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TOPOLOGICAL PRESSURE OF CONTINUOUS FLOWS WITHOUT FIXED POINTS

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Abstract. The purpose of this paper is to correct erroneous proofs of theorems of He, Lu, Wang and Zheng [1] about the topological pressure of continuous flows without fixed points and to give proofs of some results which are stated in the literature without proofs and thereby to show that their theorems are still true.

1. Introduction and preliminaries

Let (X, Φ) be a continuous flow [i.e. $\Phi : X \times \mathbb{R} \to X$ is continuous, $\Phi(x, 0) = x$ and $\Phi(x, s+t) = \Phi(\Phi(x, s), t))$] on a compact metric space (X, d). Write Φ_t for homeomorphism of X defined by $\Phi_t(x) = \Phi(x, t)$. Let $C(X, \mathbb{R})$ denote the Banach algebra of real valued continuous functions on X equipped with the supremum norm.

Definition 1.1. For $E \subset X$, we say that E is a (t, δ)*-separated set* if for every $x, y \in E$ with $x \neq y$, $d(\Phi_s x, \Phi_s y) > \delta$ for some $s \in [0, t]$.

Theorem 1.2. *Let E be a (t, δ)- separated set. Then E is a finite set.*

Proof. By the integral continuity theorem (Theorem 4.2 in [2]), for any $x \in X$, any number $t > 0$ and any $\varepsilon > 0$, there exists $\delta_x > 0$ such that $d(\Phi_s(x), \Phi_s(y)) < \varepsilon/2$ for all $y \in X$ and $s \in \mathbb{R}$ which satisfies the inequalities $d(x, y) < \delta_x$ and $0 \leq s \leq t$. The collection $\{B(x, \delta_x) : x \in X\}$ is an open covering of X. Hence, finitely many of them, say $B(x_1, \delta_{x_1}), B(x_2, \delta_{x_2}), \ldots, B(x_n, \delta_{x_n})$ cover X. To prove the set E is a finite set, assume the cardinal number of the set E is greater than n.

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Then we can take $x, y \in E$ with $x \neq y$ and $x, y \in B(x_i, \delta_{x_i})$ for some $1 \leq i \leq n$. It follows that

$$d(\Phi_s(x), \Phi_s(y)) \leq d(\Phi_s(x), \Phi_s(x_i)) + d(\Phi_s(x_i), \Phi_s(y)) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

contradicting the assumption that E is a (t, ε)-separated set of X. □

For $f \in C(X, \mathbb{R})$, $E \subset X$ and $t > 0$, put

$$S_t(E) = \sum_{x \in E} \exp \int_0^t f(\Phi_t(x)) ds.$$

Also, we define

$$P(\Phi, f, \varepsilon, t) = \sup\{S_t(E) : E \quad \text{is a (t, } \varepsilon) - \text{separated set of X}\}.$$

$$P(\Phi, f, \varepsilon) = \limsup_{t \to \infty} 1/t \log P(\Phi, f, \varepsilon, t).$$

and

$$P(\Phi, f) = \lim_{\varepsilon \to 0} P(\Phi, f, \varepsilon).$$

DEFINITION 1.3. The map $P(\Phi, \cdot) : C(X, \mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ defined above is called the *topological pressure* of Φ.

REMARK 1.4. For $f = 0$, the number $P(\Phi, f)$ is just the topological entropy $h(\Phi)$ of Φ.

Let I be any interval of real numbers containing the origin. A *reparametrization* of I is an orientation preserving homeomorphism from I onto its image fixing the origin. We put

$$Rep(I) = \{\sigma : I \to \mathbb{R} \text{ is a reparmetrization of } I\},$$

and

$$Rep(\varepsilon, I)$$
$$= \{\sigma : I \to J \text{is a diffeomorphism with} \sigma(0) = 0 \text{and} |\sigma'(s) - 1| < \varepsilon\},$$

where $\sigma : I \to J$ is a reparametrization of I.

DEFINITION 1.5. For $E \subset X$ and $\varepsilon > 0$, E is called a (t, ε)-*strongly separated set* if for every $x, y \in E$ with $x \neq y$ and $\alpha, \beta \in Rep(\varepsilon, [o, t])$, then $d(\Phi_{\alpha(s)}(x), \Phi_s(y)) > \varepsilon$ or $d(\Phi_s(x), \Phi_{\beta(s)}(y)) > \varepsilon$ for some $s \in [0, t]$.

DEFINITION 1.6. For $E, F \subset X$ and $\varepsilon > 0$, E is called a (t, ε)-*spanning set* of F if for every $x \in F$, there is $y \in E$ such that $d(\Phi_s(y), \Phi_s(x)) \leq \varepsilon$ for all $0 \leq s \leq t$.

DEFINITION 1.7. For $E, F \subset X$ and $\varepsilon > 0$, E is called a (t, ε)-*weakly spanning set* of F if for every $x \in X$, there are $y \in E$ and $\alpha \in Rep(\varepsilon, [0, t])$ such that $d(\Phi_{\alpha(s)}(x), \Phi_s(y)) \leq \varepsilon$ for all $0 \leq s \leq t$.

DEFINITION 1.8. For $E, F \subset X$ and $\varepsilon > 0$, E is called a (t, ε)-*tracing set* of F if for every $x \in F$, there are $y \in E$ and $\alpha \in Rep(\varepsilon, [0, t])$ such that $d(\Phi_{(s)}(x), \Phi_{\alpha(s)}(y)) \leq \varepsilon$ for all $0 \leq s \leq t$.

2. Topological pressure

For $f \in C(X, \mathbb{R})$, $\varepsilon > 0$ and $t > 0$, we put

$$Q(\Phi, f, \varepsilon, t) = \inf\{S_t(E) : E \quad \text{is a (t, } \varepsilon) - \text{spanning set of } X\},$$

$$P_s(\Phi, f, \varepsilon, t) = \sup\{S_t(E) : E \quad \text{is a (t, } \varepsilon)\text{−strongly separated set of } X\},$$

$$Q_w(\Phi, f, \varepsilon, t) = \inf\{S_t(E) : E \quad \text{is a (t, } \varepsilon) - \text{weakly spanning set of } X\},$$

and

$$T(\Phi, f, \varepsilon, t) = \inf\{S_t(E) : E \quad \text{is a (t, } \varepsilon) - \text{tracing set of } X\}.$$

Put

$$R(\Phi, f, \varepsilon) = \limsup_{t \to \infty} 1/t \log R(\Phi, f, \varepsilon, t),$$

and

$$R(\Phi, f) = \lim_{\varepsilon \to 0} P(\Phi, f, \varepsilon),$$

where $R(\cdot) = Q(\cdot)$, $P_s(\cdot)$, $Q_W(\cdot)$ or $T(\cdot)$.

In this paper, we shall show that if $\Phi : X \times \mathbb{R} \to X$ is a continuous flow without fixed points, then $P(\Phi, f) = Q(\Phi, f) = P_s(\Phi, f) = Q_w(\Phi, f) = T(\Phi, f)$ for any $f \in C(X, \mathbb{R})$.

LEMMA 2.1. [1] *For any $f \in C(X, \mathbb{R})$ and $c \in \mathbb{R}$, we have $R(\Phi, f + c) = R(\Phi, f) + c$.*

LEMMA 2.2. [1] *For any $\lambda > 0$, there exists $\varepsilon > 0$ such that for any $\alpha \in Rep(\varepsilon, [0,t])$, we have $|\alpha(s) - s| < s\lambda$ for all $0 \le s \le t$.*

LEMMA 3. [1] *Let $g : [a,b] \to [c,d]$ be a strictly increasing differentiable function. If there is $M > 0$ such that $|g(x) - g(y)| \le M|x - y|$ for all $x, y \in M$ and f is an integrable function on $[c,d]$, then $f(g(t))g'(t)$ is an integrable function on $[a,b]$ and $\int_\alpha^\beta f(x)dx = \int_{g^{-1}(\alpha)}^{g^{-1}(\beta)} f(g(t))g'(t)dt$ for $\alpha, \beta \in g([a,b])$.*

THEOREM A. *The topological pressure $P_s(\Phi, f)$ which is defined by a (t,ε)-strongly separated set is equal to the topological pressure $Q_w(\Phi, f)$ defined by a (t,ε)-weakly spanning set of X.*

PROOF. Suppose that $Q_w(\Phi, f) < P_s(\Phi, f)$. Choose two real numbers a, b such that

$$Q_w(\Phi, f) < a < b < P_s(\Phi, f).$$

Let

$$0 < \lambda < \min\{(1 + b + \sqrt{(1-b)^2 + 4a})/2, 1\}.$$

This inequality means that $\lambda - \lambda^2 + a < b(1 - \lambda)$. By Lemma 2.1, we can assume that $0 \le f(x) \le M$ for all $x \in X$. Choose ε_0 such that $\lambda - \lambda^2 + M\varepsilon_0 + a < b(1 - \lambda)$, and if $d(x,y) < \varepsilon_0$, then $|f(x) - f(y)| < \lambda$. If $0 < \varepsilon < \varepsilon_1$ for some ε_1, then $Q_w(\Phi, f, \varepsilon) < a$ and $P_s(\Phi, f, \varepsilon) > b$. Let $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1, 3\lambda, 1\}$. Since

$$Q_w(\Phi, f, \varepsilon/3) = \limsup_{t\to\infty} 1/t \log Q_w(\Phi, f, \varepsilon/3, t) < a,$$

there exists a $t_0 > 0$

$$1/t \log Q_w(\Phi, f, \varepsilon/3, t) < a \quad \text{for} \quad t > t_0.$$

Also, since

$$P_s(\Phi, f, \varepsilon) = \limsup_{t\to\infty} 1/t \log P_s(\Phi, f, \varepsilon, t) > b,$$

there exists $t_1 > t_0$ such that

$$1/t_1 \log P_s(\Phi, f, \varepsilon, t_1) > b.$$

Put $t = t_1/1 - \lambda$. Then we have $t_1 = (1-\lambda)t$. Since $t > t_0$, we have

$$1/t \log Q_w(\Phi, f, \varepsilon/3, t) < a, \quad \text{that is,} \quad Q_w(\Phi, f, \varepsilon/3, t) < e^{at}.$$

From the definition of $Q_w(\Phi, f, \varepsilon/3, t)$, there exists a $(t, \varepsilon/3)$-weakly spanning set F such that $S_t(F) < e^{at}$. Also, since

$$P_s(\Phi, f, \varepsilon, (1-\lambda)t) = P_s(\Phi, f, \varepsilon, t_1) > e^{bt_1} = e^{b(1-\lambda)t},$$

there exists a $((1-\lambda)t, \varepsilon)$-strongly separated set E such that $S_{(1-\lambda)t}(E) > e^{b(1-\lambda)t}$. Define a map $\xi : E \to F$ by choosing for each $x \in E$, some point $\xi(x) \in F$ and $\alpha \in Rep(\varepsilon/3, [0,t])$ such that

$$d(\Phi_{\alpha(s)}(x), \Phi_s(\xi(x))) \leq \varepsilon/3 \quad \text{for all} \quad 0 \leq s \leq t.$$

We claim that the map ξ is injective. To prove this claim, let $\xi(x) = \xi(y)$ for $x, y \in E$ satisfying

$$d(\Phi_{\alpha(s)}(x), \Phi_s(\xi(x))) \leq \varepsilon/3,$$

and

$$d(\Phi_{\beta(s)}(y), \Phi_s(\xi(y))) \leq \varepsilon/3,$$

for all $0 \leq s \leq t$, and $\alpha, \beta \in Rep(\varepsilon/3, [0,t])$. Then, we get

$$d(\Phi_{\alpha(s)}(x), \Phi_{\beta(s)}(y)) \leq 2\varepsilon/3,$$

for all $0 \leq s \leq t$ using $\xi(x) = \xi(y)$.

By virtue of the mean value theorem,

$$|\beta(t) - t|/t = |\beta(t)/t - 1| = |\beta'(\eta) - 1| < \varepsilon/3 < \lambda \quad \text{for some} \quad \eta.$$

This means $\beta(t) > (1-\lambda)t$. Let $s = \beta^{-1}(u)$ for $u \in [0, (1-\lambda)t]$. We assert that

$$\alpha\beta^{-1} \in Rep(\varepsilon, [0, (1-\lambda)t]) \quad \text{and} \quad \alpha^{-1}\beta \in Rep(\varepsilon, [0, (1-\lambda)t]).$$

Since

$$(\alpha\beta^{-1})'(u) = \alpha'(\beta^{-1}(u))(\beta^{-1})'(u)) = \alpha'(s)/\beta'(s),$$

and

$$1-\varepsilon/3<\alpha'(s)<1+\varepsilon/3, 1-\varepsilon/3<\beta'(s)<1+\varepsilon/3,$$

we get

$$(1-\varepsilon/3)/(1+\varepsilon/3)<(\alpha\beta^{-1})'(u)<(1+\varepsilon/3)/(1-\varepsilon/3).$$

That is,

$$1-(2\varepsilon/3+\varepsilon)<(\alpha\beta^{-1})'(u)<1+(2\varepsilon/3-\varepsilon).$$

Therefore,

$$|(\alpha\beta^{-1})'(u)-1|<(2\varepsilon)/(3-\varepsilon)<\varepsilon$$

implies that

$$\alpha\beta^{-1}\in Rep(\varepsilon,[0,(1-\lambda)t]).$$

Using a similar argument, one can show that

$$\beta\alpha^{-1}\in Rep(\varepsilon,[0,(1-\lambda)t]).$$

Consequently, we get $x=y$ by

$$d(\Phi_{\alpha\beta^{-1}(u)}(x),\Phi_u(y))\leq 2\varepsilon/3,$$

and

$$d(\Phi_u(x),\Phi_{\beta\alpha^{-1}(u)}(y))\leq 2\varepsilon/3$$

for all $0\leq u\leq(1-\lambda)t$.

For $x\in E$ and $0\leq s\leq(1-\lambda)t$, let $u=\alpha^{-1}(s)$. Since

$$d(\Phi_s(x),\Phi_{\alpha^{-1}(s)}(\xi(x)))=d(\Phi_{\alpha(u)}(x),\Phi_u(\xi(x)))\leq\varepsilon/3<\varepsilon_0,$$

we obtain that $|f(\Phi_s(x))-f(\Phi_{\alpha^{-1}(s)}(\xi(x)))|<\lambda$. Hence we have

$$f(\Phi_s(x))<\lambda+f(\Phi_{\alpha^{-1}(s)}(\xi(x))).$$

Now, we assert that

$$\int_0^{(1-\lambda)t} f(\Phi_s(x))ds\leq\lambda(1-\lambda)t+\varepsilon/3Mt+\int_0^t f(\Phi_u(\xi(x)))du$$

for $x\in E$ and $0\leq s\leq(1-\lambda)t$.

To prove this fact, let $\alpha(c) = (1-\lambda)t$. Then, it follows that

$$
\begin{aligned}
&\int_0^{(1-\lambda)t} f(\Phi_s(x))ds \\
&< \int_0^{(1-\lambda)t} (\lambda + f(\Phi_{\alpha^{-1}(s)}(\xi(x))))ds \\
&= \lambda(1-\lambda)t + \int_0^{(1-\lambda)t} f(\Phi_{\alpha^{-1}(s)}(\xi(x)))ds \\
&= \lambda(1-\lambda)t + \int_0^{c} f(\Phi_u(\xi(x)))\alpha'(u)du \\
&\leq \lambda(1-\lambda)t + \int_0^{t} f(\Phi_u(\xi(x)))(\varepsilon/3+1)du \\
&= \lambda(1-\lambda)t + \varepsilon/3\int_0^{t} f(\Phi_u(\xi(x)))du + \int_0^{t} f(\Phi_u(\xi(x)))du \\
&\leq \lambda(1-\lambda)t + \varepsilon/3\int_0^{t} Mdu + \int_0^{t} f(\Phi_u(\xi(x)))du \\
&= \lambda(1-\lambda)t + \varepsilon/3Mt + \int_0^{t} f(\Phi_u(\xi(x)))du.
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
\exp(b(1-\lambda)t) &< S_{(1-\lambda)t}(E) \\
&= \sum_{x\in E} \exp\int_0^{(1-\lambda)t} f(\Phi_s(x))ds \\
&\leq \sum_{x\in E} \exp((\lambda-\lambda^2+\varepsilon/3M)t)\exp\int_0^{t} f(\Phi_u(\xi(x)))du \\
&\leq \exp((\lambda-\lambda^2+\varepsilon_0 M)t)\sum_{y\in F}\int_0^{t} f(\Phi_u(y))du \\
&= \exp((\lambda-\lambda^2+\varepsilon_0 M)t)S_t(F) \\
&< \exp((\lambda-\lambda^2+\varepsilon_0 M + a)t).
\end{aligned}
$$

But this is a contradiction. Thus we have $P_s(\Phi, f) \leq Q_w(\Phi, f)$.

To prove that $P_s(\Phi, f) \geq Q_w(\Phi, f)$, assume that $P_s(\Phi, f) < Q_w(\Phi, f)$. Choose a real number a such that $P_s(\Phi, f) < a < Q_w(\Phi, f)$. Also, there exists $\varepsilon_0 > 0$ such that $P_s(\Phi, f, \varepsilon) < a < Q_w(\Phi, f, \varepsilon)$ for all $0 < \varepsilon < \varepsilon_0$. Then, for some $t_0 > 0$, if $t > t_0$, then

$$1/t \log P_s(\Phi, f, \varepsilon, t) < a \quad \text{and} \quad 1/t \log Q_w(\Phi, f, \varepsilon, t) > a.$$

Hence, there exists a maximal (t, ε)-strongly separated set E such that $S_t(E) < e^{at}$, and also E is a (t, ε) -weakly spanning set. It follows that

$$e^{at} < Q_w(\Phi, f, \varepsilon, t) \leq S_t(E) < e^{at}.$$

This is a contradiction. Hence, we get $P_s(\Phi, f) \geq Q_w(\Phi, f)$ □

THEOREM B. *The topological pressure $T(\Phi, f)$ which is defined by a (t, ε)-tracing set is equal to the topological pressure $Q_w(\Phi, f)$ defined by a (t, ε)-weakly spanning set of X.*

PROOF. Suppose on the contrary that $Q_w(\Phi, f) \neq T(\Phi, f)$. Then either $Q_w(\Phi, f) < T(\Phi, f)$ or $Q_w(\Phi, f) > T(\Phi, f)$. First, assume that $Q_w(\Phi, f) < T(\Phi, f)$. Choose two real numbers a, b satisfying $Q_w(\Phi, f) < a < b < T(\Phi, f)$. Also, since $a/b < 1$, choose a real number λ with $0 < \lambda < 1 - a/b$. Then $a < (1 - \lambda)b$. From the assumption, there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then $Q_w(\Phi, f, \varepsilon) < a$ and $b < T(\Phi, f, \varepsilon)$. Let

$$0 < \varepsilon < \min\{\lambda, \varepsilon_0/2, 1/2\}.$$

Since

$$a > Q_w(\Phi, f, \varepsilon) = \lim_{t \to \infty} \sup 1/t \log Q_w(\Phi, f, \varepsilon, t),$$

there is t_0 such that if $t > t_0$, then $1/t \log Q_w(\Phi, f, \varepsilon, t) < a$. Also, from the

$$b < T(\Phi, f, 2\varepsilon) = \lim_{t \to \infty} \sup 1/t \log T(\Phi, f, 2\varepsilon, t)$$

there is a $t_1 > t_0$ such that $b < 1/t_1 \log T(\Phi, f, 2\varepsilon, t_1)$.

Put $t = t_1/1 - \lambda$. Since $t > t_0$, we have $1/t \log Q_w(\Phi, f, \varepsilon, t) < a$, that is, $Q_w(\Phi, f, \varepsilon, t) < e^{at}$. Since $t_1 = (1-\lambda)t$ and $b < 1/t_1 \log T(\Phi, f, 2\varepsilon, t_1)$, we obtain

$$T(\Phi, f, 2\varepsilon, (1 - \lambda)t) > e^{b(1-\lambda)t}.$$

Hence, there exists a (t, ε)-weakly spanning set E such that $S_t(E) < e^{at}$. For each $x \in X$, there exist $y \in X$ and $\alpha \in Rep(\varepsilon, [0, t])$ such that $d(\Phi_{\alpha(s)}(x), \Phi_s(y)) \leq \varepsilon$ for all $s \in [0, t]$. Also, since

$$|\alpha(t) - t|/t = |\alpha(t)/t - 1| = |\alpha'(\eta) - 1| < \epsilon < \lambda \quad \text{for some} \quad \eta,$$

we obtain

$$(1-\lambda)t < \alpha(t).$$

For $u \in [0, (1-\lambda)t]$, put $\alpha^{-1}(u) = s$. Then $d(\Phi_u(x), \Phi_{\alpha^{-1}(u)}(y)) \leq \varepsilon$. Since $(\alpha^{-1})'(u) = 1/\alpha'(s)$, we get

$$1 - \varepsilon/1 + \varepsilon = 1/1 + \varepsilon < (\alpha^{-1})'(u) < \varepsilon/1 - \varepsilon < 1 + \varepsilon/1 - \varepsilon.$$

Hence,

$$|(\alpha^{-1})'(u) - 1| < \varepsilon/1 - \varepsilon < 2\varepsilon \quad \text{that is,} \quad \alpha^{-1} \in \mathit{Rep}(2\varepsilon, [0, (1-\lambda)t]).$$

Consequently, E is a $((1-\lambda)t, 2\varepsilon)$-tracing set.

Without loss of generality, we may assume that $0 \leq f(x)$ for all $x \in X$.

Then we obtain that

$$\begin{aligned} e^{b(1-\lambda)t} &< T(\Phi, f, 2\varepsilon, (1-\lambda)t) \\ &\leq S_{(1-\lambda)t}(E) \\ &= \sum_{x\in E} \exp \int_0^{(1-\lambda)t} f(\Phi_s(x))ds \\ &\leq \sum_{x\in E} \exp \int_0^{t} f(\Phi_s(x))ds \\ &= S_t(E) \\ &< e^{at}. \end{aligned}$$

This is a contradiction. Consequently, we get $T(\Phi, f) \leq Q_w(\Phi, f)$.

Secondly, assume that $T(\Phi, f) < Q_w(\Phi, f)$. Choose two real numbers a, b such that $T(\Phi, f) < a < b < Q_w(\Phi, f)$. Also, choose a real number λ such that $0 < \lambda < 1 - a/b$.

Then $a < (1-\lambda)b$. There is $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then $T(\Phi, f, \varepsilon) < a$ and $b < Q_w(\Phi, f, \varepsilon)$. Let

$$0 < \varepsilon < \min\{\lambda, \varepsilon_0/2, 1/2\}.$$

From the facts that $T(\Phi, f, \varepsilon) < a$ and $b < Q_w(\Phi, f, 2\varepsilon)$, there are t_0 and t_1, respectively, such that if $t > t_0$ and $t_1 > t_0$ then

$$1/t \log T(\Phi, f, \varepsilon, t) < a \quad \text{and} \quad b < 1/t_1 \log Q_w(\Phi, f, 2\varepsilon, t).$$

Let $t = t_1/1 - \lambda$. Since $t > t_0$, we obtain that

$$1/t \log T(\Phi, f, \varepsilon, t) < a, \quad \text{that is,} \quad T(\Phi, f, \varepsilon, t) < e^{at}.$$

Also,

$$Q_w(\Phi, f, 2\varepsilon, (1-\lambda)t) > e^{b(1-\lambda)t}$$

using $t_1 = (1-\lambda)t$ and $b < 1/t_1 \log Q_w(\Phi, f, 2\varepsilon, t_1)$.

There is a (t, ε)-tracing set E such that $S_t(E) < e^{at}$. Moreover, E is a $((1-\lambda)t, 2\varepsilon)$-weakly spanning set of X. Therefore, we get

$$\begin{aligned} e^{b(1-\lambda)t} &< Q_w(\Phi, f, \varepsilon, (1-\lambda)t) \\ &\leq S_{(1-\lambda)t}(E) \\ &= \sum_{x \in E} \exp \int_0^{(1-\lambda)t} f(\Phi_s(x))ds \\ &\leq \sum_{x \in E} \exp \int_0^{t} f(\Phi_s(x))ds \\ &= S_t(E) \\ &< e^{at}. \end{aligned}$$

We have a contradiction. Hence, we get $Q_w(\Phi, f) \leq T(\Phi, f)$. Consequently, it follows that $Q_w(\Phi, f) = T(\Phi, f)$. □

THEOREM C. *The topological pressure $P(\Phi, f)$ using a (t, ε)-separated set is equal to the topological pressure $Q(\Phi, f)$ via a (t, ε)-spanning set of X.*

PROOF. Assume that $P(\Phi, f) < Q(\Phi, f)$. Choose a real number a such that $P(\Phi, f) < a < Q(\Phi, f)$. Since $P(\Phi, f) = \lim_{\varepsilon \to 0} P(\Phi, f, \varepsilon)$ and $Q(\Phi, f) = \lim_{\varepsilon \to 0} Q(\Phi, f, \varepsilon)$, there is ε_0 such that if $0 < \varepsilon < \varepsilon_0$, then $P(\Phi, f, \varepsilon) < a < Q(\Phi, f, \varepsilon)$.

By the definitions of $P(\Phi, f, \varepsilon)$ and $Q(\Phi, f, \varepsilon)$, there exists $t_0 > 0$ such that

$$1/t \log P(\Phi, f, \varepsilon, t) < a \quad \text{and} \quad 1/t \log Q(\Phi, f, \varepsilon, t) > a \quad \text{for all} \quad t > t_0.$$

There exists a maximal (t, ε)-separated set E such that $S_t(E) < e^{at}$. Also, E is a (t, ε)-spanning set. Hence, we obtain

$$e^{at} < Q(\Phi, f, \varepsilon, t) \leq S_t(E) < e^{at}.$$

Hence, this is a contradiction. Therefore, we have $P(\Phi, f) \geq Q(\Phi, f)$.

Now, to show that $P(\Phi, f) \leq Q(\Phi, f)$, we assume that $P(\Phi, f) > Q(\Phi, f)$. Choose two numbers a, b such that $Q(\Phi, f) < a < b < P(\Phi, f)$. There exists $\varepsilon_0 > 0$ such that $Q(\Phi, f, \varepsilon) < a < b < P(\Phi, f, \varepsilon)$ for $0 < \varepsilon < \varepsilon_0$.

Choose $0 < \varepsilon < \varepsilon_0$ satisfying $|f(x) - f(y)| < b - a$ whenever $d(x, y) < \varepsilon$. By virtue of the definition of

$$P(\Phi, f, \varepsilon) \quad \text{and} \quad Q(\Phi, f, \varepsilon),$$

there exists $t_0 > 0$ such that if $t > t_0$, then

$$1/t \log Q(\Phi, f, \frac{\varepsilon}{2}, t) < a \quad \text{and} \quad 1/t \log P(\Phi, f, \varepsilon, t) > b.$$

Then, in view of $P(\Phi, f, \varepsilon, t) > e^{bt}$ and $Q(\Phi, f, \varepsilon/2, t) < e^{at}$, E is a (t, ε)-separated set with $S_t(E) > e^{bt}$ and F is a $(t, \varepsilon/2)$-spanning set with $S_t(F) < e^{at}$.

Define the the map $\xi : E \to F$ by choosing for each $x \in E$ such that

$$d(\Phi_s(x), \Phi_s(\xi(x))) \leq \varepsilon/2 \quad \text{for all} \quad 0 \leq s \leq t.$$

Then ξ is an injective map. To see this, let $\xi(x) = \xi(y)$ for $x, y \in E$. Then we obtain $x = y$ from the fact

$$\begin{aligned} d(\Phi_s(x), \Phi_s(\xi(y)) &\leq d(\Phi_s(x), \Phi_s(\xi(x))) + d(\Phi_s(\xi(y)), \Phi_s(y)) \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for all $0 \leq s \leq t$.

Also, we get $|f(\Phi_s(x)) - f(\Phi_s(\xi(x)))| < b - a$ from the inequality $d(\Phi_s(x), \Phi_s(\xi(x))) \leq \varepsilon/2 < \varepsilon$. Hence, we obtain $f(\Phi_s(x)) < b - a + f(\Phi_s(\xi(x)))$. It follows that

$$\begin{aligned} e^{bt} < S_t(E) &= \sum_{x\in E} \exp \int_0^t f(\Phi_s(x))ds \\ &\leq \sum_{x\in E} \exp \int_0^t (b-a+f(\Phi_s(\xi(x))))ds \\ &= e^{(b-a)t} \sum_{x\in E} \exp \int_0^t f(\Phi_s(\xi(x)))ds \\ &\leq e^{(b-a)t} \sum_{y\in F} \exp \int_0^t f(\Phi_s(y))ds \\ &= e^{(b-a)t} S_t(F) \\ &< e^{(b-a)t} e^{at} \\ &= b^{bt}. \end{aligned}$$

This is a contradiction. Hence, we get $P(\Phi, f) \leq Q(\Phi, f)$ and This completes the proof of the theorem. □

THEOREM D. *The topological pressure $T(\Phi, f)$ which is defined by a (t, ε)-tracing set is equal to the topological pressure $Q(\Phi, f)$ defined by a (t, ε)-spanning set of X.*

PROOF. Suppose that $T(\Phi, f) < Q(\Phi, f)$. Choose two real numbers a, b such that $T(\Phi, f) < a < b < Q(\Phi, f)$.

We can assume that $0 \leq f(x) \leq M$ for all $x \in X$.

Take

$$0 < \lambda < \{-a - M + \sqrt{(M-a)^2 + 4bM}\}/\{2M\}.$$

Then, it is clear that $\lambda(1+\lambda)M + (1+\lambda)a < b$.

For some $\varepsilon_0 > 0$, take $0 < \varepsilon < \varepsilon_0$ satisfying $T(\Phi, f, \varepsilon) < a < b < Q(\Phi, f, \varepsilon)$.

Let $0 < \varepsilon < \min\{\lambda, \varepsilon_0, 1\}$ with $\varepsilon + \lambda(1+\lambda)M + (1+\lambda)a < b$. For any $\varepsilon > 0$, choose $0 < \delta < \varepsilon/3$ satisfying $|f(x) - f(y)| \leq \varepsilon$ provided that $x, y \in X$ and $d(x, y) < \delta$.

Since $T(\Phi, f, \delta) < a$, there exists $t_0 > 0$ such that $1/t \log T(\Phi, f, \delta, t) < a$ for all $t > t_0$. Take $\tau > 0$ such that

$$(1+\lambda)/\tau \log 3 + \varepsilon + \lambda(1+\lambda)M + (1+\lambda)a < b.$$

Also, take $t > \max\{t_0, \tau\}$ such that $1/t \log Q(\Phi, f, \varepsilon, t) > b$ via $Q(\Phi, f, \varepsilon) > b$.

Put $t_1 = (1+\lambda)t$. Then, $t_1 > t_0$. Hence, we get

$$1/t_1 \log T(\Phi.f, \delta, t_1) < a.$$

Since $T(\Phi, f, \delta, t_1) < \exp(at_1)$, there is a (t_1, δ)-tracing set E such that $S_t(E) < \exp(at_1)$.

According to the proof of Proposition 14 in [3], we get a (t_1, ε)-spanning set F such that for every $x \in E$, there exists at most $3^{[t_1/\tau]}$ points in F which can be (t_1, δ)-traced by x.

Suppose $y \in F$ can be (t, δ)-traced by $x \in E$, that is, for some $\alpha \in Rep(\delta, [0, t])$ and $d(\Phi_s(y), \Phi_{\alpha(s)}(x)) \leq \alpha$ holds for all $s \in [0, t_1]$.

From $d(\Phi_s(y), \Phi_{\alpha(s)}(x)) \leq \alpha$, we get

$$|f(\Phi_s(y)) - f(\Phi_{\alpha(s)}(x))| \leq \varepsilon, \quad \text{that is,} \quad f(\Phi_s(y)) \leq \varepsilon + f(\Phi_{\alpha(s)}(x)),$$

for all $s \in [0, t]$. Hence, it follows that

$$\int_0^t f(\Phi_s(y))ds \leq \int_0^t (\varepsilon + f(\Phi_{\alpha(s)}(x)))ds = \varepsilon t + \int_0^t f(\Phi_{\alpha(s)}(x))ds.$$

Also, we have

$$1/\alpha'(s) \leq 1/(1-\delta) < 1+2\delta < 1+2\varepsilon/3 < 1+\lambda \quad \text{by} \quad 1-\delta \leq \alpha'(s) \leq 1+\delta.$$

Put $u = \alpha(s)$. Then,

$$ds = 1/\alpha'(s)du \leq (1+\lambda)du.$$

Hence, we obtain that

$$\begin{aligned}\int_0^t f(\Phi_{\alpha(s)}(x))ds &\leq \int_0^{\alpha(t)} f(\Phi_u(x))(1+\lambda)du \\ &= \lambda \int_0^{\alpha(t)} f(\Phi_u(x))du + \int_0^{\alpha(t)} f(\Phi_u(x))du.\end{aligned}$$

Also, since

$$|\alpha(t) - t|/t = |\alpha(t)/t - 1| = |\alpha'(\xi) - 1| \leq \delta < \lambda \quad \text{for some} \quad \xi,$$

we get

$$\alpha(t) < (1+\lambda)t = t_1.$$

Therefore, this show that

$$\int_0^t f(\Phi_s(y))ds \le \lambda \int_0^{t_1} M du + \int_0^{t_1} f(\Phi_u(x))du$$
$$= \lambda M t_1 + \int_0^{t_1} f(\Phi_u(x))du.$$

However, using of a (t,ε)-spanning set F, it follows that

$$\begin{aligned}
\exp(bt) &< Q(\Phi, f, \varepsilon, t) \le S_t(F) \\
&= \sum_{y\in F} \exp \int_0^t f(\Phi_s(y))ds \\
&\le 3^{[t_1/\tau]} \exp((\varepsilon + \lambda(1+\lambda)M)t) \sum_{x\in E} \exp \int_0^{t_1} f(\Phi_u(x))du \\
&\le 3^{t_1/\tau} \exp((\varepsilon + \lambda(1+\lambda)M)t) \sum_{x\in E} \exp \int_0^{t_1} f(\Phi_u(x))du \\
&= \exp((\frac{1+\lambda}{\tau} \log 3 + \varepsilon + \lambda(1+\lambda)M)t) S_{t_1}(E) \\
&< \exp((\frac{1+\lambda}{\tau} \log 3 + \varepsilon + \lambda(1+\lambda)M)t) \exp(at_1) \\
&= \exp((\frac{1+\lambda}{\tau} \log 3 + \varepsilon + \lambda(1+\lambda)M + (1+\lambda)a))t).
\end{aligned}$$

This is a contradiction. Consequently, it follows that $Q(\Phi, f) \le T(\Phi, f)$.

In order to prove that $Q(\Phi, f) \ge T(\Phi, f)$, we assume that $Q(\Phi, f) < T(\Phi, f)$.

Choose a real number a such that $Q(\Phi, f) < a < T(\Phi, f)$. Take $\varepsilon_0 > 0$ with $Q(\Phi, f, \varepsilon) < a < T(\Phi, f, \varepsilon)$ for all $0 < \varepsilon < \varepsilon_0$. From the definitions of $Q(\Phi, f, \varepsilon)$ and $T(\Phi, f, \varepsilon)$, there exists $t_0 > 0$ such that

$$1/t \log Q(\Phi, f, \varepsilon, t) < a \quad \text{and} \quad 1/t \log T(\Phi, f, \varepsilon, t) > a$$

for all $t > t_0$.

$Q(\Phi, f, \varepsilon, t) < e^{at}$ imply $S_t(E) < e^{at}$ for some a (t, ε)-spanning set E. Since E is a (t, ε)-tracing set, it follows that

$$e^{at} < T(\Phi, f, \varepsilon, t) \leq S_t(E) < e^{at}.$$

This is a contradiction. Consequently, we get $T(\Phi, f) \leq Q(\Phi, f)$, and this completes the proof of this theorem. □

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