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MODIFICATION TO A BOUND FOR RANDOM ERROR CORRECTION WITH LEE WEIGHT

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Abstract. In [1], Sharma and Goel obtained a bound for random error correcting codes with Lee weight considerations. The purpose of this paper is to first point out a discrepancy in this result and then give a correct version of the same, improving upon the bound tremendously.

1. Bound for random error correction

In Sharma and Goel [1], the following result has been proved:

Lemma. *If V^e denotes all vectors of length n and Lee weight e or less, then*

$$V^e = \sum_{N=0}^{e} P(N, n, [q/2]), \tag{1}$$

where for an integer $N \geq 0$

$$P(N, n, [q/2]) = \sum_{K=1}^{[q/2]} \sum_{r_0, r_1, \cdots, r_K} \frac{n!}{r_0! r_1! \cdots r_K!} e_1^{r_1} e_2^{r_2} \cdots e_K^{r_K} \tag{2}$$

and the r_i's are integers such that

$$r_0 + r_2 + \cdots + r_K = n, \quad r_K \geq 1, \qquad r_i \geq 0, \qquad i \neq K \tag{3}$$

$$r_1 + 2r_2 + \cdots + K r_K = N. \tag{4}$$

(*Here $[q/2]$ stands for the largest integer contained in $q/2$*).

There are few discrepancies in the above result which are as follows:

(1) The condition (3) viz. $r_0 + r_2 + \cdots + r_K = n$ should be read as $r_0 + r_1 + r_2 + \cdots + r_K = n$.

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(2) In equations (2), (3), and (4), the minimum value of K is 1 and therefore, we are enumerating vectors of Lee weight $1, 2, \cdots, N$. The vector of Lee weight 0 viz $(0, 0, \cdots, 0)$ has not been taken care of i.e. the case $N = 0$ has not been considered. Further, equations (2), (3) and (4) are valid only for $N > 0$ and not for $N = 0$.

(3) Moreover, the upper limit of K in the summation of equation (2) should be $\min(N, [q/2])$ because when we consider partitions of an integer N, the largest entry in the partition can not be greater than N. Also, over $F(q)$, the largest equivalent value is $[q/2]$. Therefore, in the partition of an integer $N (N > 0)$ over $F(q)$, the largest entry in the partition is equivalent to K where $1 \leq K \leq \min(N, [q/2])$.

For example, for enumerating vector of Lee weight 2 or less ($e = 2 \Rightarrow N = 1, 2$) over $GF(97)$, we consider the following two cases for $N = 1, 2$ respectively:

Case(i): For $N = 1$ ($\Rightarrow K$ varies from 1 *to* $[q/2] = 48$), the equations (3) and (4) for $N = 1$ and for different values of K varying from 1 to 48 are given by

(5)

$$\begin{cases} \text{For } \ K = 1 \begin{cases} r_0 + r_1 = n, \ r_1 \geq 1, \ r_i \geq 0 \ \ i \neq 1 \\ r_1 = 1, \end{cases} \\ \text{For } \ K = 2 \begin{cases} r_0 + r_1 + r_2 = n, \ r_2 \geq 1, \ r_i \geq 0 \ \ i \neq 2 \\ r_1 + 2r_2 = 1, \end{cases} \\ \text{For } \ K = 3 \begin{cases} r_0 + r_1 + r_2 + r_3 = n, \ r_3 \geq 1, \ r_i \geq 0 \ \ i \neq 3 \\ r_1 + 2r_2 + 3r_3 = 1, \end{cases} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \text{For } \ K = 48 \begin{cases} r_0 + r_1 + \cdots + r_{48} = n, \ r_{48} \geq 1, \ r_i \geq 0 \ \ i \neq 48 \\ r_1 + 2r_2 + 3r_3 + \cdots + 48r_{48} = 1. \end{cases} \end{cases}$$

Case(ii): For $N = 2$ ($\Rightarrow K$ varies from 1 to $[q/2] = 48$), the equations (3) and (4) for $N = 2$ and for different values of K varying from 1 to 48 are given by

(6)

$$\begin{cases} \text{For } K=1 \begin{cases} r_0+r_1=n,\ r_1\geq 1,\ r_i\geq 0 \quad i\neq 1 \\ r_1=2, \end{cases} \\ \text{For } K=2 \begin{cases} r_0+r_1+r_2=n,\ r_2\geq 1,\ r_i\geq 0 \quad i\neq 2 \\ r_1+2r_2=2, \end{cases} \\ \text{For } K=3 \begin{cases} r_0+r_1+r_2++r_3=n,\ r_3\geq 1,\ r_i\geq 0 \quad i\neq 3 \\ r_1+2r_2+3r_3=2, \end{cases} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \text{For } K=48 \begin{cases} r_0+r_1+\cdots+r_{48}=n,\ r_{48}\geq 1,\ r_i\geq 0 \quad i\neq 48 \\ r_1+2r_2+3r_3+\cdots+48r_{48}=2. \end{cases} \end{cases}$$

Now, for $N=1$, 48 simultaneous equations in system of equations (5) are checked. Out of which only first simultaneous equation (for $K=1$) has solution (viz. $r_0=n-1, r_1=1$) and rest of the 47 equations are not required to be satisfied.

Also, $N=2$, the first two simultaneous equations (for $K=1,2$) in system of equations (6) have solutions and rest of the 46 equations are redundant.

Therefore, for $e=2$ ($\Rightarrow 1\leq N\leq 2$), total number of redundant equations checked over $F(97)=47+46=93$.

Similarly, over $F(q)$, number of redundant equations checked for $e=2$ (i.e., $1\leq N\leq 2$)

$$\begin{aligned} &= ([q/2]-1)+([q/2]-2) \\ &= 2[q/2]-3. \end{aligned}$$

In general, number of redundant equations checked for any integer $e>0$ ($\Rightarrow 1\leq\ N\leq e$) over $F(q)$

$$\begin{aligned} &= ([q/2]-1)+([q/2]-2)+\cdots+([q/2]-e) \\ &= e[q/2]-(1+2+\cdots+e) \\ &= e[q/2]-\frac{e(e+1)}{2} \\ &= e[q/2]-\frac{(e+1)}{2}. \end{aligned}$$

When $q >> e$, this number is very large.

In view of the above, we give correct version of the Lemma with its proof.

THEOREM. *If V^e denotes all vectors of length n and Lee weight e or less, then*

$$V^e = \sum_{N=0}^{e} P(N, n, [q/2]), \tag{7}$$

where for an integer $N = 0$

$$P(N, n, [q/2]) = 1 \tag{8}$$

and for $N > 0$

$$P(N, n, [q/2]) = \sum_{K=1}^{min\ (N,[q/2])} \sum_{r_0, r_1 \cdots, r_K} \frac{n!}{r_0! r_1! \cdots r_K!} e_1^{r_1} e_2^{r_2} \cdots e_K^{r_K}, \tag{9}$$

where $r_i's$ are integers such that

$$r_0 + r_1 + r_2 + \cdots + r_K = n, \ r_K \geq 1, \ r_i \geq 0 \quad i \neq K \tag{10}$$

$$r_1 + 2r_2 + \cdots + Kr_K = N. \tag{11}$$

PROOF. We consider partitions of an integer $N(1 \leq N \leq e)$, the largest entry in which is exactly equivalent to $K(1 \leq K \leq \min\ (N, [q/2])$. If r_i is the number of times i or an entry equivalent to i occurs, then the number of vectors of length n that can be formed by filling n positions from the integers $0, 1, \cdots, K$ is given by

$$\frac{n!}{r_0! r_1! \cdots r_K!} e_1^{r_1} e_2^{r_2} \cdots e_K^{r_K}. \tag{12}$$

Conditions (10) and (11) immediately follow, as the total number of entries are n and their sum is N. Now summing (12) for all possible values of $r_i's$ and $K(1 \leq K \leq \min(N, [q/2]))$, we get equation (9). Finally, V^e is obtained by summing (8) and (9) for all possible values of N which range from 0 to e. □

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