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THE MINIMAL FREE RESOLUTION OF A CERTAIN DETERMINANTAL IDEAL

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Abstract. Let $S = R[x_{ij}|1 \le i \le m, 1 \le j \le n]$ be the polynomial ring over a noetherian commutative ring R and I_p be the determinantal ideal generated by the $p \times p$ minors of the generic matrix $(x_{ij})(1 \le p \le \min(m,n))$. We describe a minimal free resolution of S/I_p, in the case $m = n = p+2$ over $\mathbb{Z}$.

1. Introduction

Let R be a noetherian commutative ring with unity, and x_{ij} be variables with $1 \le i \le m$ and $1 \le j \le n$. If we let $S = R[x_{ij}]$ be the polynomial ring over R, then we have the generic matrix (x_{ij}) and we may form the determinantal ideal I_p of S generated by the $p \times p$ minors of this matrix for $1 \le p \le \min(m,n)$. For many years there has been considerable interest in finding minimal free resolutions of S/I_p. Eagon and Hochster [6] proved that I_p is perfect (i.e., gradeI_p =pd$_S S/I_p$) and that if R is a (normal) domain S/I_p is a (normal) domain and S/I_p is R-free. If R is Cohen-Macaulay then so is S/I_p. Therefore free resolutions of S/I_p have the property of so-called depth sensitivity. If $p = 1$, then the Koszul complex gives us such a resolution. Eagon and Northcott [7], Buchsbaum and Rim [5] constructed a minimal free resolution of S/I_p when $p = \min(m,n)$, separately. On the other hand, Roberts [12], Lascoux [11], Pragacz and Weyman [13] constructed the minimal free resolution (Lascoux's resolution) of S/I_p for any m, n, and p in the case when R contains the rational number field $\mathbb{Q}$. Their description of the resolution is based on the representation theory of general linear group.

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If we have a minimal free resolution $\mathbb{P}$ of S/I_p when $R = \mathbb{Z}$, the ring of integers, then for any ring R, $R \otimes_{\mathbb{Z}} \mathbb{P}$ is a minimal free resolution, since S/I_p is $\mathbb{Z}$-free. There have been some efforts to apply the representation theory to the case $R = \mathbb{Z}$. In 1979, Buchsbaum [3] gave another description of the Eagon-Northcott complex using multilinear algebra. In the early eighties Akin, Buchsbaum and Weyman developed characteristic-free representation theory of general linear groups and constructed a minimal free resolution (the Akin-Buchsbaum-Weyman complex) of S/I_p over $\mathbb{Z}$, in the case $p = \min(m,n) - 1$ [1].

Roberts proved that there exists a minimal free resolution of S/I_p over $\mathbb{Z}$ if and only if the Betti numbers of S/I_p are independent of characteristic of base field [4]. Using this fact, in the early nineties Hashimoto and Kurano [10] proved that there exists a minimal free resolution of S/I_p when $m = n = p+2$. Hashimoto extended this result to the case $p = \min(m,n) - 2$ [9] and proved that there is no minimal free resolution of S/I_p over $\mathbb{Z}$, in the case $2 \leq p \leq \min(m,n) - 3$ [8]. But the construction of the minimal free resolution of S/I_p over $\mathbb{Z}$ when $p = \min(m,n) - 2$ is still open.

In section 2, we review some facts on characteristic free representation theory of general linear group including Schur modules and Schur complexes.

In section 3, we define the durfee square complex and give the explicit characteristic free resolution of an ideal generated by the submaximal minors of the generic square matrix. The main result consists of finding the minimal free resolution of the ideal of $p \times p$ minors of $(p+2) \times (p+2)$ matrix in a characteristic free case.

2. Preliminaries

This section is devoted to introducing the definitions and quoting without proofs the basic facts on Schur modules and Schur complexes from([1, 2]).

We will denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{N}^\infty$ the set of sequences of elements of $\mathbb{N}$ of finite support. If $\mathbb{N}^p$ denotes the set of p-tuples of elements of $\mathbb{N}$, then $\mathbb{N}^p$ is identified with a subset of $\mathbb{N}^\infty$ by extending any p-tuple $(\lambda_1, \dots, \lambda_p)$ by zeroes. Thus $\mathbb{N}^\infty = \bigcup_{p \geqslant 0} \mathbb{N}^p$.

If $\lambda = (\lambda_1, \lambda_2, \dots)$ is an element of $\mathbb{N}^\infty$, we define the conjugate $\tilde{\lambda}$ of λ to be the element $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots)$ of $\mathbb{N}^\infty$ where $\tilde{\lambda}_j$ is the number of terms of λ which are greater than or equal to j. The conjugate $\tilde{\lambda}$

of any element $\lambda \in \mathbb{N}^\infty$ has the property that $\tilde{\lambda}_1 \geqslant \tilde{\lambda}_2 \geqslant \cdots$ and the element $\tilde{\tilde{\lambda}}$ is the sequence λ rearranged in decreasing order. Thus, if λ is a non-increasing sequence, $\lambda = \tilde{\tilde{\lambda}}$.

A partition is an element $\lambda = (\lambda_1, \lambda_2, \dots)$ of $\mathbb{N}^\infty$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$. The weight of the partition λ, denoted by $\|\lambda\|$, is defined to be the sum $\sum \lambda_i$. If $\|\lambda\| = n$, λ is said to be a partition of n. The number of non-zero terms of λ is called the length of λ.

The diagram (or shape) of element $\lambda \in \mathbb{N}^\infty$ is the set of ordered pairs (i,j) in $\mathbb{N}^2$ with $i \geqslant 1$ and $1 \leqslant j \leqslant i$, and is denoted by Δ_λ. We are adopting the convention that is used with the matrices that the row index i increases as one goes downward and the column index j increases from left to right.

Using the diagrams, one can see that if λ is a partition then $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots)$ is the partition whose jth term $\tilde{\lambda}_j$ is the number of squares in the jth column of diagram of λ, where we count the columns from left to right. It is therefore clear that $\|\lambda\| = \|\tilde{\lambda}\|$.

If F is a free module over a commutative ring R and $\lambda = (\lambda_1, \dots, \lambda_q)$ is in $\mathbb{N}^\infty$, we use the following notation:

$$\begin{aligned} \wedge_\lambda F &= \wedge^{\lambda_1} F \otimes_R \cdots \otimes_R \wedge^{\lambda_q} F; \\ S_\lambda F &= S_{\lambda_1} F \otimes_R \cdots \otimes_R S_{\lambda_q} F; \\ D_\lambda F &= D_{\lambda_1} F \otimes_R \cdots \otimes_R D_{\lambda_q} F, \end{aligned}$$

where $\wedge$, S, and D denote the exterior, symmetric and divided powers.

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_q)$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_q)$ be in $\mathbb{N}^\infty$. We define $\mu \subseteq \lambda$ if $\mu_i \leqslant \lambda_i$ for all $i \geqslant 1$, and the skew partition, denoted by λ/μ, to be $(\lambda_1 - \mu_1, \lambda_2 - \mu_2, \dots)$. It is natural to think of a sequence $(0) = (0, 0, \dots)$.

Suppose now that $\lambda/\mu = (\lambda_1, \dots, \lambda_s)/(\mu_1, \dots, \mu_s)$ is a skew partition. The shape matrix of λ/μ is an $s \times \lambda_1$ matrix $A = (a_{ij})$ defined by the rule

$$a_{ij} = \begin{cases} 1 & \text{if } \mu_i + 1 \leqslant j \leqslant \lambda_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = (a_{ij})$ be the $s \times t$ shape matrix of λ/μ. Also let $a_i = \sum_{j=1}^t a_{ij}$ and $b_j = \sum_{i=1}^s a_{ij}$. We are going to define a morphism of functors $d_{\lambda/\mu} : \wedge_A \to S_{\tilde{A}}$, where $\wedge_A = \wedge^{a_1} \otimes \cdots \otimes \wedge^{a_s}$, $S_{\tilde{A}} = S_{b_1} \otimes \cdots \otimes S_{b_t}$, and $\tilde{A}$ denotes the transpose of A. $d_{\lambda/\mu}$ is described as follows. Diagonalize $\wedge_A$ to get

$$(\wedge^{a_{11}} \otimes \cdots \otimes \wedge^{a_{1t}}) \otimes (\wedge^{a_{21}} \otimes \cdots \otimes \wedge^{a_{2t}}) \otimes \cdots \otimes (\wedge^{a_{s1}} \otimes \cdots \otimes \wedge^{a_{st}})$$

which is isomorphic, by rearranging along the columns of A, to

$$(\wedge^{a_{11}} \otimes \cdots \otimes \wedge^{a_{s1}}) \otimes (\wedge^{a_{12}} \otimes \cdots \otimes \wedge^{a_{s2}}) \otimes \cdots \otimes (\wedge^{a_{1t}} \otimes \cdots \otimes \wedge^{a_{st}}).$$

As $a_{ij} \in \{0, 1\}$, this is equal to

$$(S_{a_{11}} \otimes \cdots \otimes S_{a_{s1}}) \otimes (S_{a_{12}} \otimes \cdots \otimes S_{a_{s2}}) \otimes \cdots \otimes (S_{a_{1t}} \otimes \cdots \otimes S_{a_{st}}).$$

Now use multiplication in S to go to

$$S_{b_1} \otimes S_{b_2} \otimes \cdots \otimes S_{b_t} = S_{\tilde{A}}.$$

Similarly, one defines a morphism $d'_{\lambda/\mu} : D_A \to \wedge_{\tilde{A}}$, where $D_A = D_{a_1} \otimes \cdots \otimes D_{a_s}$ and $\wedge_{\tilde{A}} = \wedge^{b_1} \otimes \cdots \otimes \wedge^{b_t}$.

Let R be a commutative ring, F and G be free R-modules of ranks m and n, respectively, and $\phi : F \otimes G \to R$ be an R-map. We will identify $\mathrm{Hom}_R(F \otimes G, R)$ with $\mathrm{Hom}_R(G, F^*)$ via the canonical isomorphism. With this identification, we will use the same symbol ϕ to denote the corresponding map $\phi : G \to F^*$, and write $\phi^* : F \to G^*$ for the map dual to ϕ. We will also denote by c_ϕ the element of $F^* \otimes G^*$ corresponding to ϕ under the canonical isomorphism:

$$\mathrm{Hom}_R(F \otimes G, R) \cong F^* \otimes G^*.$$

DEFINITION 2.1. $\mathbb{A}(\phi)$ is the (doubly graded) complex $(\wedge F^* \otimes DG, \partial_\phi)$, where the differential ∂_ϕ is the given by the action of $c_\phi \in \wedge F^* \otimes SG^*$ on $\wedge F^* \otimes DG$. The subcomplex

$$0 \to D_l G \to F^* \otimes D_{l-1} G \to \cdots \to \wedge^{l-i} F^* \otimes D_i G \to \cdots \to \wedge^l F^* \to 0$$

of $\mathbb{A}(\phi)$ will be denoted by $\mathbb{A}^l(\phi)$. The component of $\mathbb{A}^l(\phi)$ of degree i, denoted by $A_i^l(\phi)$, is the term $\wedge^{l-i} F^* \otimes D_i G$.

We note that $\mathbb{A}^l(\phi)$ is isomorphic to the complex

$$0 \to \wedge^m F \otimes D_l G \to \cdots \to \wedge^{m-l+i} F \otimes D_i G \to \cdots \to \wedge^{m-l} F \to 0$$

with the isomorphism being induced by $\wedge^m F \otimes \wedge^{l-i} F^* \cong \wedge^{m-l+i} F$. We will use the same notation, $\mathbb{A}^l(\phi)$, to denote either one of these two isomorphic complexes.

DEFINITION 2.2. $\mathbb{B}(\phi)$ is the (doubly graded) complex $(DF^* \otimes \wedge G, \delta_\phi)$, where the differential δ_ϕ is given by the action of $c_\phi \in DF^* \otimes \wedge G^*$ on $DF^* \otimes \wedge G$. The subcomplex

$$0 \to \wedge^l G \to F^* \otimes \wedge^{l-1} G \to \cdots \to D_i F^* \otimes \wedge^{l-i} G \to \cdots \to D_l F^* \to 0$$

of $\mathbb{B}(\phi)$ will be denoted by $\mathbb{B}_l(\phi)$. The component of $\mathbb{B}_l(\phi)$ of degree i, denoted by $B_l^i(\phi)$, is the term $D_i F^* \otimes \wedge^{l-i} G$.

DEFINITION 2.3. The tensor product of the complexes $\mathbb{A}^{m-p}(\phi)$ and $\mathbb{A}^{n-p}(\phi^*)$ is again a complex, $\mathbb{A}^{m-p}(\phi) \otimes \mathbb{A}^{n-p}(\phi^*)$, with its customary boundary map. We denote by $\mathbb{U}^p(\phi)$ this complex with its degree shifted by one, i.e., $\mathbb{U}^p(\phi) = \{U_{k+1}^p(\phi)\}$ with

$$U_{k+1}^p(\phi) = (\mathbb{A}^{m-p}(\phi) \otimes \mathbb{A}^{n-p}(\phi^*))_k \quad for \quad k \geq 0.$$

We will denote the boundary map of $\mathbb{U}^p(\phi)$ by ∂_ϕ^u.

Explicitly, we have

$$U_{k+1}^p(\phi) = \sum_{a+b=k} \wedge^{p+a} F \otimes D_a G \otimes \wedge^{p+b} G \otimes D_b F.$$

To describe the boundary map explicitly, we let $\{f_i\}, i = 1, \ldots, m$ and $\{g_j\}, j = 1, \ldots, n$, be bases of F and G, respectively. Let $\{\rho_i\}$, $\{\gamma_j\}$ be their respective dual bases. Then for $x \otimes y \otimes u \otimes v \in \wedge^{p+a} F \otimes D_a G \otimes \wedge^{p+b} G \otimes D_b F$ we have:

$$\partial_\phi^u(x \otimes y \otimes u \otimes v) = \sum \phi(f_i \otimes g_j)\{\rho_i(x) \otimes \gamma_j(y) \otimes u \otimes v + (-1)^a x \otimes y \otimes \gamma_j(u) \otimes \rho_i(v)\}.$$

DEFINITION 2.4. Let $1_F : F \to F$ be the identity map. Instead of writing $\mathbb{A}^l(1_F)$ or $\mathbb{B}_l(1_F)$ we shall denote these complexes by $\mathbb{A}^l(F)$ and $\mathbb{B}_l(F)$, and write ∂_F, δ_F for their boundary maps. Notice that $\mathbb{A}_i^l(F) = \mathbb{B}_l^i(F) = \wedge^{l-i} F \otimes D_i F$.

PROPOSITION 2.5 [1, PROPOSITION 1.5]. *For any free R-module F we have:*

(1) *$\mathbb{A}^l(F)$ is exact for $l > 0$, i.e., $H_i(\mathbb{A}^l(F)) = 0$ for $i \geq 0$.*
(2) *$H^i(\mathbb{B}_l(F)) = 0$ for $i \leq (l-1)/2$.*
(3) *$Coker(B_l^i(F) \to B_l^{i+1}(F))$ is free for $i \leq (l-1)/2$.*

$U_k^p(\phi)$ was defined to be the sum

$$\sum_{a+b=k} \wedge^{p+a} F \otimes D_a G \otimes \wedge^{p+b} G \otimes D_b F.$$

If $x \otimes y \otimes u \otimes v \in \wedge^{p+a} F \otimes D_a G \otimes \wedge^{p+b} G \otimes D_b F$ with $a + b = k$, then $x \otimes v \in \mathbb{A}_b^{p+k}(F)$ and $u \otimes y \in \mathbb{A}_a^{p+k}(G)$. Thus we may apply the map ∂_F to $(x \otimes v)$ and we have $\partial_F(x \otimes v) \otimes y \otimes u \in \wedge^{p+a+1} F \otimes D_{b-1} F \otimes D_a G \otimes \wedge^{p+b} G \cong \wedge^{p+a+1} F \otimes D_a G \otimes \wedge^{p+b} G \otimes D_{b-1} F$. But $\wedge^{p+a+1} F \otimes D_a G \otimes \wedge^{p+b} G \otimes D_{b-1} F$ is in $U_k^{p+1}(\phi)$, so with this identification understood, we may consider $\partial_F(x \otimes v) \otimes (y \otimes u)$ to be an element of $U_k^{p+1}(\phi)$. Similarly, $u \otimes y \in \mathbb{A}_a^{p+k}(G)$, we can apply ∂_G to $u \otimes y$, and after rearrangement of terms we may consider $(x \otimes v) \otimes \partial_G(u \otimes y)$ to be an element of $U_k^{p+1}(\phi)$. In the exactly same way, $x \otimes v \in \mathbb{B}_{p+k}^b(F)$ and $u \otimes y \in \mathbb{B}_{p+k}^a(G)$, and we may consider the elements of $U_{k+2}^{p-1}(\phi)$. With this conventions in mind, we make the following definitions:

DEFINITION 2.6. We define the maps

$$\partial_{k+1}^p(F, G) : U_{k+1}^p(\phi) \to U_k^{p+1}(\phi)$$

and

$$\delta_{k+1}^p(F, G) : U_{k+1}^p(\phi) \to U_{k+2}^{p-1}(\phi)$$

as follows. If $x \otimes y \otimes u \otimes v \in \wedge^{p+a} F \otimes D_a G \otimes \wedge^{p+b} G \otimes D_b F$,

$$\partial_{k+1}^p(F, G)(x \otimes y \otimes u \otimes v) = (x \otimes v) \otimes \partial_G(u \otimes y) + (-1)^a \partial_F(x \otimes v) \otimes (y \otimes u)$$

$$\delta_{k+1}^p(F, G)(x \otimes y \otimes u \otimes v) = (x \otimes v) \otimes \delta_G(u \otimes y) + (-1)^{a+1} \delta_F(x \otimes v) \otimes (y \otimes u).$$

The notations $\partial(F, G)$ and $\delta(F, G)$ underscore the fact that these maps depend only on the modules F and G; they are completely dependent of the map ϕ.

PROPOSITION 2.7 [1, PROPOSITION 1.8].

(1)

$$U_{k+2}^{p-1}(\phi) \xrightarrow{\partial_{k+2}^{p-1}(F,G)} U_{k+1}^{p}(\phi) \xrightarrow{\partial_{k+1}^{p}(F,G)} U_{k}^{p+1}(\phi)$$

is an exact sequence if $p+k>0$.

(2)

$$U_{k}^{p+1}(\phi) \xrightarrow{\delta_{k}^{p+1}(F,G)} U_{k+1}^{p}(\phi) \xrightarrow{\delta_{k+1}^{p}(F,G)} U_{k+2}^{p-1}(\phi)$$

is an exact sequence if $p \geqslant 0$.

DEFINITION 2.8. For $p \geqslant 1$ and $k \geqslant 0$, we define

$$Z_{k+1}^{p}(F,G) = Coker(U_{k}^{p+1}(\phi) \xrightarrow{\delta_{k}^{p+1}(F,G)} U_{k+1}^{p}(\phi)).$$

LEMMA 2.9 [1, LEMMA 1.12]. *The diagram*

$$\begin{array}{ccc} U_{k}^{p+1}(\phi) & \xrightarrow{\partial(F,G)} & U_{k-1}^{p+2}(\phi) \\ \downarrow{\scriptstyle \delta(F,G)} & & \downarrow{\scriptstyle \delta(F,G)} \\ U_{k+1}^{p}(\phi) & \xrightarrow{\partial(F,G)} & U_{k}^{p+1}(\phi) \end{array}$$

is anticommutative.

By the above lemma, we see that the maps

$$\partial_{k+1}^{p}(F,G) : U_{k+1}^{p}(\phi) \to U_{k}^{p+1}(\phi)$$

induce unique maps, which we will denote by

$$\bar{\partial}_{k+1}^{p}(F,G) : Z_{k+1}^{p}(F,G) \to Z_{k}^{p+1}(F,G).$$

DEFINITION 2.10. We define $X_{k+1}^{p}(1,F,G)$ to be the kernel of the map $\bar{\partial}_{k+1}^{p}(F,G)$.

PROPOSITION 2.11 [1, PROPOSITION 1.14]. *For $p \geqslant 1$, the sequence*

$$Z_{k+2}^{p-1}(F,G) \to Z_{k+1}^{p}(F,G) \to Z_{k}^{p+1}(F,G)$$

is exact. We therefore have an exact sequence

$$0 \to X_{k+1}^{p}(1,F,G) \to Z_{k+1}^{p}(F,G) \to X_{k}^{p+1}(1,F,G) \to 0$$

for all $p \geqslant 0$. It follows from this that $X_{k+1}^{p}(1,F,G)$ is universally free for $p \geqslant 2$.

LEMMA 2.12 [1, LEMMA 1.15]. *The following diagrams are commutative :*

$$\begin{array}{ccc} U_{k+1}^{p}(\phi) & \xrightarrow{\partial_{k+1}^{p}(F,G)} & U_{k}^{p+1}(\phi) \\ {\scriptstyle \partial_{\phi}^{u}(F,G)}\downarrow & & \downarrow{\scriptstyle \partial_{\phi}^{u}(F,G)} \\ U_{k}^{p}(\phi) & \xrightarrow{\partial_{k}^{p}(F,G)} & U_{k-1}^{p+1}(\phi) \end{array} \qquad \begin{array}{ccc} U_{k+1}^{p}(\phi) & \xrightarrow{\delta_{k+1}^{p}(F,G)} & U_{k+2}^{p-1}(\phi) \\ {\scriptstyle \partial_{\phi}^{u}(F,G)}\downarrow & & \downarrow{\scriptstyle \partial_{\phi}^{u}(F,G)} \\ U_{k}^{p}(\phi) & \xrightarrow{\delta_{k}^{p}(F,G)} & U_{k+1}^{p-1}(\phi) \end{array}$$

DEFINITION 2.13. The complex $\{Z_{k+1}^{p}(F,G), \partial_{\phi}^{Z}\}$ will be denoted by $\mathbb{Z}^{p}(\phi)$.

Lemma 2.9 and 2.12 show us that the map $\bar{\partial}(F,G) : \mathbb{Z}^{p}(\phi) \to \mathbb{Z}^{p+1}(\phi)$ (sending $Z_{k+1}^{p}(F,G)$ to $Z_{k}^{p+1}(F,G)$) is a map of complexes, and thus the map $\partial_{\phi}^{Z} : Z_{k+1}^{p}(F,G) \to Z_{k}^{p}(F,G)$ induces a map $\partial_{\phi}^{X} : X_{k+1}^{p}(1,F,G) \to X_{k}^{p}(1,F,G)$.

DEFINITION 2.14. The complexes $\{X_{k+1}^{p}(1,F,G), \partial_{\phi}^{X}\}$ will be denoted by $\mathbb{X}^{p}(1,\phi)$.

Clearly, the complex $\mathbb{X}^{p}(1,\phi)$ is the kernel of the map of complexes

$$\bar{\partial}(F,G) : \mathbb{Z}^{p}(\phi) \to \mathbb{Z}^{p+1}(\phi).$$

Let F and G be free R-modules of ranks m and n, where $m \geqslant n$, respectively, and let p be a positive integer less than or equal to n. Let (p^{r}) denote the partition

$$\underbrace{(p,\ldots,p)}_{r}.$$

We let I_p denote the ideal in $S(F\otimes G)$ generated by $\wedge^p F\otimes\wedge^p G$, and I_p^r denotes the rth power of I_p. Because $p\le \operatorname{rank} G$, we have a canonical injection of the tensor product of Schur functors $L_{(p^r)}F\otimes L_{(p^r)}G\to I_p^r$ (by the standard basis theorem). Consequently, we have a map

$$\eta_k^p(r,F,G):\wedge^k(F\otimes G)\otimes L_{(p^r)}F\otimes L_{(p^r)}G\to\wedge^{k-1}(F\otimes G)\otimes I_p^r$$

which is the composition of the maps

$$\begin{aligned}&\wedge^k(F\otimes G)\otimes L_{(p^r)}F\otimes L_{(p^r)}G\\ &\xrightarrow{\alpha\otimes 1}\wedge^{k-1}(F\otimes G)\otimes(F\otimes G)\otimes L_{(p^r)}F\otimes L_{(p^r)}G\\ &\xrightarrow{1\otimes\beta}\wedge^{k-1}(F\otimes G)\otimes I_p^r,\end{aligned}$$

where α is obtained by diagonalizing $\wedge^k(F\otimes G)$ and β is obtained by multiplying the image of $L_{(p^r)}F\otimes L_{(p^r)}G$ in I_p^r by $F\otimes G$ in $S(F\otimes G)$.

DEFINITION 2.15. For $k\geqslant 0$, we denote the kernel of the map $\eta_k(r,F,G)$ by $Y_{k+1}(r,F,G)$.

The followings are directly induced from the definition

$$Y_1^p(r,F,G)=L_{(p^r)}F\otimes L_{(p^r)}G$$

and

$$0\to Y_2^p(r,F,G)\to Y_1^p(r,F,G)\otimes F\otimes G\to I_p^r$$

is exact.

PROPOSITION 2.16 [1, PROPOSITION 3.2]. *For each $k\ge 2$, there is an exact sequence*

$$0\to Y_{k+1}^p(r,F,G)\to Y_k^p(r,F,G)\otimes(F\otimes G)\to Y_{k-1}^p(r,F,G)\otimes S_2(F\otimes G).$$

DEFINITION 2.17. The complex $\{Y_{k+1}^p(r,F,G),\partial_\phi^Y\}$ will be denoted by $\mathbb{Y}^p(r,\phi)$. When ϕ is the generic map, this complex will be denoted by $\mathbb{Y}(r,F,G)$.

THEOREM 2.18 [1, THEOREM 3.4]. *Let F and G be free R-modules of ranks m and n, where $m\ge n$. Let $\phi: F\otimes G\to R$ be a map and suppose that for each $j=1,\dots,n$ the ideal $I_j(\phi)$ generated by the minors of ϕ of order j has grade $\ge(n+1-j)(m-n)+1$. Let p be a positive integer where $0\le p\le n$. Then $\mathbb{Y}_p(r,\phi)$ is a free resolution of $I_p^r(\phi)$ and this resolution can be augmented to give the resolution of $R/I_p^r(\phi)$.*

3. Minimal free resolution

Let R be a commutative noetherian ring, and let F and G be free R-modules of ranks m and n, where $p \leqslant \min(m,n)$. Then we have the following map

$$\begin{aligned}\xi_k^p : \wedge^k(F \otimes G^*) \otimes L_{((p+r-1)^r)}F \otimes L_{((p+r-1)^r)}G^* \\ \longrightarrow \wedge^{k-1}(F \otimes G^*) \otimes S_{r(p+r-1)+1}(F \otimes G^*).\end{aligned}$$

The above map is the composition

$$\begin{aligned}&\wedge^k (F \otimes G^*) \otimes L_{((p+r-1)^r)}F \otimes L_{((p+r-1)^r)}G^* \\ \xrightarrow{1\otimes\alpha} &\wedge^k (F \otimes G^*) \otimes S_{r(p+r-1)}(F \otimes G^*) \\ \xrightarrow{\sigma} &\wedge^{k-1} (F \otimes G^*) \otimes S_{r(p+r-1)+1}(F \otimes G^*),\end{aligned}$$

where α is the canonical embedding

$$L_{((p+r-1)^r)}F \otimes L_{((p+r-1)^r)}G^* \rightarrow S_{r(p+r-1)}(F \otimes G^*)$$

and σ is the Koszul map.

DEFINITION 3.1. For $k \geqslant 0$, we denote the kernel of the map $\xi_k^p(r, F, G)$ by $X_{k+1}^p(r, F, G)$.

The followings are directly induced from the definition

$$X_1^p(r, F, G) = L_{((p+r-1)^r)}F \otimes L_{((p+r-1)^r)}G^*$$

and

$$0 \rightarrow X_2^p(r, F, G) \rightarrow X_1^p(r, F, G) \otimes F \otimes G^* \rightarrow S_2(F \otimes G^*)$$

is exact.

PROPOSITION 3.2. *For each $k \geqslant 2$, there is an exact sequence*

$$0 \rightarrow X_{k+1}^p(r, F, G) \rightarrow X_k^p(r, F, G) \otimes (F \otimes G^*) \rightarrow X_{k-1}^p(r, F, G) \otimes S_2(F \otimes G^*).$$

PROOF. To obtain the map

$$\partial^X : X_{k+1}^p(r, F, G) \rightarrow X_k^p(r, F, G) \otimes (F \otimes G^*)$$

we observe that the diagonal map

$$\wedge^k(F \otimes G^*) \to \wedge^{k-1}(F \otimes G^*) \otimes (F \otimes G^*)$$

tensored with the identity on $L_{(p+r-1)^r}F \otimes L_{(p+r-1)^r}G^*$ induces a unique map from $X^p_{k+1}(r, F, G)$ to $X^p_k(r, F, G)$. The map

$$X^p_k(r, F, G) \otimes (F \otimes G^*) \to X^p_{k-1}(r, F, G) \otimes S_2(F \otimes G^*)$$

is just the composition

$$\begin{aligned} X^p_k(r, F, G) \otimes (F \otimes G^*) &\xrightarrow{\partial^X \otimes 1} X^p_{k-1}(r, F, G) \otimes (F \otimes G^*) \otimes (F \otimes G^*) \\ &\xrightarrow{1 \otimes \mu} X^p_{k-1}(r, F, G) \otimes S_2(F \otimes G^*), \end{aligned}$$

where μ is multiplication in $S(F \otimes G^*)$. The exactness is a consequence of the acyclicity of the Koszul complex $\wedge(F \otimes G^*) \otimes S(F \otimes G^*)$. □

If $\phi : F \otimes G^* \to R$ is a map, then we obtain a map

$$\partial^X_\phi : X^p_{k+1}(r, F, G) \to X^p_k(r, F, G)$$

$k \geq 1$ which is the composition

$$\begin{aligned} X^p_{k+1}(r, F, G) &\xrightarrow{\partial^X} X^p_k(r, F, G) \otimes (F \otimes G^*) \\ &\xrightarrow{1 \otimes \phi} X^p_k(r, F, G) \otimes R = X^p_k(r, F, G). \end{aligned}$$

It is easy to check that $\partial^X_\phi \circ \partial^X_\phi = 0$, so we obtain a complex $\{X^p_k(r, F, G), \partial^X_\phi\}$.

Definition 3.3. The complex $\{X^p_{k+1}(r, F, G), \partial^X_\phi\}$ will be denoted by $\mathbb{X}(r, \phi)$. When ϕ is the generic map, this complex will be denoted by $\{\mathbb{X}(r, F, G)\}$ and called Durfee squre r complex.

Let R be a commutative noetherian ring, and let F and G be free R-modules of ranks $p + 2$. We will assume that ϕ is the generic map and that F and G are fixed. Thus we will write $\mathbb{U}^p$, $\mathbb{Z}^p$ and $\mathbb{X}(1)$ for $\mathbb{U}^p(\phi)$, $\mathbb{Z}^p(\phi)$, and $\mathbb{X}(1, \phi)$, and we will also write Z^p_{k+1}, $X^p_{k+1}(1)$, for $Z^p_{k+1}(F, G)$ and $X^p_{k+1}(1, F, G)$.

PROPOSITION 3.4 [1, PROPOSITION 3.5]. *We can obtain followings when* $rankF = rankG = p+2$.

(a) $H_i(\mathbb{U}^{p+1}) = 0$ *for* $i \geqslant 3$.
(b) $\mathbb{X}^{p+2}(1) \cong \mathbb{Z}^{p+2}$ *is a resolution of* $I_{p+2}(\phi)$.
(c) $H_i(\mathbb{X}^{p+1}(1)) \cong H_i(\mathbb{Z}^{p+1})$ *for* $i \geqslant 2$.
(d) $H_i(\mathbb{X}^{p+1}(1)) = 0$ *for* $i > 3$.
(e) *We have the exact sequence*

$$0 \to H_3(\mathbb{Z}^{p+1}) \to I_{p+2}(\phi) \to H_2(\mathbb{U}^{p+1}) \to H_2(\mathbb{Z}^{p+1}) \to 0.$$

(f) $H_3(\mathbb{X}^{p+1}(1)) = I^2_{p+2}(\phi)$ *and* $H_2(\mathbb{X}^{p+1}(1)) = 0$.

PROPOSITION 3.5. *We can obtain followings when* $rankF = rankG = p+2$.

(a) $H_i(\mathbb{U}^p) = 0$ *for* $i \geqslant 3$.
(b) $H_i(\mathbb{X}^p(1)) \cong H_i(\mathbb{Z}^p)$ *for* $i > 4$.
(c) $H_2(\mathbb{X}^p(1)) = 0$.
(d) $H_4(\mathbb{X}^p(1)) = 0$.
(e) $H_5(\mathbb{X}^p(1)) = I^2_{p+2}(\phi)$.
(f) *There exists an inclusion map from* $I^2_{p+2}(\phi)$ *to* $H_3(\mathbb{X}^p(1))$.

PROOF. First we will prove (a). we use the fact that $\mathbb{U}^p$ is the tensor product of two complexes:

$$(\wedge^{p+2}F \otimes D_2G \to \wedge^{p+1}F \otimes G \to \wedge^p F)$$
$$\otimes\ (\wedge^{p+2}G \otimes D_2F \to \wedge^{p+1}G \otimes F \to \wedge^p G)$$

and both complexes are acyclic [4]. By acyclic assembly lemma [2], $\mathbb{U}^p$ is acyclic. Since the indexing of our complex $\mathbb{U}^p$ starts with 1, it accounts for the condition $i \geqslant 3$.

We recall the two exact sequences in Section 2:

$$0 \to Z^{p+1}_k \to U^p_{k+1} \to Z^p_{k+1} \to 0$$

$$0 \to X^p_{k+1}(1) \to Z^p_{k+1} \to X^{p+1}_k(1) \to 0.$$

In fact the above sequences are exact sequences of complexes

$$0 \to \mathbb{Z}^{p+1} \to \mathbb{U}^p \to \mathbb{Z}^p \to 0 \tag{1}$$

$$(2) \qquad 0 \to \mathbb{X}^p(1) \to \mathbb{Z}^p \to \mathbb{X}^{p+1}(1) \to 0$$

provided we keep in mind the dimension shifts. We use the sequence (2) to prove (b). From the associated homology sequence we get

$$H_i(\mathbb{X}^p(1)) \cong H_i(\mathbb{Z}^p) \text{ for } i > 4.$$

To prove (c) we use the sequence (2). From the associated homology sequence of complex we can obtain the exact sequence

$$H_2(\mathbb{X}^{p+1}(1)) \to H_2(\mathbb{X}^p(1)) \to H_2(\mathbb{Z}^p) \to H_1(\mathbb{X}^{p+1}(1)) \to H_1(\mathbb{X}^p(1)).$$

We already know $H_1(\mathbb{X}^{p+1}(1)) = I_{p+1}(\phi)$ and $H_1(\mathbb{X}^p(1)) = I_p(\phi)$. But $I_{p+1}(\phi) \subset I_p(\phi)$ which means $H_2(\mathbb{Z}^p) = 0$. Since $H_2(\mathbb{X}^{p+1}(1)) = 0$, $H_2(\mathbb{X}^p(1)) = 0$. To prove (d) we use the sequence (1). From the associated homology sequences of complex we can obtain the exact sequence

$$H_4(\mathbb{U}^p) \to H_4(\mathbb{Z}^p) \to H_2(\mathbb{Z}^{p+1}) \to H_3(\mathbb{U}^p).$$

Since $H_4(\mathbb{U}^p) = H_3(\mathbb{U}^p) = 0$ we can say $H_4(\mathbb{Z}^p) \cong H_2(\mathbb{Z}^{p+1})$. From the fact $H_2(\mathbb{Z}^{p+1}) = 0$, we get $H_4(\mathbb{Z}^p) = 0$. From the associated homology sequence of (2), we get

$$H_4(\mathbb{X}^{p+1}(1)) \to H_4(\mathbb{X}^p(1)) \to H_4(\mathbb{Z}^p).$$

Since $H_4(\mathbb{X}^{p+1}(1)) = 0 = H_4(\mathbb{Z}^p)$, $H_4(\mathbb{X}^p(1)) = 0$. To prove (e) we use sequence (1). From the associated homology sequence of the complex, we get

$$H_5(\mathbb{U}^p) \to H_5(\mathbb{Z}^p) \to H_3(\mathbb{Z}^{p+1}) \to H_4(\mathbb{U}^p).$$

Since $H_5(\mathbb{U}^p) = 0 = H_4(\mathbb{U}^p)$, we get $H_5(\mathbb{Z}^p) \cong H_3(\mathbb{Z}^{p+1})$. From the associated homology sequence of (2), we get

$$H_5(\mathbb{X}^{p+1}(1)) \to H_5(\mathbb{X}^p(1)) \to H_5(\mathbb{Z}^p) \to H_4(\mathbb{X}^{p+1}(1)).$$

Since $H_5(\mathbb{X}^{p+1}(1)) = 0 = H_4(\mathbb{X}^{p+1}(1))$, $H_5(\mathbb{X}^p(1)) \cong H_5(\mathbb{Z}^p)$. Also we already know $H_3(\mathbb{Z}^{p+1}) \cong H_3(\mathbb{X}^{p+1}(1))$ and $H_3(\mathbb{X}^{p+1}(1)) = I^2_{p+2}(\phi)$. Hence $H_5(\mathbb{X}^p(1)) = I^2_{p+2}(\phi)$. To prove (f) we use sequence (2). From the associated homology sequence of the complex, we get

$$H_4(\mathbb{Z}^p) \to H_3(\mathbb{X}^{p+1}(1)) \to H_3(\mathbb{X}^p(1)).$$

Since $H_4(\mathbb{Z}^p) = 0$ and $H_3(\mathbb{X}^{p+1}(1)) = I^2_{p+2}(\phi)$, there is an inclusion map from $I^2_{p+2}(\phi)$ to $H_3(\mathbb{X}^p(1))$. □

We have the Durfee square 1 complex $\mathbb{X}^p(1)$ which may be a half part of the resolution of the $p \times p$ minors when $rankF = rankG = p + 2$ as followings:

$$0 \longrightarrow X^p_5(1) \xrightarrow{\partial^p_5(1)} X^p_4(1) \xrightarrow{\partial^p_4(1)} X^p_3(1) \xrightarrow{\partial^p_3(1)} X^p_2(1) \xrightarrow{\partial^p_2(1)} X^p_1(1).$$

DEFINITION 3.6. The Durfee square 2 complex $\mathbb{X}^p(2)$ is defined to be the dual complex of above Durfee square 1 complex. That is, $\mathbb{X}^p(2) = \{X_k^p(1)^*, \partial(1)^*\}$, where

$$X_k^p(2) = X_{6-k}^p(1)^*$$

for $k \geqslant 1$, and $\partial(2)$ is the dual boundary map of $\mathbb{X}(1)$.

PROPOSITION 3.7. *There exists a map*

$$\psi_1 : X_1^p(2) \longrightarrow X_3^p(1)$$

and

$$\psi_2 : X_3^p(2) \longrightarrow X_5^p(1).$$

PROOF. We know by Proposition 3.5 $H_5(\mathbb{X}^p(1)) = I_{p+2}^2(\phi)$ and there exist the inclusion map from $I_{p+2}^2(\phi)$ to $H_3(\mathbb{X}^p(1))$. Since $X_1^p(2) = X_5^p(1)^*$, we have the inclusion map $\psi_1 : X_1^p(2) \to X_3^p(1)$. Thus there is the dual map $\psi_1^* : X_3^p(1)^* \to X_1^p(2)^*$. Since $X_3^p(1)^* = X_3^p(2)$ and $X_1^p(2)^* = X_5^p(1)$, we define the map ψ_2 by ψ_1^*. □

DEFINITION 3.8. The $X_1^p(3)$ is defined to be the dual of cokernel of the map $\partial_2^p(1)$.

REMARK. Since the cokernel of two universally free modules are universally free, $X_1^p(3)$ is universally free[4].

Now, we show that the complex $\mathbb{X}^p$ is a minimal free resolution of determinantal ideal $I_p(\phi)$:

$$\begin{array}{ccccccccccc}
X_1^p(3) & & & & & & & & & & \\
\downarrow & & & & & & & & & & \\
X_5^p(2) & \longrightarrow & X_4^p(2) & \longrightarrow & X_3^p(2) & \longrightarrow & X_2^p(2) & \longrightarrow & X_1^p(2) & & \\
& & & \swarrow & & \swarrow & & \swarrow & & & \\
& & X_5^p(1) & \longrightarrow & X_4^p(1) & \longrightarrow & X_3^p(1) & \longrightarrow & X_2^p(1) & \longrightarrow & X_1^p(1)
\end{array}$$

THEOREM 3.9. *The complex $\mathbb{X}^p$ is a minimal free resolution of $I_p(\phi)$.*

PROOF. If we augment the complex $\mathbb{X}^p$ by mapping $X_1^p(1) = \wedge^p F \otimes \wedge^p G$ to R, what we want to show is that this augmented complex is a resolution of $R/I_p(\phi)$. The augmented complex has length 10, so by the acyclicity lemma it suffices to localize at primes of height less than 10. (Because of universality, we may assume that R is Cohen-Macaulay so that we need not distinguish between height and grade of ideals.) However, $I_p(\phi)$ is of height 8, so that under localization by such a prime, $I_p(\phi)$ blows up. It therefore suffices to prove acyclicity after inverting a $p \times p$ minor of ϕ and we may assume that $\phi = id + \phi' : H \oplus G' \to H \oplus F'^*$ where rank $H = p$, rank $F' = 2$, rank $G' = 2$, and $\phi' : G' \to F'^*$ is generic. From Proposition 3.5 (a) we see that it is enough to show that $H_2(\mathbb{U}^p) = I_{p+1}(\phi)/I_{p+2}(\phi)$ since the map $I_{p+1}(\phi) \to H_2(\mathbb{U}^p)$ will be easily seen to be the canonical surjection [1]. By the usual argument reducing $\mathbb{U}^p$ modulo homotopy equivalance, we may assume that

$$\mathbb{U}^p = (\wedge^2 F' \otimes D_2 G' \to F' \otimes G' \to R) \otimes (\wedge^2 G' \otimes D_2 F' \to F' \otimes G' \to R)$$

where $I_{p+1}(\phi) = I_1(\phi') = (X_1, X_2, X_3, X_4)$, and X_1, X_2, X_3, X_4 is a regular sequence. A simple argument shows that

$$0 \longrightarrow H_2(\mathbb{U}^p) \longrightarrow R/I_1(\phi') \otimes F' \otimes G' \xrightarrow{1 \otimes \phi'} R/I_1(\phi') \otimes R$$

is exact. But $1 \otimes \phi'$ is the zero map and it is easy to see $F' \otimes G'/I_1(\phi') \cong I_2(\phi')$. From the associated homology sequence of complex we can obtain the exact sequence

$$H_3(\mathbb{U}^p) \to H_3(\mathbb{Z}^p) \to H_1(\mathbb{Z}^{p+1}) \to H_2(\mathbb{U}^p) \to H_2(\mathbb{Z}^p).$$

Since $H_3(\mathbb{U}^p) = 0 = H_2(\mathbb{Z}^p)$ and $H_1(\mathbb{Z}^{p+1}) \cong I_{p+1}(\phi)/I_{p+2}(\phi) \cong H_2(\mathbb{U}^p)$, $H_3(\mathbb{Z}^p) = 0$. From the associated homology sequence of complex we can obtain the exact sequence

$$H_4(\mathbb{Z}^p) \to H_3(\mathbb{X}^{p+1}(1)) \to H_3(\mathbb{X}^p(1)) \to H_3(\mathbb{Z}^p).$$

We already know that $H_3(\mathbb{X}^{p+1}(1)) = I^2_{p+2}(\phi)$ and $H_3(\mathbb{Z}^p) = 0 = H_4(\mathbb{Z}^p)$. Therefore $H_3(\mathbb{X}^p(1)) = I^2_{p+2}(\phi)$. This proves acyclicity of our complex $\mathbb{X}^p$.

Now we need to show the minimality of the complex. Since we are not looking over a local, but over a graded ring, by the minimality we mean that the coefficients of the boundary maps of $\mathbb{X}^p$ are in the ideal of $S(F \otimes G)$ generated by $F \otimes G$. We have already seen that the complexes $\mathbb{X}^p(1)$ and $\mathbb{X}^p(2)$ are linear, so the only maps that need to be examined are the maps $\psi_k(1)$ and $\psi_k(2)$. To do this, recall that all our modules, complexes, etc., are graded, and so, therefore, is the homology of all of these complexes. We see that $H_3(\mathbb{Z}^p) = 0$ in the degrees less than p (since $I_{p+1}(\phi)$ has its first non-zero component in degree $p+1$, and the map $H_3(\mathbb{Z}^p) \to I_{p+1}(\phi)$ is of degree one). This permits us to define $\psi_1(1)$ as a map of degree p, and the degrees of the other are therefore also seen to be p. □

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