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UNDERSTANDING OF NAVIER-STOKES EQUATIONS VIA A MODEL FOR BLOOD FLOW

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ABSTRACT: A pedagogic model for blood flow is introduced to help medicine majors understand a simplified version of Navier-Stokes equations which is known to be a good tool for interpreting the phenomena in blood flow. The pressure gradient consists of a time-independent part known as Hagen-Poiseuille's gradient and a time-dependent part known as Sexl's, and the model formula for the volume rate of blood flow is reduced to a very simple form. For demonstration, the blood rate in human aorta system is analyzed in connection with the time-dependence of pressure gradient. It is shown for Sexl's part that the flow rate lags the pressure gradient by $\pi/2$, which is thought to be due to the relaxation process involved.[1)]

1. INTRODUCTION

Navier Stokes equations[NSE] [1] are a model example of Newton's law of motion and is a good tool for interpreting some interesting phenomena appearing in engineering flows. Nevertheless, the coverage is dealt with in rather limited scheme in physiology classes, since equations are difficult to solve analytically with few exceptions. Nowadays, however, due to the widespread use of computers, obtaining any numerical solutions is feasible. The most simple one will be finite element method[2] which includes several versions. Another reason for limiting the coverage lies in difficulty in finding easy and interesting examples beyond Hegen-Poiseuille's law [HP] for quasi-static pressure gradient[3]. In 1930 Sexl [4] introduced an example with sinusoidally varying pressure gradient and with no-slip condition for viscous mechanical fluids flowing in a circular duct. The solution is given in a Bessel function with complex arguments, and thus has drawn little attention among applied scientists who usually dislike mathematics. In 1956 Uchida [5] paved the way for easy

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access to this model by giving numerical calculations.

Blood is a good example of viscous fluid and blood tubes are the counterpart of mechanical tubes in approximation. And thus the blood flow in veins can be dealt with as viscous flow with constant pressure gradient and that in arteries can be approximated to that with sinusoidal pressure gradient. The former one, which shall be called the primitive HP stating that rate of flow in a pipe with circular cross-section is proportional to the fourth power of the radius of the pipe, is covered in regular physiology classes. But the latter one is not covered, to the knowledge of the present authors, since the Bessel functions of the first and second kinds are not so popular even among physiology professors.

This pedagogic article introduces a model for flood flow which helps medicine majors understand the NSE. The model consists of the traditional constant pressure gradient and sinusoidal pressure gradient in blood tubes.

2. A Model for Blood Flow

For incompressible fluids, the NSE in cylindrical coordinates (r, θ, z) are given as [6]

r-component :

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)$$

$$=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right\}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right]+\rho g_r \quad (1)$$

θ-component :

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right\}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]+\rho g_\theta \quad (2)$$

z-component :

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)$$
$$=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]+\rho g_z \tag{3}$$

where p and $\vec{v}$, respectively, are the pressure and velocity and ρ is the mass density, μ the coefficient of viscosity, and $\vec{g}$ the acceleration due to gravity.

For viscous flows in a circular pipe of radius R, we can put

$$v_r=0,\quad v_\theta=0,\quad v_z=v_z(r,\ t) \tag{4}$$

in Eqs. (1)-(3) along with the following boundary conditions

$$\left(\frac{\partial v_z}{\partial r}\right)_{r=0}=0 \tag{5a}$$

$$v_z(r=R)=0 \tag{5b}$$

And choosing the pressure gradient with a constant term and a sinusoidal term as [3, 4]

$$-\frac{1}{\rho}\frac{\partial p}{\partial z}=A_0+A_1\sin\omega t=A_0+A_1\mathrm{Im}\left(e^{i\omega t}\right) \tag{6}$$

the NSE for v_z becomes

$$\frac{\partial v_z}{\partial t}=A_0+A_1\mathrm{Im}\left(e^{i\omega t}\right)+\nu\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}\right)+g_z \tag{7}$$

where A_0 and A_1 have dimension of acceleration, Im denotes "the imaginary part of" and $\nu=\mu/\rho$ is the coefficient of kinetic viscosity. Eq. (7) is a simplified version of NSE for physiology teaching. Note that this model is reduced to Sexl's if $A_0=0$ and to Poiseuille's if $A_1=0$. The solution $v_z(r,\ t)$ of the partial differential equation (7) can be obtained in the following form :

$$v_z(r,\ t)=v_z^{(1)}(r)+\mathrm{Im}\left[e^{i\omega t}v_z^{(2)}(r)\right] \tag{8}$$

Then, substituting this trial form into Eq. (7) and applying the boundary conditions (5a-5b), we have [7]

$$v_z(r,\ t)=\rho\frac{A_0+g_z}{4\mu}\left(R^2-r^2\right)-\frac{A_1}{\omega}\mathrm{Re}\left[\left\{1-\frac{J_0\left(\sqrt{-i\omega/\nu}\cdot r\right)}{J_0\left(\sqrt{-i\omega/\nu}\cdot R\right)}\right\}e^{i\omega t}\right] \tag{9}$$

and the expression for volume rate of flow, defined by [8]

$Q(t) = \int_0^R v_z(r, t) 2\pi r dr$, is given as

$$Q(t) = \text{Poiseuille's part} + \text{Sexl's part}$$
$$= \frac{\pi R^4 \rho (A_0 + g_z)}{8\mu} - \frac{\pi R^2 A_1}{\omega} \mathrm{Re}\left[e^{i\omega t} \left\{ 1 - \left(\frac{2}{R^*} \right) \frac{J_1(R^*)}{J_0(R^*)} \right\} \right]. \quad (10)$$

Here Re denotes "the real part of" and $R^* \equiv R\sqrt{-i\omega/\nu} = R\sqrt{-i\omega\rho/\mu}$, which shall be called the reduced radius hereafter, and $J_0(R^*)$ and $J_1(R^*)$ are the Bessel functions of the first kind of order zero and order one, respectively [9-10]. Note that the argument R^* is complex. Here the first part is Hagen-Poiseuille's result with inclusion of the gravitational effect and the second one is the counterpart obtained from Sexl's velocity distribution.

If we are interested in blood flow in the human system, we can take the asymptotic approximation $J_1(R^*)/J_0(R^*) \to \tan(R^*)$ since $|R^*| \fallingdotseq 10$ which is large enough for this criterion to apply. We then have the Sexl term simplified for large R^* as

$$\text{Sexl's part} = -\left(\frac{\pi R^2 A_1}{\omega} \right) \mathrm{Re}\left[\left(1 - 2\frac{\tan R^*}{R^*} \right) e^{i\omega t} \right]$$
$$= -\frac{\pi R^2 A_1}{\omega} \cos\omega t = \frac{\pi R^2 A_1}{\omega} \sin\left(\omega t - \frac{\pi}{2} \right) \quad (11)$$

for the human blood flow system since $\mathrm{Re}(2\tan R^*/R^*) \ll 1$ [See below]. We see that $Q(t)$ for Sexl's part is out of phase with the pressure gradient by $\pi/2$. The fact that $Q(t)$ lags the pressure gradient by 90° comes simply from the relaxation process involved.

3. Discussions and Concluding Remarks

It is to be noted that Eq. (10) along with Eq. (11) holds for laminar flow. For that purpose, the Reynolds number $R_e \equiv 2R\rho\bar{v}/\mu$, $\bar{v}$ being the average velocity, should be smaller than 2300. Otherwise, the transition and/or turbulence will be set up.

The above can be summarized as follows : The combined Poiseuille-Sexl's formula for rate of follow in human blood systems is reduced to

$$Q(t) = \frac{\pi R^4}{8\mu}\rho(A_0 + g_z) - \frac{\pi R^2}{\omega} A_1 \cos \omega t \tag{12}$$

in the above approximation.

In order to get into details, considering only Poiseuille's part and neglecting the gravitational effects, we take the following experimental data for normal human aorta system. $\rho = 1.05 \times 10^3 \mathrm{kg/m^3}$, $\mu = 4 \times 10^{-3} \mathrm{Pa \cdot s}$, $R = 0.01\,\mathrm{m}$, and $\bar{v} = 0.4\,\mathrm{m/s}$ [11]. We then have $R_e = 2100$, implying that the blood flow in the aorta is laminar. Taking the angular frequency of the heart · beat $\omega = 2\pi f = 7.5\,\mathrm{Hz}$, $f = 1.2\,\mathrm{Hz}$ being the frequency, we obtain $|R^*| \risingdotseq 10$ which is large enough for the critera of our approximation to hold. Thus we have

$$Q(t) = 9 \times 10^{-4}(A_0 + g_z - 0.05 A_1 \cos \omega t)\ [\mathrm{L^3/s}], \tag{13}$$

where A_0 and A_1 are given in the SI-unit$[\mathrm{m/s^2}]$.

We now compare the Poiseuille term and Sexl term for the human aorta system. In order to adopt this model, the two factors A_0 and A_1 should compete with each order in the almost same order. Note that we cannot claim that our A_0 is identical with that in Hagen-Poiseuille's formula. A_0 and A_1 can be obtained by fitting the theory to the available experiment. It is regretful that the fitting cannot be accomplished due to lack of experimental data. We will give only qualitative analysis instead.

We consider the aorta of length L and assume that the oscillation disappears at $z = L$. We further assume that $g_z = 0$, which means that the system is laid in the horizontal plane. Then the pressure can be expressed as

$$P_z(z) = P_0 - A_0 z + A_1 (L - z) \sin \omega t \tag{14}$$

which yields our pressure gradient $\partial P_z / \partial z = -(A_0 + A_1 \sin \omega t)$.

Roughly we have $\rho(A_0 - 0.05A_1\sin\omega t) \approx 100\,\mathrm{Pa/m}$ and $Q(t) = 10^{-4}\,\mathrm{m^3/s}$ [14]. Thus once either A_0 or A_1 is known, the whole behavior of the system can be exactly analyzed. A rough pictorial analysis is shown in the following figures.

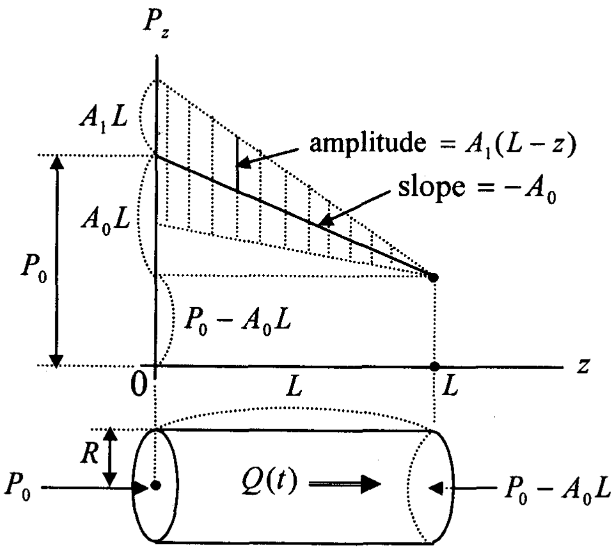

Figure 1. Pressure gradient and flow rate

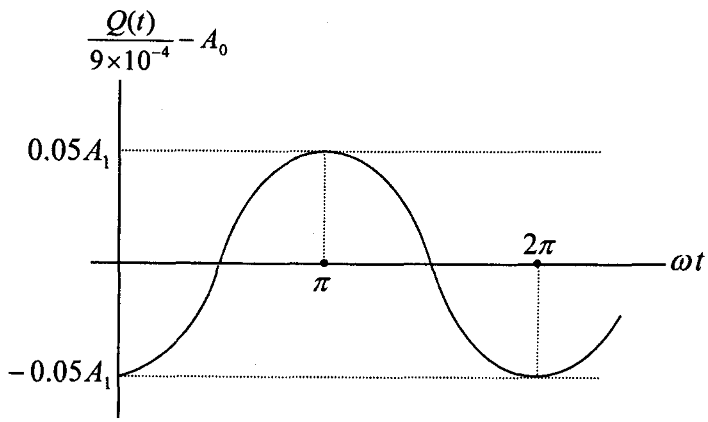

Figure 2. Flow rate versus ωt

In conclusion, the flow rate of the human circulatory system is also affected by Sexl's pressure gradient, which is physical as expected. So far we have introduced a simple model for flow rate on the basis of the simplified version of Navier-Stokes equation. We hope the medicine majors would be helped in understanding the Naviers-Stokes equation via this model. This model theory will be helpful in investigating other similar problems, too. One possible problem will be the flood flow in a pipe of elliptical cross-section [6]. Another interesting problem will be in the blood flow in curved vessels [13]. If Sexl's part is combined with these models, more meaningful results will be obtained. These works are in progress and will be reported later.

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