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Euler-Maruyama Numerical solution of some stochastic functional differential equations

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Abstract

In this paper we study the numerical solutions of the stochastic functional differential equations of the following form

$$du(x,t) = f(x,t,u_t)dt + g(x,t,u_t)dB(t),\ t > 0$$

with initial data $u(x,0) = u_0(x) = \xi \in L^p_{F_0}([-\tau,0];R^n)$.
Here $x \in R^n$,(R^n is the $\nu - dimenional$ Euclidean space),
$f : C([-\tau,0];R^n) \times R^{\nu+1} \to R^n$, $g : C([-\tau,0];R^n) \times R^{\nu+1} \to R^{n\times m}$,
$u(x,t) \in R^n$ for each t, $u_t = u(x,t+\theta) : -\tau \leq \theta \leq 0 \in C([-\tau,0];R^n)$, and B(t) is an m-dimensional Brownian motion.

Keywards: Euler-Maruyama, stochastic functional differential equations,local Lipschitz condition, linear growth condition, convergence theory.

AMS Subject Classifications: 65C30, 60H20, 65C20 .

1-Introduction

The numerical solutions of the stochastic differential equations studied in many papers (see [1],[2],[3],[4], [5], [6],[7],[8],[9]). In this paper we study the Euler-Maruyama numerical solution of the SFDE

$$du(x,t) = f(x,t,u_t)dt + g(x,t,u_t)dB(t),\ \ t > 0$$

with initial data $u(x,0) = u_0(x) = \xi \in L^p_{F_0}([-\tau,0]; R^n)$.
Here $x \in R^n$,(R^n is the $\nu - dimenional$ Euclidean space),
$f : C([-\tau,0]; R^n) \times R^{\nu+1} \to R^n$, $g : C([-\tau,0]; R^n) \times R^{\nu+1} \to R^{n\times m}$,
$u(x,t) \in R^n$ for each t, $u_t = u(x,t+\theta) : -\tau \leq \theta \leq 0 \in C([-\tau,0]; R^n)$, and B(t) is an m-dimensional Brownian motion (see [10],[11],[12],[13]). The initial data ξ is an F_0-measurable $C([-\tau,0]; R^n)$-valued random variable such that $E \parallel \xi \parallel^p < \infty$ for some $p > 2$. In the next section we introduce the Euler-Maruyama method for SFDEs, and we state our main result that the Euler-Maruyama numerical solutions convergence strongly to the exact solution if f and g satisfy local Lipschitz condition and the linear growth condition.

2- The Euler-Maruyama Method

Throughout this paper we use the following notations. Let

$$\sup_x |u(x,t)| = \parallel u(.,t) \parallel,$$

$|.|$ be the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{trace(A^TA)}$. Let $R_+ = [0,\infty)$, and let $\tau > 0$. Denote by $C([-\tau,0]; R^n)$ the family of continuous functions from $[-\tau,0]$ to R^n with norm

$$\parallel \phi \parallel = \sup_{-\tau\leq\theta\leq 0} \parallel \phi(.,\theta) \parallel$$

Let $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual condition (that is, it is increasing and right continuous, while F_0 contains all P-null sets). Let $B(t) = (B_1(t), ..., B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $p > 0$, and denote by $L^p_{F_0}([-\tau, 0]; R^n)$ the family of F_0-measurable $C([-\tau, 0]; R^n)$-valued random variables such that $E \parallel \xi \parallel^p < \infty$. If u(x,t) is an R^n-valued stochastic process on $t \in [-\tau, \infty)$, we let $u_t = \{u(x, t+\theta) : -\tau \leq \theta \leq 0\}$ for $t \geq 0$.Let

$$f : C([-\tau, 0]; R^n) \times R^{\nu+1} \to R^n, \quad g : C([-\tau, 0]; R^n) \times R^{\nu+1} \to R^{n\times m}.$$

In this paper we impose the following hypotheses.

Assumption 2.1 (The local Lipschitz condition).For each integer $j \geq 1$, there is a right-continuous nondecreasing function $\mu_j : [-\tau, 0] \to R_+$ such that

$$\parallel f(.,t,\phi) - f(.,t,\psi) \parallel^2 \vee \parallel g(.,t,\phi) - g(.,t,\psi) \parallel^2 \leq \int_{-\tau}^{0} \parallel \phi(.,\theta) - \psi(.,\theta) \parallel^2 d\mu_j(\theta)$$

for those ϕ, $\psi \in C([-\tau, 0]; R^n)$ with $\parallel \phi \parallel \vee \parallel \psi \parallel \leq j$, where the integral is of the Lesbesgue Stieltjes type.

Assumption 2.2 (The linear growth condition).There is a constant $K > 0$ such that

$$\parallel f(.,t,\phi) - g(.,t,\psi) \parallel^2 \leq K(1 + \parallel \phi \parallel^2)$$

for all $\phi \in C([-\tau, 0]; R^n)$.

Consider the n-dimensional SFDE :

$$du(x,t) = f(x,t,u_t)dt + g(x,t,u_t)dB(t), \ t > 0 \qquad (2.1)$$

with initial data $u(x, 0) = u_0(x) = \xi$. We impose the following condition on the initial data.

Assumption 2.3. **$\xi \in L^p_{F_0}([-\tau, 0]; R^n)$ for some $p > 2$. We can therefore state the**

following theorem.

Theorem 2.1. Under assumptions 2.1 - 2.3, for any $T > 0$ there is a constant $C > 0$ such that equation (2.1) has a unique continuous solution $u(x,t)$ on $t \geq -\tau$. Moreover, the solution has the property that

$$E(\sup_{-\tau \leq t \leq T} \| u(.,t) \|^P) \leq 2^{(p+4)/2}(1 + E \| \xi \|^p)e^{CT}. \qquad (2.2)$$

In other words, pth moment of the solution is finite .

Let us now introduce a numerical scheme for the SFDE (2.1); we refer to it as the Euler Maruyama method. Let the step size $\Delta \in (0,1)$ be a fraction of τ, namely $\Delta = \tau/N$ for some integer $N > \tau$. The discrete Euler-Maruyama approximate solution $\overline{v}(x, k\Delta)$, $k \geq -N$ is defined as follows:

$$\begin{cases} \overline{v}(x, k\Delta) = \xi(k\Delta), & -N \leq k \leq 0 \\ \overline{v}(x,(k+1)\Delta) = \overline{v}(x,k\Delta) + f(x,k\Delta,\overline{v}_{k\Delta})\Delta + g(x,k\Delta,\overline{v}_{k\Delta})\Delta B_k, & k \geq 0 \end{cases} \qquad (2.3)$$

where $\Delta B_k = B((k+1)\Delta) - B(k\Delta)$ and $\overline{v}_{k\Delta} = \{\overline{v}_{k\Delta}(x,\theta) : -\tau \leq \theta \leq 0\}$ is a $C([-\tau,0];R^n)$-valued random variable defined as follows:

$$\overline{v}_{k\Delta}(x,\theta) = \overline{v}_{k\Delta}(x,(k+1)\Delta) + \frac{\theta - i\Delta}{\Delta}[\overline{v}(x,(k+i+1)\Delta) - \overline{v}(x,(k+i)\Delta)]$$

for

$$i\Delta \leq \theta(i+1)\Delta, \ i = -N, -(N-1), ..., -1. \qquad (2.4)$$

That is $\overline{v}(x,.)$ is the linear interpolation of $\overline{v}(x,(k-N)\Delta)$, $\overline{v}(x,(k-N+1)\Delta), ..., \overline{v}(x,k\Delta)$. We can rewrite (2.4) as

$$\overline{v}_{k\Delta}(x,\theta) = \frac{\Delta - (\theta - i\Delta)}{\Delta}\overline{v}(x,(k+i)\Delta) + \frac{\theta - i\Delta}{\Delta}\overline{v}(x,(k+i+1)\Delta),$$

which yields

$$\| \overline{v}_{k\Delta}(.,\theta) \| = \frac{\Delta-(\theta-i\Delta)}{\Delta} \| \overline{v}(.,(k+i)\Delta) \| + \frac{\theta-i\Delta}{\Delta} \| \overline{v}(.,(k+i+1)\Delta) \|$$

$$\leq \| \overline{v}(.,(k+i)\Delta) \| \vee \| \overline{v}(.,(k+i+1)\Delta) \| .$$

We therefore have

$$\| \overline{v}_{k\Delta} \| = \max_{-N\leq i\leq 0} \| \overline{v}(.,(k+i)\Delta) \| \quad for\ all\ k\geq 0. \quad (2.5)$$

In our analysis it will be more convenient to use continuous-time approximations. We hence introduce the $C([-\tau,0];R^n)$-value step process

$$\overline{v}_t = \Sigma_{k=0}^{\infty} \overline{v}_{k\Delta} 1_{[k\Delta,(k+1)\Delta)}(x,t), \quad t\geq 0, \qquad (2.6)$$

and we define the continuous Euler-Maruyama approximate solution as follows:

$$v(x,t) = \begin{cases} \xi(x,t), & -\tau\leq t\leq 0 \\ \xi_0(x) + \int_0^t f(x,s,\overline{v}_s)ds + \int_0^t g(x,s,\overline{v}_s)dB(s), & t\geq 0. \end{cases} \qquad (2.7)$$

It should be pointed out that the $C([-\tau,0];R^n)$-value process $\overline{v}_t$ is simply defined by (2.6), but we do not define here an R^n-valued continuous process $\overline{v}(x,t)$ from which $\overline{v}_t$ is then induced by

$$\overline{v}_t = \{\overline{v}(x,t+\theta) : -\tau\leq\theta\leq 0\}.$$

It should also be pointed out that the reason why we do not use the linear interpolation of $\overline{v}(x,k\Delta)$ as a continuous-time approximation for $u(x,t)$, instead using $v(x,t)$ from (2.7) is because the linear interpolation of $\overline{v}(x,k\Delta)$ is not F_t-adapted. It follows from (2.7) that for any $t\geq 0$ that satisfy $k\Delta\leq t$,

$$v(x,t) = \xi_0(x) + \int_0^{k\Delta} f(x,s,\overline{v}_s)ds + \int_0^{k\Delta} g(x,s,\overline{v}_s)dB(s) + \int_{k\Delta}^t f(x,s,\overline{v}_s)ds + \int_{k\Delta}^t g(x,s,\overline{v}_s)dB(s)$$

$$= \overline{v}(x,k\Delta) + \int_{k\Delta}^t f(x,s,\overline{v}_s)ds + \int_{k\Delta}^t g(x,s,\overline{v}_s)dB(s). \qquad (2.8)$$

In particular, we observe that $v(x,k\Delta)=\overline{v}(x,k\Delta)$ for all $k\geq -N$. That is, the discrete and continuous Euler-Maruyama approximate solutions coincide at the gridpoints. It is then obvious that

$$\| \overline{v}_{k\Delta} \| \leq \| v_{k\Delta} \|, \; for\ all\ k\geq 0. \quad (2.9)$$

Moreover, for any $t\geq 0$, let $[t/\Delta]$ be the integer part of t/Δ. Then

$$\| \overline{v}_t \| = \| \overline{v}_{[t/\Delta]\Delta} \| \leq \| v_{[t/\Delta]\Delta} \| \leq \sup_{-\tau\leq s\leq t} \| v(.,s) \| . \qquad (2.10)$$

This property will be used frequently in what follows, without further explanation. To illustrate our numerical scheme, as well as to see why we call it the Euler-Maruyama method, let us consider a special SFDE

$$du(x,t)=F(x,t,D(u_t))dt+G(x,t,D(u_t))dB(t), \qquad (2.11)$$

where

$$F: R^{n+\nu+1}\to R^n, \quad G: R^{n+\nu+1}\to R^{n\times m},$$

and is a linear operator from $C([-\tau,0];R^n)$ to R^n given by

$$D(\phi)=\frac{1}{\tau}\int_{-\tau}^{0}\phi(x,\theta)d\theta,$$

$\phi\in C([-\tau,0];R^n)$; that is, D is an average operator. In this case, the discrete approximate solution (2.3) takes the following simple form

$$\begin{cases} \overline{v}(x,k\Delta)=\xi(k\Delta),, & -N\leq k\leq 0 \\ \overline{v}(x,(k+1)\Delta)=\overline{v}(x,k\Delta)+F(x,k\Delta,D(\overline{v}_{k\Delta}))\Delta+G(x,k\Delta,D(\overline{v}_{k\Delta}))\Delta B_k, & k\geq 0 \end{cases}$$

where

$$D(\overline{v}_{k\Delta}) = \frac{1}{\tau}\int_{-\tau}^{0} \overline{v}_{k\Delta}(\theta)d\theta = \frac{1}{\tau}\Sigma_{i=-N}^{-1}\frac{\Delta}{2}[\overline{v}(x,(k+i)\Delta) + \overline{v}(x,(k+i+1)\Delta)]$$

$$= \frac{1}{N}(\frac{1}{2}\overline{v}(x,(k-N)\Delta)) + \overline{v}(x,(k-N+1)\Delta) + ... + \overline{v}(x,(k-1)\Delta) + \frac{1}{2}\overline{v}(x,k\Delta).$$

We see clearly from this simple form that the discrete approximate solution (2.3) is a natural generalization of the classical Euler-Maruyama numerical scheme for SDEs, and that is why we call (2.3) the Euler-Maruyama approximate solution. The primary aim of this paper is to establis the following main result.

Theorem 2.2. Under assumptions 2.1 - 2.3,

$$\lim_{\Delta\to 0} E(\sup_{0\le t\le T} \| u(.,t) - v(.,t) \|^2) = 0 \ for\ all\ T > 0. \quad (2.12)$$

The proof of this theorem is very technical, so we present some lemmas.

Lemma 2.1. Let assumption 2.3 hold. Define $\alpha : (0,T] \to R_+$ by

$$\alpha(z) = \sup_{t,s\in[-\tau,0],\ |t-s|<z} E \| \xi(.,t) - \xi(.,s) \|^2 .$$

Then α is nondecreasing and has the property that $\alpha(z) \to 0$ as $z \to 0$. Moreover,

$$E \| \xi(.,t) - \xi(.,s) \|^2 \le \alpha|t-s|. \ -\tau \le s \le t \le 0 \quad (2.13)$$

Proof: From the definition of α we see clearly that α is nondecreasing and (2.13) holds. We therefore need only to show that $\alpha(z) \to 0$ as $u \to 0$. If this is not true, then

$$\lim_{z\to 0} \alpha(z) = \varepsilon_0 > 0. \quad (2.14)$$

From the definition of α we observe that for each integer $k \ge 1$ we can find a pair of t_k and s_k in $[-\tau, 0]$ with $|t_k - s_k| < \frac{1}{k}$ for which

$$E \| \xi(.,t_k) - \xi(.,s_k) \|^2 \ge \frac{\varepsilon_0}{2}. \quad (2.15)$$

Since $\{t_k\}$ is a sequence in the bounded interval $[-\tau, 0]$, it must have a convergent subsequence. Without any loss of generality, we may assume that $\{t_k\}$ is already a convergent sequence, and that it converges to $\bar{t} \in [-\tau, 0]$. Clearly, $\{s_k\}$ converges to $\bar{t}$ too .Now, by the continuity of $\xi(.,.)$,

$$\lim_{k \to \infty} \| \xi(., t_k) - \xi(., \bar{t}) \|^2 = 0$$

almost surely.

Moreover

$$\| \xi(., t_k) - \xi(., \bar{t}) \|^2 \leq 2 \| \xi(., t_k) \|^2 + \| \xi(., \bar{t}) \|^2 \leq 4 \| \xi \|^2,$$

while (by assumption 2.3 and the Holder inequality)

$$E \| \xi \|^2 \leq (E \| \xi \|^p)^{2/p} < \infty.$$

We can then apply the dominated convergence theorem to obtain

$$\lim_{k \to \infty} E \| \xi(., t_k) - \xi(., \bar{t}) \|^2 = 0.$$

Similarly, we can show that

$$\lim_{k \to \infty} E \| \xi(., s_k) - \xi(., \bar{t}) \|^2 = 0.$$

Consequently, we have

$$\lim_{k \to \infty} E \| \xi(., t_k) - \xi(., s_k) \|^2 = 0,$$

but this is in contradiction to (2.15). We therefore must have

$$\lim_{z \to 0} \alpha(z) = 0$$

the proof is therefore complete.

Lemma 2.2. Under assumption 2.2 and 2.3,

$$E(\sup_{-\tau \leq t \leq T} \| v(., t) \|^p) \leq H, \ for \ all \ T > 0, \qquad (2.16)$$

where H is a positive number dependent on ξ, K, p and T, but independent of Δ.

Proof. By the Holder inequality, it is easy to see from (2.7) that

$$\| v(.,t) \|^p < 3^{p-1}[\| \xi_0(.) \|^p + t^{p-1} \int_0^t \| f(.,s,\overline{v}_s) \|^p ds + \| \int_0^t g(.,s,\overline{v}_s) dB(s) \|^p].$$

Hence, for any $t_1 \in [0,T]$,

$$E(\sup_{0\le t\le t_1} \| v(.,t) \|^p) < 3^{p-1}[\| \xi_0(.) \|^p + T^{p-1} \int_0^{t_1} \| f(.,s,\overline{v}_s) \|^p ds +$$
$$E(\sup_{0\le t\le t_1} \| \int_0^t g(.,s,\overline{v}_s) dB(s) \|^p]. \quad (2.17)$$

By assumptio 2.2, we compute that

$$E\int_0^{t_1} \| f(.,s,\overline{v}_s) \|^p ds \le 2^{(p-2)/2} K^{p/2} E\int_0^{t_1} (1 + \| \overline{v}(.,s) \|^p) ds$$
$$\le 2^{(p-2)/2} K^{p/2} [T + \int_0^{t_1} E(\sup_{-\tau\le t\le s} \| v(.,t) \|^p) ds]. \quad (2.18)$$

We also compute, using the Burkholder-Davis-Gundy inequality,

$$E(\sup_{0\le t\le t_1} \| \int_0^t g(.,s,\overline{v}_s) dB(s) \|^p) \le c_p E(\int_0^{t_1} g(.,s,\overline{v}_s) \|^2 ds)^{p/2} \le c_p T^{(p-2)/2} E\int_0^{t_1} \| g(.,s,\overline{v}_s) \|^p ds,$$

where c_p is a constant dependent only on p. In the same way as (2.18) was obtained, we can then show that

$$E(\sup_{0\le t\le t_1} \| \int_0^t g(.,s,\overline{v}_s) dB(s) \|^p) \le c_p (2T)^{(p-2)/2} K^{p/2} [T + \int_0^{t_1} E(\sup_{-\tau\le t\le s} \| v(.,t) \|^p) ds]. \quad (2.19)$$

Substituting (2.18) and (2.19) into (2.17) yields

$$E(\sup_{0\le t\le t_1} \| v(.,t) \|^p) \le 3^{p-1} E \| \xi_0(.) \| + C_1 + C_2 \int_0^{t_1} E(\sup_{-\tau\le t\le s} \| v(.,t) \|^p) ds, \quad (2.20)$$

where C_1 and C_2 are two positive numbers dependent only on K,p and T. We then derive the following inequalities :

$$E(\sup_{-\tau\le t\le t_1} \| v(.,t) \|^p) \le E \| \xi \|^p + E(\sup_{0\le t\le t_1} \| v(.,t) \|^p)$$

$$(1+3^{p-1})E \| \xi \|^p + C_1 + C_2 \int_0^{t_1} E(\sup_{-\tau\le t\le s} \| v(.,t) \|^p)ds. \qquad (2.21)$$

By the Gronwall inequality we find that

$$E(\sup_{-\tau\le t\le T} \| v(.,t) \|^p) \le [(1+3^{p-1})E \| \xi \|^p + C_1 e^{C_2 T}],$$

and hence the required assertion must hold .

Lemma 3.3. Let assumptions 2.1 - 2.3 hold, let $T > 0$. Then there is a nondecreasing function $\beta : (0,T] \to R_+$ that has the property that $\beta(z) = 0$ as $z \to 0$, such that

$$E \| v(.,s+\theta) - \overline{v}_s(.,\theta) \|^2 \le \beta(\Delta), \ s \in [0,T], \ \theta \in [-\tau, 0]. \qquad (2.22)$$

Proof. Fix $s \in [0,T]$ and $\theta \in [-\tau,0]$. Let k_s and k_θ be the integers for which $s \in [k_s\Delta, (k_s+1)\Delta]$ and $\theta \in [k_\theta\Delta, (k_\theta+1)\Delta]$, respectively. (When θ/Δ is an integer, the choice for k_θ may not be unique, but this will not affect the proof below .) Clearly, $0 \le s - k_s\Delta < \Delta$ and $0 \le \theta - k_\theta\Delta \le \Delta$, so

$0 \le s + \theta - (k_s + k_\theta)\Delta < 2\Delta. \qquad (2.23)$

Moreover, it follows from (2.4) and (2.6) that

$$\overline{v}_s(x,\theta) = \overline{v}_{k_s\Delta}(x,\theta) = \overline{v}(x,(k_s+k_\theta)\Delta) + \frac{\theta - k_\theta\Delta}{\Delta}[\overline{v}(x,(k_s+k_\theta+1)\Delta) - \overline{v}(x,(k_s+k_\theta)\Delta)].$$

Hence

$$E \| v(.,s+\theta) - \overline{v}_s(.,\theta) \|^2 \le 2E \| v(.,s+\theta) - \overline{v}(.,(k_s+k_\theta)\Delta) \|^2$$

$$+ 2E \| \overline{v}(.,(k_s+k_\theta+1)\Delta) - \overline{v}(.,(k_s+k_\theta)\Delta) \|^2 \qquad (2.24)$$

If $k_s + k_\theta \le -1$, then lemma 2.1,

$$E \| \overline{v}(.,(k_s+k_\theta+1)\Delta) - \overline{v}(.,(k_s+k_\theta)\Delta) \|^2 \le \alpha(\Delta).$$

If $k_s + k_\theta \ge 0$, and lemma 2.2 we compute from (2.3) that

$$E \parallel \overline{v}(.,(k_s+k_\theta+1)\Delta)-\overline{v}(.,(k_s+k_\theta)\Delta) \parallel^2 = \Delta^2 E \parallel f(.,(k_s+k_\theta)\Delta,\overline{v}(.,(k_s+k_\theta)\Delta)) \parallel^2$$

$$+\Delta E \parallel g(.,(k_s+k_\theta)\Delta,\overline{v}(.,(k_s+k_\theta)\Delta)) \parallel^2 \leq 2\Delta K(1+E \parallel \overline{v}(.,(k_s+k_\theta)\Delta) \parallel^2$$

$$\leq 2\Delta K(1+E[\sup_{-\tau\leq u\leq (k_s+k_\theta)\Delta} \parallel v(.,z) \parallel^2])$$

$$\leq 2\Delta K\{(1+E[\sup_{-\tau\leq u\leq (k_s+k_\theta)\Delta} \parallel v(.,y) \parallel^p])^{2/p}\} \leq 2K(1+H^{2/p})\Delta,$$

where H is the constant specified in lemma 2.2. We hence always have

$$E \parallel \overline{v}(.,(k_s+k_\theta+1)\Delta)-\overline{v}(.,(k_s+k_\theta)\Delta) \parallel^2 \leq 2K(1+H^{2/p})\Delta+\alpha(\Delta).$$

Using this bounded in (2.24) gives

$$E \parallel v(.,s+\theta)-\overline{v}(.,\theta) \parallel^2 \leq 2E \parallel v(.,s+\theta)-\overline{v}(.,(k_s+k_\theta)\Delta) \parallel^2 +4K(1+H^{2/p})\Delta+2\alpha(\Delta). \quad (2.25)$$

To bound the first term on the right hand side, let us discuss the following possible cases.

Case 1: $k_s+k_\theta\geq 0$. **It follows from (2.8) that**

$$v(x,s+\theta)-\overline{v}(x,(k_s+k_\theta)\Delta)=\int_{(k_s+k_\theta)\Delta}^{s+\theta} f(x,r,\overline{v}_r)dr+\int_{(k_s+k_\theta)\Delta}^{s+\theta} g(x,r,\overline{v}_r)dB(r).$$

By assumption 2.2 and lemma 2.2, we compute that

$$E \parallel v(.,s+\theta)-\overline{v}(.,(k_s+k_\theta)\Delta \parallel^2 \leq 2[\Delta E\int_{(k_s+k_\theta)\Delta}^{s+\theta} \parallel f(.,r,\overline{v}_r) \parallel^2 dr+E\int_{(k_s+k_\theta)\Delta}^{s+\theta} \parallel g(.,r,\overline{v}_r) \parallel^2 dr]$$

$$\leq 6KE\int_{(k_s+k_\theta)\Delta}^{s+\theta}(1+ \parallel \overline{v} \parallel^2)dr \leq 6K\int_{(k_s+k_\theta)\Delta}^{s+\theta}(1+E[\sup_{-\tau\leq z\leq r} \parallel v(.,z) \parallel^2])dr$$

$$\leq 6K\int_{(k_s+k_\theta)\Delta}^{s+\theta}\{1+(E[\sup_{-\tau\leq z\leq r} \parallel v(.,z) \parallel^p])^{2/p}\}dr \leq 12K(1+H^{2/p})\Delta. \quad (2.26)$$

Case 2 : $k_s+k_\theta=-1$ **and** $\Delta<s+\theta-(k_s+k_\theta)\Delta<2\Delta$.

In this case,

$$0\leq \Delta+(k_s+k_\theta)\Delta<s+\theta<2\Delta+(k_s+k_\theta)\Delta=\Delta.$$

So

$$E \parallel v(.,s+\theta) - \overline{v}(.,(k_s+k_\theta)\Delta) \parallel^2 = E \parallel v(.,s+\theta) - \overline{v}(.,-\Delta) \parallel^2$$

$$\leq E \parallel v(.,s+\theta) - \xi_0(.) \parallel^2 + 2E \parallel \xi_0(.) - \xi(.,-\Delta) \parallel^2 .$$

It can be shown in the same way as in case 1 that

$$E \parallel v(.,s+\theta) - \xi_0(.) \parallel^2 \leq 4K(1+H^{2/p})\Delta,$$

while by lemma 2.1,

$$E \parallel \xi_0(.) - \xi(.,-\Delta) \parallel^2 \leq \alpha(\Delta).$$

We therefore that

$$E \parallel v(.,s+\theta) - \overline{v}(.,(k_s+k_\theta)\Delta) \parallel^2 \leq 8K(1+H^{2/p})\Delta + 2\alpha(\Delta). \quad (2.27)$$

Case 3 :$k_s + k_\theta = -1$ **and** $0 \leq s+\theta-(k_s+k_\theta)\Delta \leq \Delta$. **In this case**

$$-\Delta \leq (k_s+k_\theta)\Delta < s+\theta < \Delta + (k_s+k_\theta)\Delta = 0.$$

So

$$E \parallel v(.,s+\theta) - \overline{v}(.,(k_s+k_\theta)\Delta) \parallel^2 = E \parallel \xi(.,s+\theta) - \overline{v}(.,-\Delta) \parallel^2 = E \parallel \xi(.,s+\theta) - \xi(.,-\Delta) \parallel^2 .$$

By lemma 2.1, we then have

$$E \parallel v(.,s+\theta) - \overline{v}_s(.,\theta) \parallel^2 \leq \alpha\Delta \qquad (2.28)$$

Case 4 : $k_s + k_\theta \leq -2$. **In this case** $s+\theta \leq 0$. **So**

$$E \parallel v(.,s+\theta) - \overline{v}(.,(k_s+k_\theta)\Delta) \parallel^2 = E \parallel \xi(.,s+\theta) - \xi(.,(k_s+k_\theta)\Delta) \parallel^2 .$$

By lemma 2.1 and (2.23), we then have

$$E \parallel v(.,s+\theta) - \overline{v}(.,(k_s+k_\theta)\Delta) \parallel^2 \leq \alpha(2\Delta) \qquad (2.29)$$

Combining the four cases above together, we can conclude that we always have

$$E \parallel v(., s+\theta) - \overline{v}(., (k_s + k_\theta)\Delta) \parallel^2 \leq 12K(1 + H^{2/p})\Delta + 2\alpha(2\Delta) \qquad (2.30)$$

Now, we define $\beta : (0, \tau] \rightarrow R_+$ **by**

$$\beta(z) = 28K(1 + H^{2/p})z + 6\alpha(2z).$$

Clearly, β **is nondecreasing. Moreover, it follows from (2.25) and (2.30) that**

$$E \parallel v(., s+\theta) - \overline{v}_s(., \theta) \parallel^2 \leq \beta(\Delta).$$

Which is the required assertion. The proof is complete.

Proof of theorem 2.2

Let us now being to prove theorem 2.2. We first note from theorem 2.1 and lemma 2.2 that there is a positive constant $\overline{H}$ **such that**

$$E(\sup_{-\tau \leq t \leq T} \parallel u(., t) \parallel^p) \vee E(\sup_{-\tau \leq t \leq T} \parallel v(., t) \parallel^p) \leq \overline{H}. \qquad (2.31)$$

Let j be sufficiently large integer. Define the stopping times

$$p_j =: inf\{t \geq 0 : \parallel u(., t) \parallel \geq j\}, \quad q_j =: inf\{t \geq 0 : \parallel v(., t) \parallel \geq j\}, \quad \rho_j = p_j \wedge q_j,$$

where we set inf $\emptyset = \infty$. **Let** $e(x,t) = u(x,t) - v(x,t)$ **obviously,**

$$E[\sup_{0 \leq t \leq T} \parallel e(., t) \parallel^2] = E[\sup_{0 \leq t \leq T} \parallel e(., t) \parallel^2 1_{\{p_j \leq T \ or \ q_j \leq T\}}].$$

Recall the following elementary inequality :

$$a^\gamma b^{1-\gamma} \leq \gamma a + (1-\gamma)b, \forall a, b > 0, \gamma \in [0, 1].$$

We thus have, for any $\delta > 0$,

$$E[\sup_{0\le t\le T} \| e(.,t) \|^2 1_{\{p_j\le T \; or \; q_j\le T\}}] = E[(\delta \sup_{0\le t\le T} \| e(.,t) \|^p)^{2/p}(\delta^{-2/(p-2)} 1_{\{p_j\le T \; or \; q_j\le T\}})^{(p-2)/p}]$$

$$\le \frac{2\delta}{p} E[\sup_{0\le t\le T} \| e(.,t) \|^p] + \frac{p-2}{p^{\delta^2/(p-2)}} P(p_j \le T \; or \; q_j \le T).$$

Hence

$$E[\sup_{0\le t\le T} \| e(.,t) \|^2]$$

$$\le E[\sup_{0\le t\le T} \| e(.,t) \|^2 1_{\{\rho_j>1\}}] + \frac{2\delta}{p} E[\sup_{0\le t\le T} \| e(.,t) \|^p] + \frac{p-2}{p^{\delta^2/(p-2)}} P(p_j \le T \; or \; q_j \le T). \quad (2.32)$$

Now

$$P(q_j \le T) = E[1_{\{p_j\le T\}} \frac{\| u_{p_j}(.,t) \|^p}{j^p}] \le \frac{1}{j^p} E[\sup_{-\tau\le t\le T} \| u(.,t) \|^p] \le \frac{\overline{H}}{j^p},$$

using (2.31). Similarly, we have $P(q_j \le T) \le \frac{\overline{H}}{j^p}$ **Thus**

$$P(p_j \le T \; or \; q_j \le T) \le P(p_j \le T) + P(q_j \le T) \le \frac{2\overline{H}}{j^p}.$$

We also have

$$E[\sup_{0\le t\le T} \| e(.,t) \|^2 1_{\{\rho_j>T\}}] = E[\sup_{0\le t\le T} \| e(.,t\wedge\rho_j) \|^2 1_{\{\rho_j>T\}}] \le E[\sup_{0\le t\le T} \| e(.,t\wedge\rho_j) \|^2].$$

Using these bounds in (2.32) yields

$$E[\sup_{0\le t\le T} \| e(.,t) \|^2] \le E[\sup_{0\le t\le T} \| e(.,t\wedge\rho_j) \|^2] + \frac{2^{p+1}\delta\overline{H}}{p} + \frac{(p-2)2\overline{H}}{p^{\delta^2/(p-2)j^p}}. \quad (2.33)$$

Now

$$\| e(.,t\wedge\rho_j) \|^2 = \| u(.,t\wedge\rho_j) - v(.,t\wedge\rho_j) \|^2 =$$

$$\| \int_0^{t\wedge\rho_j} [f(.,s,u_s) - f(.,s,\overline{v}_s)]ds + \int_0^{t\wedge\rho_j} [g(.,s,u_s) - g(.,s,\overline{v}_s)]dB(s) \|^2$$

$$\le 2[T\int_0^{t\wedge\rho_j} \| f(.,s,u_s) - f(.,s,\overline{v}_s) \|^2 ds + \| \int_0^{t\wedge\rho_j} [g(.,s,u_s) - g(.,s,\overline{v}_s)]dB(s) \|^2].$$

By the Doob martingale inequality we have, for any $t_1 \le T$,

$$E[\sup_{0\le t\le t_1} \| e(.,t\wedge\rho_j) \|^2] \le 2[TE\int_0^{t_1\wedge\rho_j} \| f(.,s,u_s) - f(.,s,\overline{v}_s) \|^2 \, ds$$

$$+ 4\int_0^{t_1\wedge\rho_j} \| g(.,s,u_s) - g(.,s,\overline{v}_s) \|^2 \, ds]$$

$$= 4(T+4)E\int_0^{t_1\wedge\rho_j} \| [f(.,s,u_s)-f(.,s,\overline{v}_s) \|^2 \wedge \| g(.,s,u_s)-g(.,s,\overline{v}_s) \|^2]ds.$$

But, by assumption 2.1, we derive that, for $s \in (0, t_1\wedge\rho_j]$,

$$\| f(.,s,u_s)-f(.,s,\overline{v}_s) \|^2 \le 2 \| f(.,s,u_s)-f(.,s,v_s) \|^2 \le \int_{-\tau}^0 \| u(.,s+\theta)-v(.,s+\theta) \|^2 \, d\mu_j(\theta)$$

$$+2\int_{-\tau}^0 \| v(.,s+\theta) - \overline{v}_s(.,\theta) \|^2 \, d\mu_j(\theta) \le 2\int_{-\tau}^0 [\sup_{-\tau\le\theta\le 0} \| u(.,s+\theta) - v(.,s+\theta) \|^2] d\mu_j(\theta)$$

$$+2\int_{-\tau}^0 \| v(.,s+\theta) - \overline{v}_s(.,s+\theta) \|^2 \, d\mu_j(\theta) \le 2(\mu_j(0) - \mu_j(-\tau))[\sup_{0\le t\le s} \| u(.,t) - v(.,t) \|^2]$$

$$+ 2\int_{-\tau}^0 \| v(.,s+\theta) - \overline{v}_s(.,\theta) \|^2 \, d\mu_j(\theta).$$

a similar result can be obtained for $\| g(.,s,u_s) - g(.,s,\overline{v}_s) \|^2$, **so that**

$$E[\sup_{0\le t\le t_1} \| e(.,t\wedge\rho_j) \|^2] \le 8(T+4)(\mu_j(0) - \mu_j(-\tau))E\int_0^{t_1\wedge\rho_j} [\sup_{0\le t\le s} \| e(.,s) \|^2]ds$$

$$+ 8(T+4)E\int_0^{t_1\wedge\rho_j} [\int_{-\tau}^0 \| v(.,s+\theta) - \overline{v}_s(.,\theta) \|^2 \, d\mu_j(\theta)]ds \quad (2.34)$$

$$\le 8(T+4)(\mu_j(0) - \mu_j(-\tau)) \int_0^{t_1} E[\sup_{0\le t\le s} \| e(.,s\wedge\rho_j) \|^2]ds$$

$$+ 8(T+4)\int_0^T [\int_{-\tau}^0 \| v(.,s+\theta) - \overline{v}_s(.,\theta) \|^2 \, d\mu_j(\theta)]ds. (2.35)$$

By lemma 2.3 we therefore find that

$$E[\sup_{0\le t\le t_1} \| e(.,t\wedge\rho_j) \|^2] \le 8(T+4)(\mu_j(0) - \mu_j(-\tau)) \int_0^{t_1} E[\sup_{0\le t\le s} \| e(.,s\wedge\rho_j) \|^2]ds$$

$$+ 8T(T+4)(\mu_j(0) - \mu_j(-\tau))\beta(\Delta).$$

The Gronwall inequality implies that

$$E[\sup_{0\le t\le T} \| e(.,t\wedge\rho_j) \|^2] \le C_j\beta(\Delta),$$

$$C_j = 8T(T+4)(\mu_j(0) - \mu_j(-\tau))exp[8T(T+4)(\mu_j(0) - \mu_j(-\tau))].$$

Substituting this into (2.33) gives

$$E[\sup_{0\leq t\leq T} \| e(.,t) \|^2] \leq C_j\beta(\Delta) + \frac{2^{p+1}\overline{H}}{p} + \frac{(p-2)2\overline{H}}{p^{\delta^2/(p-2)}j^p}. \quad (2.36)$$

Given $\varepsilon > 0$ **we can now choose** δ **sufficiently small for** $(2^{p+1}\delta\overline{H})/p < \varepsilon/3$, **then choose j sufficiently large for** $\frac{(p-2)2\overline{H}}{p^{\delta^2/(p-2)}j^p} < \varepsilon/3$ **and finally choose** Δ **so that** $C_j\beta(\Delta) < \varepsilon/3$. **Thus (2.36),**

$$E[\sup_{0\leq t\leq T} \| e(.,t) \|^2] < \varepsilon,$$

as required.

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