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LINEAR MAPS THAT PRESERVE COMMUTING PAIRS OF MATRICES OVER GENERAL BOOLEAN ALGEBRA

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ABSTRACT. We consider the set of commuting pairs of matrices and their preservers over binary Boolean algebra, chain semiring and general Boolean algebra. We characterize those linear operators that preserve the set of commuting pairs of matrices over a general Boolean algebra and a chain semiring.

1. Introduction

One of the most active and fertile subjects in matrix theory during the past one hundred years is the linear preserver problem, which concerns the characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. Although the linear operators concerned are mostly linear operators on matrix spaces over some fields or rings, the same problem has been extended to matrices over various semirings ([1]-[8]).

Let $\mathbb{M}_n(\mathbb{S})$ denote the set of $n \times n$ matrices over a nonempty set $\mathbb{S}$. A set of commuting pairs of matrices, $\mathbb{C}$, is the set of (unordered) pairs of matrices (A, B) such that $AB = BA$. A linear operator T on $\mathbb{M}_n(\mathbb{S})$ is said to *preserve* $\mathbb{C}$ (or T *preserves commuting pairs of matrices*) if $(T(A), T(B)) \in \mathbb{C}$ for all $(A, B) \in \mathbb{C}$.

Commutativity of matrices play a central role in the theory of matrices. There are many papers on the linear transformations that preserve the set of commuting pairs of matrices over various algebraic structure (see [1]-[3], [5]-[6], [8]). Song and Beasley [6] obtained characterizations of linear operators that preserve commuting pairs of matrices over the

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nonnegative reals, $\mathbb{R}_+$. Watkins [8] showed that if $n \geq 4$ and $\mathbb{S}$ is an algebraically closed field of characteristic 0, and T is a nonsingular linear operator on $\mathbb{M}_n(\mathbb{S})$ which preserves commuting pairs of matrices, then there exist an invertible matrix U, a nonzero scalar α and a linear functional $f : \mathbb{M}_n(\mathbb{S}) \to \mathbb{S}$ such that either

(1) $T(X) = \alpha UXU^{-1} + f(X)I_n$ for all $X \in \mathbb{M}_n(\mathbb{S})$, or
(2) $T(X) = \alpha UX^tU^{-1} + f(X)I_n$ for all $X \in \mathbb{M}_n(\mathbb{S})$.

Beasley [1] extended this to the case $n = 3$. The real symmetric and complex Hermitian cases were investigated by Chan and Lim [2]. Further extensions to more general fields were obtained by Radjavi [5] and Choi, Jafarian and Radjavi [3].

In this paper, we study the semigroup of linear operators that preserve commuting pairs of matrices over a Boolean algebra and a chain semiring. In Theorem 3.3 we show that it is generated by transpositions and similarity operators over the binary Boolean algebra and a chain semiring. Also in Theorem 4.3 we extend these results to the general Boolean case and obtain an linear operator which is not generated by transpositions and similarity operators.

2. Preliminaries

A *semiring* is a nonempty set $\mathbb{S}$ on which operations of addition(+) and multiplication($\cdot$) have been defined such that the following conditions are satisfied:

(a) $(\mathbb{S}, +)$ is a commutative monoid with identity element 0;
(b) $(\mathbb{S}, \cdot)$ is a monoid with identity element 1;
(c) multiplication distributes over addition from either side;
(d) $s0 = 0 = 0s$ for all $s \in \mathbb{S}$.

For a fixed positive integer k, let $\mathbb{B}_k$ be the *Boolean algebra* of subsets of a k-element set $\mathbb{S}_k$ and $\sigma_1, \sigma_2, \ldots, \sigma_k$ denote the singleton subsets of $\mathbb{S}_k$. Union is denoted by + and intersection by juxtaposition; 0 denotes the null set and 1 the set $\mathbb{S}_k$. Under these two operations, $\mathbb{B}_k$ is a commutative antinegative semiring(that is, only 0 has an additive inverse); all of its elements, except 0 and 1, are zero-divisors. In particular, if $k = 1$, $\mathbb{B}_1$ is called the *binary Boolean algebra.*

Let $\mathbb{K}$ be any set of two or more elements. If $\mathbb{K}$ is totally ordered by $<$ (i.e., $x < y$ or $y < x$ for all distinct elements x, y in $\mathbb{K}$), then define $x+y$ as $\max(x, y)$ and xy as $\min(x, y)$ for all $x, y \in \mathbb{K}$. If $\mathbb{K}$ has a universal lower bound and a universal upper bound, then $\mathbb{K}$ becomes a semiring, and called a *chain semiring.* The following are interesting examples of a chain semiring.

Let $\mathbb{H}$ be any nonempty family of sets nested by inclusion, $0 = \cap_{x \in \mathbb{H}} x$, and $1 = \cup_{x \in \mathbb{H}} x$. Then $\mathbb{S} = \mathbb{H} \cup \{0, 1\}$ is a chain semiring.

Let α, w be real numbers with $\alpha < w$. Define $\mathbb{S} = \{\beta \in \mathbb{R} : \alpha \leq \beta \leq w\}$. Then $\mathbb{S}$ is a chain semiring with $\alpha = 0$ and $w = 1$. It is isomorphic to the chain semiring in the previous example with $\mathbb{H} = \{[\alpha, \beta] : \alpha \leq \beta \leq w\}$. Furthermore, if we choose the real numbers 0 and 1 as α and w in the previous example, then $m \times n$ matrices over $\mathbb{F} \equiv \{\beta : 0 \leq \beta \leq 1\}$ is called *fuzzy matrices.*

In particular, if we take $\mathbb{H}$ to be a singleton set, say $\{a\}$, and denote $\emptyset$ by 0 and $\{a\}$ by 1, the resulting chain semiring becomes the binary Boolean algebra $\mathbb{B} = \{0, 1\}$, and it is a subsemiring of every chain semiring. Since any general Boolean algebra $\mathbb{B}_k (k \geq 2)$ is not totally ordered under inclusion, it is not a chain semiring.

Hereafter, unless otherwise specified, $\mathbb{S}$ denote an arbitrary semiring, and $\mathbb{B}_k$ a general Boolean algebra, and $\mathbb{K}$ a chain semiring.

Let $\mathbb{M}_n(\mathbb{S})$ denote the set of all $n \times n$ matrices with entries in $\mathbb{S}$. Then algebraic operations on $\mathbb{M}_n(\mathbb{S})$ are defined as if $\mathbb{S}$ were a field. The identity matrix and the zero matrix are denoted by I_n and 0_n. Since $\mathbb{M}_n(\mathbb{S})$ is a semiring, we can consider the invertible members of its multiplicative monoid. A matrix $A \in \mathbb{M}_n(\mathbb{S})$ is said to be *invertible* if there exists a matrix $B \in \mathbb{M}_n(\mathbb{S})$ such that $AB = BA = I_n$. It is well known [4] that the permutation matrices are the only invertible matrices on $\mathbb{M}_n(\mathbb{B}_1)$.

For any matrix $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{B}_k)$, the l^{th} *constituent*, A_l, of A is the $n \times n$ binary Boolean matrix whose (i, j)-th entry is 1 if and only if $a_{ij} \supseteq \sigma_l$. Via the constituents, A can be written uniquely as

$$A = \sum_{l=1}^{k} \sigma_l A_l,$$

which is called the *canonical form* of A (see [4]).

It follows from the uniqueness of the decomposition and the fact that the singletons are mutually orthogonal idempotents that for all matrices $A, B \in \mathbb{M}_n(\mathbb{B}_k)$ and all $\alpha \in \mathbb{B}_k$,

(2.1) $(AB)_l = A_l B_l$;
(2.2) $(A + B)_l = A_l + B_l$;
(2.3) $(\alpha A)_l = \alpha_l A_l$.

for all $1 \leq l \leq k$. Thus each l^{th} constituent operator, T_l, is a homomorphism of $\mathbb{M}_n(\mathbb{B}_k)$ onto $\mathbb{M}_n(\mathbb{B}_1)$, and preserves the products of matrices in $\mathbb{M}_n(\mathbb{B}_k)$.

LEMMA 2.1. *For any matrix $A \in \mathbb{M}_n(\mathbb{B}_k)$ with $k \geq 1$, A is invertible if and only if all its constituents are permutation matrices. In particular, if A is invertible, then $A^{-1} = A^t$, where A^t denote the transpose of A.*

Proof. If A is an invertible matrix in $\mathbb{M}_n(\mathbb{B}_k)$, then there exists a matrix $B \in \mathbb{M}_n(\mathbb{B}_k)$ such that $AB = I_n$. The equality (2.1) implies that $(AB)_l = A_l B_l = I_n$ for all $l = 1, \ldots, k$. It follows that all constituents of A are permutation matrices. Conversely, assume that each l^{th} constituent, A_l, of A is a permutation matrix. Then we have $A_l A_l^t = I_n$ for all $l = 1, \ldots, k$, and hence

$$AA^t = \left(\sum_{l=1}^{k} \sigma_l A_l\right)\left(\sum_{l=1}^{k} \sigma_l A_l\right)^t = \sum_{l=1}^{k} \sigma_l A_l A_l^t = \sum_{l=1}^{k} \sigma_l I_n = I_n.$$

Therefore A is invertible. □

Lemma 2.1 shows that all permutation matrices are only invertible matrices on $\mathbb{M}_n(\mathbb{B}_1)$, while there exists an invertible matrix on $\mathbb{M}_n(\mathbb{B}_k)(k \geq 2)$ which is not a permutation. For example, consider

$$A = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_2 & \sigma_3 & \sigma_1 \\ \sigma_3 & \sigma_1 & \sigma_2 \end{bmatrix} \in \mathbb{M}_3(\mathbb{B}_3).$$

Then $AA^t = I_3$, but A is not a permutation matrix in $\mathbb{M}_3(\mathbb{B}_3)$.

For each $x \in \mathbb{K}$, define

$$x^* = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0. \end{cases}$$

Then the mapping $x \to x^*$ is a homomorphism of $\mathbb{K}$ onto $\mathbb{B}_1 = \{0, 1\}$. Its entrywise extension to a mapping $A \to A^*$ of $\mathbb{M}_n(\mathbb{K})$ onto $\mathbb{M}_n(\mathbb{B}_1)$ preserves matrix sums and products and multiplication by scalars.

LEMMA 2.2. *The permutation matrices are the only invertible members of $\mathbb{M}_n(\mathbb{K})$.*

Proof. Let A be an invertible matrix in $\mathbb{M}_n(\mathbb{K})$. Then there exists a matrix $B \in \mathbb{M}_n(\mathbb{K})$ such that $AB = I_n$. This implies $A^*B^* = I_n$, and thus A^* and B^* are permutation matrices with $B^* = A^{*t}$. Since any product of two element in $\mathbb{K}$ is their minimum, the nonzero entries in A are 1's. Therefore A is a permutation matrix. □

3. The binary Boolean case

In this section we obtain characterizations of the linear operators that preserve the set of commuting pairs of matrices over the binary Boolean algebra $\mathbb{B}_1 = \{0, 1\}$ and a chain semiring $\mathbb{K}$.

The $n \times n$ matrix all of whose entries are zero except its (i, j)-th, which is 1, is denoted E_{ij}. We call E_{ij} a *cell.* Let $\mathbb{E}_n = \{E_{ij} \mid i, j = 1, \ldots, n\}$ denote the set of all cells. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices in $\mathbb{M}_n(\mathbb{S})$, we shall use the notation $A \leq B$(or $B \geq A$) if $b_{ij} = 0$ implies $a_{ij} = 0$ for all i, j. This provides a reflexive and transitive relation on $\mathbb{M}_n(\mathbb{S})$. If A and B are matrices in $\mathbb{M}_n(\mathbb{S})$ with $A \geq B$, it follows from the linearity of T that $T(A) \geq T(B)$ for any linear operator T on $\mathbb{M}_n(\mathbb{S})$.

LEMMA 3.1. *Let $\mathbb{S} = \mathbb{B}_1$ or $\mathbb{S} = \mathbb{K}$. Then for a linear operator T on $\mathbb{M}_n(\mathbb{S})$, T is invertible if and only if T permutes $\mathbb{E}_n$.*

Proof. Suppose that T is invertible on $\mathbb{M}_n(\mathbb{S})$. Let E_{ij} be any element in $\mathbb{E}_n$. By invertibility of T, there exists at least one cell E_{rs} in $\mathbb{E}_n$ such that $T(E_{ij}) \geq E_{rs}$. Thus we have $E_{ij} \geq T^{-1}(E_{rs})$ because T^{-1} is also linear. This implies that $T^{-1}(E_{rs}) = \alpha E_{ij}$ for some nonzero scalar $\alpha \in \mathbb{S}$, equivalently $E_{rs} = \alpha T(E_{ij})$.

If $\mathbb{S} = \mathbb{B}_1$, we have $\alpha = 1$ so that $T(E_{ij}) = E_{rs}$. If $\mathbb{S} = \mathbb{K}$, the (r, s)-th entry of E_{rs} is 1, while that of $\alpha T(E_{ij})$ is α. Thus we have $\alpha = 1$, and hence $T(E_{ij}) = E_{rs}$.

Since E_{ij} is an arbitrary cell, T permutes $\mathbb{E}_n$. The converse is immediate. □

For $\mathbb{S} = \mathbb{B}_1$ or $\mathbb{S} = \mathbb{K}$, let T be a linear operator on $\mathbb{M}_n(\mathbb{S})$ that preserves commuting pairs of matrices. Then the following example shows that T may be not invertible.

EXAMPLE 3.2. For $\mathbb{S} = \mathbb{B}_1$, let T be an operator on $\mathbb{M}_n(\mathbb{S})$ defined by

$$T(X) = X + I_n$$

for all $X \in \mathbb{M}_n(\mathbb{S})$. Then we can easily show that T is linear and preserves commuting pairs of matrices. But Lemma 3.1 implies that T is not invertible.

THEOREM 3.3. *For $\mathbb{S} = \mathbb{B}_1$ or $\mathbb{S} = \mathbb{K}$, let T be a linear operator on $\mathbb{M}_n(\mathbb{S})$. Then T is an invertible linear operator that preserves commuting pairs of matrices if and only if there exists a permutation matrix P such that either*

(a) $T(X) = PXP^t$ *for all* $X \in \mathbb{M}_n(\mathbb{S})$, *or*

(b) $T(X) = PX^tP^t$ *for all* $X \in \mathbb{M}_n(\mathbb{S})$.

Proof. Let T be an invertible linear operator on $\mathbb{M}_n(\mathbb{S})$ which preserves commuting pairs of matrices. Note that if $AX = XA$ for all $X \in \mathbb{M}_n(\mathbb{S})$, then we have $A = 0_n$ or I_n. Thus we have $T(I_n) = I_n$ because T is invertible.

By Lemma 3.1, T permutes $\mathbb{E}_n$. It follows from $T(I_n) = I_n$ that there is a permutation σ of $\{1, \ldots, n\}$ such that $T(E_{ii}) = E_{\sigma(i)\sigma(i)}$ for each $i = 1, \ldots, n$. Define an operator L on $\mathbb{M}_n(\mathbb{S})$ by

$$L(X) = P^tT(X)P$$

for all $X \in \mathbb{M}_n(\mathbb{S})$, where P is the permutation matrix corresponding to σ so that $L(E_{ii}) = E_{ii}$ for each $i = 1, \ldots, n$. Then we can easily show that L is an invertible linear operator on $\mathbb{M}_n(\mathbb{S})$ which preserves commuting pairs of matrices. By Lemma 3.1, L permutes $\mathbb{E}_n$. Therefore for any cell E_{rs} in $\mathbb{E}_n$, there exists exactly one cell E_{pq} in $\mathbb{E}_n$ such that $L(E_{rs}) = E_{pq}$.

Suppose that $r \neq s$. Since L is injective, we have $p \neq q$ because $L(E_{ii}) = E_{ii}$ for each $i = 1, \ldots, n$. Assume that $p \neq r$ and $p \neq s$. Then

$$E_{rs}(E_{rr} + E_{ss} + E_{pp}) = (E_{rr} + E_{ss} + E_{pp})E_{rs},$$

so that

$$L(E_{rs})L(E_{rr} + E_{ss} + E_{pp}) = L(E_{rr} + E_{ss} + E_{pp})L(E_{rs}),$$

equivalently

$$E_{pq}(E_{rr} + E_{ss} + E_{pp}) = (E_{rr} + E_{ss} + E_{pp})E_{pq}.$$

It follows that $q = r$ or $q = s$. Since

$$E_{rs}(E_{rr} + E_{ss}) = (E_{rr} + E_{ss})E_{rs},$$

we have

$$L(E_{rs})L(E_{rr} + E_{ss}) = L(E_{rr} + E_{ss})L(E_{rs}),$$

or equivalently

$$E_{pq}(E_{rr} + E_{ss}) = (E_{rr} + E_{ss})E_{pq}.$$

Since $q = r$ or $q = s$, we have

$$E_{pq}(E_{rr} + E_{ss}) = E_{pr} \quad \text{or} \quad E_{ps},$$

but $(E_{rr} + E_{ss})E_{pq} = 0$, a contradiction. Hence we have $p = r$ or $p = s$. Similarly we obtain $q = r$ or $q = s$. Therefore, for each $E_{rs} \in \mathbb{E}_n$, we have

$$L(E_{rs}) = E_{rs} \quad \text{or} \quad L(E_{rs}) = E_{sr}. \tag{3.1}$$

Suppose that $L(E_{rs}) = E_{rs}$ with $r \neq s$ and $L(E_{rt}) = E_{tr}$ for some $t \neq r, s$. It follows from (3.1) and invertibility of L that $L(E_{st} + E_{ts}) = E_{st} + E_{ts}$. Let $A = E_{rr} + E_{st} + E_{ts}$ so that $L(A) = E_{rr} + E_{st} + E_{ts}$. Then

$$(E_{rs} + E_{rt})A = A(E_{rs} + E_{rt}),$$

and hence

$$L(E_{rs} + E_{rt})L(A) = L(A)L(E_{rs} + E_{rt}).$$

But $L(E_{rs}+E_{rt})L(A) = E_{rt}+E_{tr}$, while $L(A)L(E_{rs}+E_{rt}) = E_{rs}+E_{sr}$. Thus we have $t = s$, a contradiction. It follows that if $L(E_{ij}) = E_{ij}$ for some $E_{ij} \in \mathbb{E}_n$ with $i \neq j$, then we have $L(E_{rs}) = E_{rs}$ for all $E_{rs} \in \mathbb{E}_n$. Similarly, if $L(E_{ij}) = E_{ji}$ for some $E_{ij} \in \mathbb{E}_n$ with $i \neq j$, then we have $L(E_{rs}) = E_{sr}$ for all $E_{rs} \in \mathbb{E}_n$. Consequently, we have established that $L(X) = X$ or $L(X) = X^t$ for all $X \in \mathbb{M}_n(\mathbb{S})$.

Let $L(X) = X$ for all $X \in \mathbb{M}_n(\mathbb{S})$. By the definition of L, we have

$$P^tT(X)P = X,$$

or equivalently

$$T(X) = PXP^t$$

for all $X \in \mathbb{M}_n(\mathbb{S})$. Similarly, if $L(X) = X^t$, we obtain that $T(X) = PX^tP^t$ for all $X \in \mathbb{M}_n(\mathbb{S})$.

The converse is immediate. □

4. The general Boolean case

In this section, we extend the results of the binary Boolean case to those of general Boolean case. Also we obtain another linear operator that preserves commuting pairs of matrices over a general Boolean algebra, which is neither a transposition operator nor a similarity operator.

If T is a linear operator on $\mathbb{M}_n(\mathbb{B}_k)$ with $k \geq 1$, for each $1 \leq l \leq k$ define its l^{th} *constituent operator*, T_l, by

$$T_l(B) = (T(B))_l$$

for all $B \in \mathbb{M}_n(\mathbb{B}_1)$. By the linearity of T, we have

$$T(A) = \sum_{l=1}^{k} \sigma_l T_l(A_l)$$

for any matrix $A \in \mathbb{M}_n(\mathbb{B}_k)$.

LEMMA 4.1. *If T is an invertible linear operator on $\mathbb{M}_n(\mathbb{B}_k)$ with $k \geq 1$, then each l^{th} constituent operator, T_l, commutes $\mathbb{E}_n$.*

Proof. It follows from Lemma 3.1 and the definition of a constituent operator. □

For any fixed invertible matrix U in $\mathbb{M}_n(\mathbb{S})$, the operator $A \to UAU^t$ is called a *similarity operator.* We can easily show that any similarity operator on $\mathbb{M}_n(\mathbb{S})$ is an invertible linear operator and preserves commuting pairs of matrices. Also, we can restate Theorem 3.3 as follows: the semigroup of linear operators that preserve commuting pairs of matrices over $\mathbb{B}_1$ is generated by transpositions and similarity operators. But for general Boolean algebra $\mathbb{B}_k$ with $k \geq 2$, the following example shows that there exists another invertible linear operator preserving commuting pairs of matrices which is neither a transposition operator nor a similarity operator.

EXAMPLE 4.2. Let

$$U = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_1 \end{bmatrix} \in \mathbb{M}_2(\mathbb{B}_2).$$

By Lemma 2.1, U is an invertible matrix in $\mathbb{M}_2(\mathbb{B}_2)$ with $U^{-1} = U^t$. Define an operator T on $\mathbb{M}_2(\mathbb{B}_2)$ by

$$T(X) = U(\sigma_1 X_1 + \sigma_2 X_2^t)U^t$$

for all $X = \sum_{l=1}^{2} \sigma_l X_l \in \mathbb{M}_2(\mathbb{B}_2)$. Then we can easily show that T is a linear operator on $\mathbb{M}_2(\mathbb{B}_2)$ which is neither a transposition nor a similarity.

Let $T(X) = T(Y)$. Since U is invertible, we have

$$\sigma_1 X_1 + \sigma_2 X_2^t = \sigma_1 Y_1 + \sigma_2 Y_2^t,$$

and hence $X_l = Y_l$ for each $l = 1, 2$. By the uniqueness of canonical form of a matrix, $X = Y$ and thus T is injective.

Let $Y = \sum_{l=1}^{2} \sigma_l Y_l$ be any matrix in $\mathbb{M}_2(\mathbb{B}_2)$. Then we can take the matrix $X = \sigma_1 Y_1 + \sigma_2 Y_2^t \in \mathbb{M}_2(\mathbb{B}_2)$, so that $T(X) = Y$. This implies that T is surjective. Therefore T is invertible. It follows from the canonical form of a matrix in $\mathbb{M}_n(\mathbb{B}_k)$ that T preserves commuting pairs of matrices.

THEOREM 4.3. *Let T be a linear operator on $\mathbb{M}_n(\mathbb{B}_k)$ with $k \geq 1$. Then T is an invertible linear operator preserving commuting pairs of matrices if and only if there exists an invertible matrix U in $\mathbb{M}_n(\mathbb{B}_k)$*

such that

$$T(X) = U\Big(\sum_{l=1}^{k} \sigma_l Y_l\Big) U^t$$

for all $X \in \mathbb{M}_n(\mathbb{B}_k)$*, where* $Y_l = X_l$ *or* $Y_l = X_l^t$ *for each* $l = 1, \ldots, k$*.*

Proof. Assume that T is an invertible linear operator on $\mathbb{M}_n(\mathbb{B}_k)$ preserving commuting pairs of matrices. Then we can easily show that all its constituent operators, T_l , are invertible linear operators on $\mathbb{M}_n(\mathbb{B}_1)$ and preserve commuting pairs of matrices for each $l = 1, \ldots, k$.

Let $X = \sum_{l=1}^{k} \sigma_l X_l$ be any matrix in $\mathbb{M}_n(\mathbb{B}_k)$. Then we have $T(X) = \sum_{l=1}^{k} \sigma_l T_l(X_l)$. By Theorem 3.3, each l^{th} constituent operator, T_l, has the form

$$T_l(X_l) = P_l X_l P_l^t \quad \text{or} \quad T_l(X_l) = P_l X_l^t P_l^t,$$

where each P_l is a permutation matrix for all $l = 1, \ldots, k$. Thus we have

$$T(X) = \sum_{l=1}^{k} \sigma_l P_l Y_l P_l^t,$$

where $Y_l = X_l$ or $Y_l = X_l^t$ for each $l = 1, \ldots, k$, equivalently

$$T(X) = \Big(\sum_{l=1}^{k} \sigma_l P_l\Big)\Big(\sum_{l=1}^{k} \sigma_l Y_l\Big)\Big(\sum_{l=1}^{k} \sigma_l P_l\Big)^t.$$

If we let $U = \Big(\sum_{l=1}^{k} \sigma_l P_l\Big)$, then U is invertible in $\mathbb{M}_n(\mathbb{B}_k)$ by Lemma 2.1, and hence the result is satisfied.

The converse is immediate. □

Thus we obtain characterizations of invertible linear operators which preserve commuting pairs of matrices over general Boolean algebra or chain semiring.

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