by successively transforming (i) appropriate oriented triples in one of the following ways,

either (a) by changing an intransitive oriented triple x(1−0)y(1−0)z (1−0)x to a transitive oriented triple x(0−0)y(0−0)z(0−0)x, which has the same mark sequences, or vice versa,

or (b) by changing an intransitive oriented triple x(1−0)y(1−0)z(0−0)x to a transitive oriented triple x(0−0)y(0−0)z(0−1)x, which has the same mark sequences, or vice versa,

or (ii) by changing a pair of symmetric arcs x(1−1)y to x(0−0)y, which has the same mark sequences, or vice versa.

Proof. This result follows from Theorem 2.2[1].

The following result follows from Theorem 2.1.

Corollary 2.1. Among all the 3-partite 2-digraphs with given mark sequences, those with the fewest arcs are transitive.

A transmitter is a vertex with indegree zero. We assume without loss of generality that transitive 3-partite 2-digraphs have no pair of symmetric arcs. For, if there is a pair of symmetric arcs x(1−1)y then it can be changed to x(0−0)y with the same mark sequences. Thus, in a transitive 3-partite 2-digraph with mark sequences $P = [p_1, p_2,..., p_l]$, $Q = [q_1, q_2,..., q_m]$ and $R = [r_1, r_2,..., r_n]$, any of the vertex with mark p_l, or q_m, or r_n can act as a transmitter.

The following result provides a useful recursive test whether the sequences of non-negative integers form the mark sequences of some 3-partite 2-digraph.

Theorem 2.2. Let $P = [p_1, p_2,..., p_l]$, $Q = [q_1, q_2,..., q_m]$ and $R = [r_1, r_2,..., r_n]$ be the sequences of non-negative integers in non-decreasing order with $p_l \geq 2(m + n)$, $q_m \leq 4(l + n) - 2$ and $r_n \leq 4(l + m) - 2$. Let P' be obtained from P by deleting one entry p_l, and let Q' and R' be obtained as follows.

(i) If $p_l \geq 3(m + n)$, then reducing $4(m + n) - p_l$ largest entries of Q and R by one each,

or (ii) If $p_l < 3(m + n)$, then reducing $3(m + n) - p_l$ largest entries of Q and R by two each and $p_l - 2(m + n)$ remaining entries by one each.

Then, P, Q and R are the mark sequences of some 3-partite 2-digraph if and only if P', Q' and R' are.

Proof. Let P', Q' and R' be the mark sequences of some 3-partite 2-digraph D' with parts X', Y' and Z'. If Q' and R' be obtained from Q and R as in (i), then a 3-partite 2-digraph D with mark sequences P, Q and R can be obtained by adding a vertex x in X' such that x(1−0)v for those vertices v of Y' and Z'

whose marks are reduced by one in going from P, Q and R to P', Q' and R', and x(2–0)v for those vertices v of Y' and Z' whose marks are not reduced in going from P, Q and R to P', Q' and R'.

If Q' and R' be obtained from Q and R as in (ii), then again a 3-partite 2-digraph D with mark sequences P, Q and R are obtained by adding a vertex x in X' such that x(1–0)v for those vertices v of Y' and Z' whose marks are reduced by one in going from P, Q and R to P', Q' and R'.

Conversely, suppose P, Q and R be the mark sequences of a 3-partite 2-digraph D with parts X, Y and Z. By Corollary 2.1, any of the vertex x, or y, or z with mark p_l, or q_m, or r_n respectively can be a transmitter. Let the vertex x with mark p_l be a transmitter. Clearly, $p_l \geq 2(m + n)$, $q_m \leq 4(l + n) - 2$ and $r_n \leq 4(l + m) - 2$ because (a) if $p_l < 2(m + n)$, then by deleting p_l we have to reduce more than m + n entries from Q and R, which is absurd, (b) if $q_m > 4(l + n) - 2$ and $r_n > 4(l + m) - 2$, then on reduction $q'_m = q_m - 1 > 4(l + n) - 3 = 4(l - 1 + n) + 1$, or $q'_m = q_m - 2 > 4(l + n) - 4 = 4(l - 1 + n)$ and $r'_n = r_n - 1 > 4(l + m) - 3 = 4(l - 1 + m) + 1$, or $r'_n = r_n - 2 > 4(l + m) - 4 = 4(l - 1 + m)$, which in all cases is impossible.

(i) If $p_l \geq 3(m + n)$, let V be the set of $4(m + n) - p_l$ vertices of largest marks in Y and Z, and let $W = (Y \cup Z) - V$. Construct D such that x(1–0)v for all $v \in V$, and x(2–0)w for all $w \in W$. Clearly, D–x realizes P', Q' and R' (arranged in non-decreasing order).

(ii) If $p_l < 3(m + n)$, let V be the set of $3(m + n) - p_l$ vertices of largest marks in Y and Z, and let $W = (Y \cup Z) - V$. Construct D such that x(1–1)v

(or $x(0\!-\!0)v$) for all $v \in V$, and $x(1\!-\!0)w$ for all $w \in W$. Then, again $D\!-\!x$ realizes P', Q' and R' (arranged in non-decreasing order).

Theorem 2.2 provides an algorithm for determining whether or not the sequences P, Q and R of non-negative integers in non-decreasing order are the mark sequences, and for constructing a corresponding 3-partite 2-digraph. Let $P = [p_1, p_2, \ldots, p_l]$, $Q = [q_1, q_2, \ldots, q_m]$ and $R = [r_1, r_2, \ldots, r_n]$, where $p_l \geq 2(m + n)$, $q_m \leq 4(l + n) - 2$ and $r_n \leq 4(l + m) - 2$, be the mark sequences of a 3-partite 2-digraph with parts $X = \{x_1, x_2, \ldots, x_l\}$, $Y = \{y_1, y_2, \ldots, y_m\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$ respectively. Deleting p_l and performing (i) or (ii) of Theorem 2.2 according as $p_l \geq 3(m + n)$ or $p_l < 3(m + n)$, we get $Q' = [q_1', q_2', \ldots, q_m']$ and $R' = [r_1', r_2', \ldots, r_n']$. If the marks of the vertices y_j and z_k were decreased by one in this process, then the construction yielded $x_l(1\!-\!0)y_j$ and $x_l(1\!-\!0)z_k$, and if these were decreased by two, then the construction yielded $x_l(1\!-\!1)y_j$ and $x_l(1\!-\!1)z_k$ (or $x_l(0\!-\!0)y_j$ and $x_l(0\!-\!0)z_k$). For vertices y_s and z_t whose marks remained unchanged, the construction yielded $x_l(2\!-\!0)y_s$ and $x_l(2\!-\!0)z_t$. Note that if at least one of the conditions $p_l \geq 2(m + n)$, or $q_m \leq 4(l + n) - 2$, or $r_n \leq 4(l + m) - 2$ does not hold, then we delete q_m, or r_n for which the conditions get satisfied and the same argument is used for defining arcs. If this process is applied recursively, then it tests whether or not P, Q and R are the mark sequences, and if P, Q and R are the mark sequences, then a 3-partite 2-digraph Δ(P, Q, R) with mark sequences P, Q and R is constructed.

We illustrate this reduction and the resulting construction with the following example, beginning with the sequences P_1, Q_1 and R_1.

$P_1 = [7, 12, 17]$ $\quad Q_1 = [6, 12]$ $\quad R_1 = [8, 11, 11]$

$P_2 = [7, 12]$ $Q_2 = [6, 11]$ $R_2 = [8, 10, 10]$

$x_3(1-0)y_2$, $x_3(1-0)z_3$, $x_3(1-0)z_2$, $x_3(2-0)y_1$, $x_3(2-0)z_1$

$P_3 = [7]$ $Q_3 = [5, 9]$ $R_3 = [7, 8, 8]$

$x_2(0-0)y_2$, $x_2(0-0)z_3$, $x_2(0-0)z_2$, $x_2(1-0)y_1$, $x_2(1-0)z_1$

$P_4 = [6]$ $Q_4 = [5]$ $R_4 = [5, 6, 6]$

$y_2(0-0)z_2$, $y_2(0-0)z_1$, $y_2(1-0)x_1$

$P_5 = [5]$ $Q_5 = [4]$ $R_5 = [5, 6]$

$z_3(1-0)x_1$, $z_3(1-0)y_1$

$P_6 = [4]$ $Q_6 = [3]$ $R_6 = [5]$

$z_2(1-0)x_1$, $z_2(1-0)y_1$

$P_7 = \phi$ $Q_7 = [1]$ $R_7 = [3]$

$x_1(0-0)y_1$, $x_1(0-0)z_1$

$P_8 = \phi$ $Q_8 = [0]$ $R_8 = \phi$

$z_1(1-0)y_1$

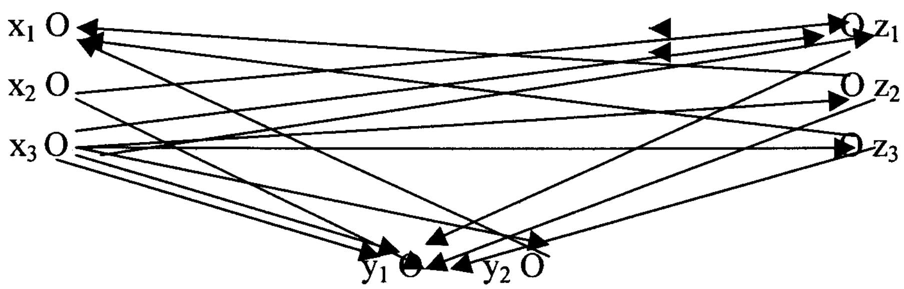

Figure 3

The next result follows by using the argument as in Theorem 2.2.

Theorem 2.3. $P = [p_1, p_2,\ldots, p_l]$, $Q = [q_1, q_2,\ldots, q_m]$ and $R = [r_1, r_2,\ldots, r_n]$ be the sequences of non-negative integers in non-decreasing order with $p_l \geq 2(m + n)$, $q_m \leq 4(l + n) - 2$ and $r_n \leq 4(l + m) - 2$. Let P' be obtained from P by deleting one entry p_l, and Q' and R' be obtained as follows.

(i) If p_l is even, then reducing $\frac{4(m+n)-p_l}{2}$ largest entries of Q and R by two each,

or (ii) If p_l is odd, then reducing $\frac{4(m+n)-p_l-1}{2}$ largest entries of Q and R by two each, and reducing the largest among the remaining entries of Q and R by one.

Then, P, Q and R are the mark sequences of some 3-partite 2-digraph if and only if P', Q' and R' are.

Theorem 2.3 also provides an algorithm of checking whether or not the sequences P, Q and R of non-negative integers in non-decreasing order are the mark sequences, and for constructing a corresponding 3-partite 2-digraph. Let $P = [p_1, p_2,\ldots, p_l]$, $Q = [q_1, q_2,\ldots, q_m]$ and $R = [r_1, r_2,\ldots, r_n]$, where $p_l \geq 2(m + n)$, $q_m \leq 4(l + n) - 2$ and $r_n \leq 4(l + m) - 2$, be the mark sequences of a 3-partite 2-digraph with parts $X = \{x_1, x_2,\ldots, x_l\}$, $Y = \{y_1, y_2,\ldots, y_m\}$ and $Z = \{z_1, z_2,\ldots, z_n\}$ respectively. Deleting p_l and performing (i) or (ii) of Theorem 2.3 according as p_l is even or odd, we get $Q' = [q_1', q_2',\ldots, q_m']$ and $R' = [r_1', r_2',\ldots, r_n']$. If the marks of the vertices y_j and z_k were decreased by one in this process, then the construction yielded $x_l(1-0)y_j$ and $x_l(1-0)z_k$, and if these were decreased by two, then the construction yielded $x_l(1-1)y_j$ and $x_l(1-1)z_k$ (or $x_l(0-0)y_j$ and $x_l(0-0)z_k$). For vertices y_s and z_t whose marks remained unchanged, the construction yielded $x_l(2\text{–}0)y_s$ and $x_l(2\text{–}0)z_t$. Note that if at least one of the conditions $p_l \geq 2(m + n)$, or $q_m \leq 4(l + n) - 2$, or $r_n \leq 4(l + m) - 2$ does not hold, then we delete q_m, or r_n for which the conditions get satisfied and the same argument is used for defining arcs. If this process is applied recursively, then it tests whether or not P, Q and R are the mark sequences, and if P, Q and R are the mark sequences, then a 3-partite 2-digraph $\Delta(P, Q, R)$ with mark sequences P, Q and R is constructed.

We illustrate this reduction and the resulting construction with the following example, beginning with the sequences P_1, Q_1 and R_1.

$P_1 = [7, 9, 12]$ $Q_1 = [8, 12]$ $R_1 = [7, 9]$

$P_2 = [7, 9]$ $Q_2 = [8, 10]$ $R_2 = [7, 7]$

$x_3(0\text{–}0)y_2$, $x_3(0\text{–}0)z_2$, $x_3(2\text{–}0)y_1$, $x_3(2\text{–}0)z_1$

$P_3 = [7]$ $Q_3 = [6, 8]$ $R_3 = [5, 6]$

$x_2(0\text{–}0)y_2$, $x_2(0\text{–}0)y_1$, $x_2(0\text{–}0)z_2$, $x_2(1\text{–}0)z_1$

$P_4 = [5]$ $Q_4 = [6]$ $R_4 = [4, 5]$

$y_2(0\text{–}0)x_1$, $y_2(0\text{–}0)z_1$, $y_2(2\text{–}0)z_2$

$P_5 = [3]$ $Q_5 = \phi$ $R_5 = [2, 3]$

$y_2(0\text{–}0)x_1$, $y_2(0\text{–}0)z_2$, $y_2(0\text{–}0)z_1$

$P_6 = [2]$ $Q_6 = \phi$ $R_6 = [2]$

$z_2(1\text{–}0)x_1$

$P_7 = \phi$ $Q_7 = \phi$ $R_7 = [0]$

$x_1(0\text{–}0)z_1$

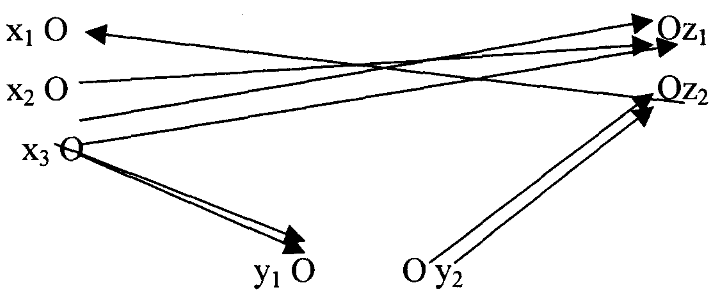

Figure 4

The next result gives a simple criterion for determining whether three sequences of non-negative integers in non-decreasing order are realizable as marks.

Theorem 2.4. Let $P = [p_1, p_2, \ldots, p_l]$, $Q = [q_1, q_2, \ldots, q_m]$ and $R = [r_1, r_2, \ldots, r_n]$ be the sequences of non-negative integers in non-decreasing order. Then, P, Q and R are the mark sequences of some 3-partite 2-digraph if and only if

$$\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j + \sum_{k=1}^{h} r_k \geq 4(fg + gh + hf), \tag{2.4.1}$$

for $1 \leq f \leq l$, $1 \leq g \leq m$ and $1 \leq h \leq n$, with equality when $f = l$, $g = m$ and $h = n$.

Proof. A sub-3-partite 2-digraph induced by f vertices from the first part, g vertices from the second part and h vertices from the third part has a sum of marks $4(fg + gh + hf)$. This proves the necessity.

For sufficiency, assume that $P = [p_1, p_2, \ldots, p_l]$, $Q = [q_1, q_2, \ldots, q_m]$ and $R = [r_1, r_2, \ldots, r_n]$ are the sequences of non-negative integers in non-decreasing order satisfying the conditions (2.4.1) but are not mark sequences of any 3-partite 2-digraph. Let these sequences be chosen in such a way that l, m and n are the smallest possible and p_1 is the least with that choice of l, m and n. We have the following two cases.

Case (a). Suppose equality in (2.4.1) holds for some f < l, g ≤ m and h ≤ n, so that

$$\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j + \sum_{k=1}^{h} r_k = 4(fg + gh + hf).$$

By the minimality of l, m and n, $P_1 = [p_1, p_2, \ldots, p_f]$, $Q_1 = [q_1, q_2, \ldots, q_g]$ and $R_1 = [r_1, r_2, \ldots, r_h]$ are the mark sequences of some 3-partite 2-digraph $D_1(X_1, Y_1, Z_1)$. Let $P_2 = [p_{f+1} - 4(g + h), p_{f+2} - 4(g + h), \ldots, p_l - 4(g + h)]$, $Q_2 = [q_{g+1} - 4(f + h), q_{g+2} - 4(f + h), \ldots, q_m - 4(f + h)]$ and $R_2 = [r_{h+1} - 4(f + g), r_{h+2} - 4(f + g), \ldots, r_n - 4(f + g)]$. Now,

$$\begin{aligned}
&\sum_{i=1}^{F}\left(p_{f+i} - 4(g+h)\right) + \sum_{j=1}^{G}\left(q_{g+j} - 4(f+h)\right) + \sum_{k=1}^{H}\left(r_{h+k} - 4(f+g)\right) \\
&= \sum_{i=1}^{f+F} p_i + \sum_{j=1}^{g+G} q_j + \sum_{k=1}^{h+H} r_k - \left(\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j + \sum_{k=1}^{h} r_k\right) \\
&\qquad - 4F(g+h) - 4G(f+h) - 4H(f+g) \\
&\geq 4\left((f+F)(g+G) + (g+G)(h+H) + (h+H)(f+F)\right) \\
&\qquad - 4(fg + gh + hf) - 4F(g+h) - 4G(f+h) - 4H(f+g) \\
&= 4(fg + fG + Fg + FG + gh + gH + Gh + GH + hf + hF + Hf + HF \\
&\qquad - fg - gh - hf - Fg - Fh - Gf - Gh - Hf - Hg) \\
&= 4(FG + GH + HF),
\end{aligned}$$

for $1 \leq F \leq l - f$, $1 \leq G \leq m - g$ and $1 \leq H \leq n - h$, with equality when $F = l - f$, $G = m - g$ and $H = n - h$. So, by the minimality for l, m and n, the sequences P_2 Q_2 and R_2 form the mark sequences of some 3-partite 2-digraph $D_2(X_2, Y_2, Z_2)$. Now, construct a new 3-partite 2-digraph D(X , Y, Z) as follows.

Let $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$, $Z = Z_1 \cup Z_2$ with $X_1 \cap X_2 = \varphi$, $Y_1 \cap Y_2 = \varphi$, $Z_1 \cap Z_2 = \varphi$. Let $x_2(2\!-\!0)y_1$, $x_2(2\!-\!0)z_1$, $y_2(2\!-\!0)x_1$, $y_2(2\!-\!0)z_1$, $z_2(2\!-\!0)x_1$ and

$z_2(2-0)y_1$ for all $x_i \in X_i$, $y_i \in Y_i$, $z_i \in Z_i$ where $1 \le i \le 2$, so that we get the 3-partite 2-digraph D(X , Y, Z) with mark sequences P, Q and R, which is a contradiction.

Case (b). Suppose that the strict inequality holds in (2.4.1) for $f \neq l$, $g \neq m$ and $h \neq n$.. Assume that $p_1 > 0$. Let $P_1 = [p_1 - 1, p_2, \ldots, p_{l-1}, p_l + 1]$, $Q_1 = [q_1, q_2, \ldots, q_m]$ and $R_1 = [r_1, r_2, \ldots, r_n]$, so that P_1, Q_1 and R_1 satisfy the conditions (2.4.1). Thus, by the minimality of p_1, the sequences P_1, Q_1 and R_1 are the mark sequences of some 3-partite 2-digraph $D_1(X_1, Y_1, Z_1)$. Let $p_{x_1} = p_1 - 1$ and $p_{x_l} = p_l + 1$. Since $p_{x_l} > p_{x_1} + 1$, therefore there exists a vertex v either in Y_1 or in Z_1 such that $x_l(0-0)v(2-0)x_1$ (or $x_l(1-1)v(2-0)x_1$), or $x_l(1-0)v(2-0)x_1$, or $x_l(2-0)v(2-0)x_1$, or $x_l(1-0)v(1-0)x_1$, or $x_l(2-0)v(1-0)x_1$, or $x_l(2-0)v(0-0)x_1$ (or $x_l(2-0)v(1-1)x_1$) in $D_1(X_1, Y_1, Z_1)$, and if these are changed to $x_l(0-1)v(1-0)x_1$, or $x_l(0-0)v(1-0)x_1$, or $x_l(1-0)v(1-0)x_1$, or $x_l(0-0)v(0-0)x_1$, or $x_l(1-0)v(0-0)x_1$, or $x_l(1-0)v(0-1)x_1$ respectively, the result is a 3-partite 2-digraph with mark sequences P, Q and R, which is a contradiction. This proves the result.

REFERENCES

[1] P. Avery, Score sequences of oriented graphs, J. Graph Theory, Vol. 15, No. 3 (1991) 251-257.

[2] S. Pirzada and Merajuddin, Score lists of oriented tripartite graphs, Novi Sad J. Math., Vol. 26, No. 2 (1996) 1-9.

[3] S. Pirzada and T.A. Naikoo, Inequalities for marks in digraphs, Mathematical Inequalities and Applications, Vol. 9, No. 2 (2006) 189-198.

[4] S. Pirzada and U. Samee, Mark sequences in digraphs, Seminare de Lotharingien Combinatoire, 55 (2006) Art. B 55c.

[5] S. Pirzada, Merajuddin and J. Yin, On the scores of oriented bipartite graphs, J. Mathematical Study, Vol. 33, No. 4 (2000) 354-359.

[6] U. Samee, Merajuddin, S. Pirzada, and T. A. Naikoo, Mark sequences in bipartite 2-digraphs, International J. of Mathematical Sciences, To appear.