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EXPLICIT EVALUATIONS OF SPECIAL MULTIPLE ZETA VALUES, $\zeta(\{4l+2\}_n)$ AND $\zeta(\{4l\}_n)$

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ABSTRACT. In this paper we calculate two special types of multiple zeta values, $\zeta(\{4l+2\}_n)$ and $\zeta(\{4l\}_n)$ using the primitive roots of unity, which may be simpler and easier.

1. Introduction

The Euler-Zagier's multiple zeta values is defined by

$$\zeta(s_1, s_2 \cdots, s_k) = \sum_{0<m_1<m_2<\cdots<m_k} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_k^{s_k}}$$

with complex variables s_i $(i = 1, 2, \cdots, k)$. When $\Re(s_i) \geq 1$ for $i = 1, 2, \cdots, k-1$ and $\Re(s_k) > 1$, multiple zeta values are absolutely convergent.

These values occur in the knot theory and quantum field theory. M. E. Hoffman [2] studied some relations and presented two conjectures, so called, sum and duality conjectures. These are first proved by Zagier [5] and extensively studied and generalized by Ohno [4].

One of the remarkable properties of the Riemann zeta function $\zeta(s)$ is that $\zeta(2n)$ can be evaluated in terms of the Bernoulli numbers as follows:
For a nonnegative integer n,

$$\zeta(2n) = (-1)^{n+1} \frac{2^{2n-1} B_{2n}}{(2n)!} \times \pi^{2n},$$

which is due to L. Euler.

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There are deep relations between multiple zeta values and Bernoulli numbers,

$$\zeta(2,2,\cdots,2) = \sum_{0<m_1<m_2<\cdots<m_k} \frac{1}{m_1^2 m_2^2 \cdots m_k^2} = \frac{\pi^{2k}}{(2k+1)!}$$
$$= (2\pi)^{2k} \sum_{m_1+2m_2+\cdots+km_k=k} \frac{1}{m_1!\cdots m_k!}\left(\frac{B_2}{2\cdot 2!}\right)^{m_1}\cdots\left(\frac{B_{2k}}{2k(2k)!}\right)^{m_k}$$

which are due to M. E. Hoffman [2]. J. M. Borwein and D. M. Bradley and D. J. Broadhurst [1] recommended further study of multiple zeta functions and other related functions. Here, we consider the case $(2l,\cdots,2l)$ for indices $(s_1,\cdots,s_n)$ of the multiple zeta values. Let $\mathbb{N}$ be the set of positive integers and $\mathbb{Z}_+ = \mathbb{N}\cup\{0\}$, and let $S(a;k)$ be the set of compositions of k length of k, i.e.,

$$S(a;k) = \{(a_1,\cdots,a_k) | a_1 + 2a_2 + \cdots + ka_k = k, \quad a_j \in \mathbb{Z}_+ \text{ for all } j = 1,2,\cdots,k\}.$$

The aim of this paper is to prove the case $(2l,\cdots,2l)$ for indices $(s_1,\cdots,s_n)$ of the Euler-Zagier's multiple zeta values which generalizes the following two theorems.

We denote n repetitions of a substring by $\{\cdots\}_n$ and ω_l the l-th primitive root of unity.

Now, we will state our results.

THEOREM 1. *For $n\in\mathbb{N}$ and $l\in\mathbb{Z}_+$ we have*

$$\zeta(\{4l+2\}_n) = (2\pi)^{4ln+2n} \sum_{1\le k\le 2l}\sum_{i_k=0}^{1}(-1)^{i_1+\cdots+i_{2l}} \times \left(1+\sum_{k=1}^{2l}(-1)^{i_k}\omega_{2l+1}^k\right)^{(2l+1)(2n+1)} \times \sum_{(a_1,\cdots,a_{2ln+l+n})\in S(a;2ln+l+n)} \prod_{j=1}^{2ln+l+n}\frac{1}{a_j!}\left(\frac{B_{2j}}{2j(2j)!}\right)^{a_j},$$

where the Bernoulli numbers B_j is defined by $\frac{t}{e^t-1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}$.

This theorem can be regarded as a generalization of Hoffman [2], if $l=0$ then we obtain the Proposition 2.4 in Hoffman [2].

THEOREM 2. *For $n, l \in \mathbb{N}$ we have*

$$\zeta(\{4l\}_n) = (-1)^{n+l} 2i(2\pi)^{4ln} \times \sum_{1\le k\le 2l-1} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l-1}} \left(1 + \sum_{k=1}^{2l-1} (-1)^{i_k} \omega_{4l}^{k}\right)^{2l(2n+1)} \times \sum_{(a_1,\cdots,a_{2ln+l})\in S(a;2ln+l)} \prod_{j=1}^{2ln+l} \frac{1}{a_j!} \left(-\frac{G_{2j}}{4j(2j)!}\right)^{a_j},$$

where the Genocchi number G_j is defined by $\frac{2t}{e^t+1} = \sum_{j=0}^{\infty} G_j \frac{t^j}{j!}$ [3].

2. Proof of results

LEMMA 1. *Let $n \in \mathbb{Z}_+$, $\alpha_i \in \mathbb{C}$ for all i, and let $[x]$ be the greatest integer not exceeding x.*

(1) *If n is odd integers, then*

$$\prod_{i=1}^{n} \sin \alpha_i = \frac{(-1)^{[\frac{n}{2}]}}{2^{n-1}} \sum_{2\le k\le n} \sum_{i_k=0}^{1} (-1)^{i_2+\cdots+i_n} \sin\left(\alpha_1 + \sum_{k=2}^{n} (-1)^{i_k} \alpha_k\right).$$

(2) *If n is even integers, then*

$$\prod_{i=1}^{n} \sin \alpha_i = \frac{(-1)^{[\frac{n}{2}]}}{2^{n-1}} \sum_{2\le k\le n} \sum_{i_k=0}^{1} (-1)^{i_2+\cdots+i_n} \cos\left(\alpha_1 + \sum_{k=2}^{n} (-1)^{i_k} \alpha_k\right).$$

PROOF. First we assume that $n \ge 1$. We use induction on n. If $n = 1$, then the formula (1) and if $n = 2$, then the formula (2) here are trivial. Assume, then, that the formula (2) in Lemma 1 holds true for $n = 2, 4, \cdots, 2l$, i.e.,

$$\prod_{i=1}^{2l} \sin \alpha_i = \frac{(-1)^{l}}{2^{2l-1}} \sum_{2\le k\le 2l} \sum_{i_k=0}^{1} (-1)^{i_2+\cdots+i_{2l}} \cos\left(\alpha_1 + \sum_{k=2}^{2l} (-1)^{i_k} \alpha_k\right).$$

We must show that the formula (2) in Lemma 1 is true when $n = 2l+2$.

Set $n = 2l+2$. We have

$$\begin{aligned}
&\prod_{i=1}^{2l+2} \sin \alpha_i \\
&= \left\{ \frac{(-1)^l}{2^{2l-1}} \sum_{2\le k\le 2l} \sum_{i_k=0}^{1} (-1)^{i_2+\cdots+i_{2l}} \cos\left(\alpha_1 + \sum_{k=2}^{2l} (-1)^{i_k}\alpha_k\right)\right\} \\
&\quad \times \sin\alpha_{2l+1}\sin\alpha_{2l+2} \\
&= \frac{(-1)^{l+1}}{2^{2l+1}} \sum_{2\le k\le 2l+2} \sum_{i_k=0}^{1} (-1)^{i_2+\cdots+i_{2l+2}} \cos\left(\alpha_1 + \sum_{k=2}^{2l+2} (-1)^{i_k}\alpha_k\right).
\end{aligned}$$

Thus, the formula (2) is true for $n = 2l+2$. Similarly we can prove the formula (1) in Lemma 1. □

Now let's look at the product formula for the sine function:

$$\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \frac{\sin \pi z}{\pi z}, \quad z \in \mathbb{C}.$$

For $s \in \mathbb{N}$ and $z \in \mathbb{C}$, it is well known [1] that

$$\prod_{n=1}^{\infty}\left(1-\frac{z^s}{n^s}\right) = \sum_{n=0}^{\infty}(-1)^n\zeta(\{s\}_n)z^{ns}. \tag{2.1}$$

Let $s = 4l+2$ in (2.1). We obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^n\zeta(\{4l+2\}_n)z^{(4l+2)n} \\
&= \prod_{n=1}^{\infty}\left(1-\frac{z^{4l+2}}{n^{4l+2}}\right) \\
&= \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\left(1-\omega_{2l+1}^2\frac{z^2}{n^2}\right)\cdots\left(1-\omega_{2l+1}^{4l}\frac{z^2}{n^2}\right) \\
&= \frac{1}{(\pi z)^{2l+1}}\prod_{i=1}^{2l+1}\sin(\pi\omega_{2l+1}^i z),
\end{aligned}$$

where $\omega_l = \cos(2\pi/l) + i\sin(2\pi/l)$ is the l-primitive root of unity.

By Lemma 1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^n\zeta(\{4l+2\}_n)z^{(4l+2)n} \\
&= \frac{(-1)^l}{2^{2l}(\pi z)^{2l+1}} \sum_{1\le k\le 2l}\sum_{i_k=0}^{1}(-1)^{i_1+\cdots+i_{2l}} \sin\left\{\left(1+\sum_{k=1}^{2l}(-1)^{i_k}\omega_{2l+1}^k\right)\pi z\right\} \\
&= \frac{(-1)^l}{2^{2l}(\pi z)^{2l+1}} \sum_{1\le k\le 2l}\sum_{i_k=0}^{1}(-1)^{i_1+\cdots+i_{2l}} \\
&\quad\times \sum_{m=0}^{\infty}(-1)^m\frac{1}{(2m+1)!}\left(1+\sum_{k=1}^{2l}(-1)^{i_k}\omega_{2l+1}^k\right)^{2m+1}(\pi z)^{2m+1} \\
&= \sum_{m=0}^{\infty}\frac{(-1)^{m+l}(\pi z)^{2m-2l}}{2^{2l}(2m+1)!} \\
&\quad\times \sum_{1\le k\le 2l}\sum_{i_k=0}^{1}(-1)^{i_1+\cdots+i_{2l}}\left(1+\sum_{k=1}^{2l}(-1)^{i_k}\omega_{2l+1}^k\right)^{2m+1}.
\end{aligned}$$

Therefore, we have the formula

$$\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^n\zeta(\{4l+2\}_n)z^{(4l+2)n} \\
&= \sum_{m=0}^{\infty}\frac{(-1)^{m+l}(\pi z)^{2m-2l}}{2^{2l}(2m+1)!} \\
&\quad\times \sum_{1\le k\le 2l}\sum_{i_k=0}^{1}(-1)^{i_1+\cdots+i_{2l}}\left(1+\sum_{k=1}^{2l}(-1)^{i_k}\omega_{2l+1}^k\right)^{2m+1}.
\end{aligned}$$

Let us compare the coefficients of the both sides of the above formula. Then we have

$$\begin{aligned}
&\zeta(\{4l+2\}_n) \\
&= \frac{2^{-2l}\pi^{(4l+2)n}}{((2n+1)(2l+1))!} \\
&\quad\times \sum_{1\le k\le 2l}\sum_{i_k=0}^{1}(-1)^{i_1+\cdots+i_{2l}}\left(1+\sum_{k=1}^{2l}(-1)^{i_k}\omega_{2l+1}^k\right)^{(2l+1)(2n+1)},
\end{aligned} \tag{2.2}$$

since $(4l+2)n = 2m-2l$ for $n \in \mathbb{N}$.

If $(4l+2)n \neq 2m-2l$ for $n \in \mathbb{N}$, then we have

$$\sum_{1\le k\le 2l} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l}} \left(1+\sum_{k=1}^{2l}(-1)^{i_k}\omega_{2l+1}^{k}\right)^{2m+1} = 0. \tag{2.3}$$

Let $s = 4l$ in (2.1). We obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^n \zeta(\{4l\}_n) z^{4ln} \\
&= \prod_{n=1}^{\infty}\left(1-\frac{z^{4l}}{n^{4l}}\right) \\
&= \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\left(1-\omega_{4l}^{2}\frac{z^2}{n^2}\right)\cdots\left(1-\omega_{4l}^{4l-2}\frac{z^2}{n^2}\right) \\
&= \frac{(-1)^l i}{(\pi z)^{2l}} \prod_{i=0}^{2l-1} \sin(\pi\omega_{4l}^{i} z).
\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^n \zeta(\{4l\}_n) z^{4ln} \\
&= \frac{2i}{(2\pi z)^{2l}} \sum_{1\le k\le 2l-1} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l-1}} \cos\left\{\left(1+\sum_{k=1}^{2l-1}(-1)^{i_k}\omega_{4l}^{k}\right)\pi z\right\} \\
&= \frac{2i}{(2\pi z)^{2l}} \sum_{1\le k\le 2l-1} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l-1}} \\
&\quad \times \sum_{m=0}^{\infty}(-1)^m \frac{(\pi z)^{2m}}{(2m)!}\left(1+\sum_{k=1}^{2l-1}(-1)^{i_k}\omega_{4l}^{k}\right)^{2m} \\
&= \sum_{m=0}^{\infty}(-1)^m \frac{(\pi z)^{2m-2l} i}{2^{2l-1}(2m)!} \\
&\quad \times \sum_{1\le k\le 2l-1} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l-1}} \left(1+\sum_{k=1}^{2l-1}(-1)^{i_k}\omega_{4l}^{k}\right)^{2m}.
\end{aligned}$$

Therefore we have the formula:

$$\sum_{n=0}^{\infty}(-1)^n\zeta(\{4l\}_n)z^{4ln}$$
$$= \sum_{m=0}^{\infty}(-1)^m\frac{(\pi z)^{2m-2l}i}{2^{2l-1}(2m)!}$$
$$\times \sum_{1\le k\le 2l-1}\sum_{i_k=0}^{1}(-1)^{i_1+\cdots+i_{2l-1}}\left(1+\sum_{k=1}^{2l-1}(-1)^{i_k}\omega_{4l}^k\right)^{2m}.$$

Let us compare the coefficients of the both sides of the above formula. Then we have

$$\zeta(\{4l\}_n) = (-1)^{n+l}\frac{2^{1-2l}\pi^{4ln}i}{(2l(2n+1))!}$$
(2.4)
$$\times \sum_{1\le k\le 2l-1}\sum_{i_k=0}^{1}(-1)^{i_1+\cdots+i_{2l-1}}\left(1+\sum_{k=1}^{2l-1}(-1)^{i_k}\omega_{4l}^k\right)^{2l(2n+1)},$$

since $4ln = 2m-2l$ for $n\in\mathbb{N}$.

Let $4ln \neq 2m-2l$ for $n\in\mathbb{N}$, then we have

$$\sum_{1\le k\le 2l-1}\sum_{i_k=0}^{1}(-1)^{i_1+\cdots+i_{2l-1}}\left(1+\sum_{k=1}^{2l-1}(-1)^{i_k}\omega_{4l}^k\right)^{2m}=0. \tag{2.5}$$

By (2.2) and (2.4), we have the following proposition.

PROPOSITION 1. *Let $n\in\mathbb{N}$.*

(1) *If $l\in\mathbb{Z}_+$, then we have*

$$\zeta(\{4l+2\}_n) = \frac{2^{-2l}\pi^{(4l+2)n}}{((2n+1)(2l+1))!}\sum_{1\le k\le 2l}\sum_{i_k=0}^{1}(-1)^{i_1+\cdots+i_{2l}}$$
$$\times\left(1+\sum_{k=1}^{2l}(-1)^{i_k}\omega_{2l+1}^k\right)^{(2l+1)(2n+1)}.$$

(2) *If $l \in \mathbb{N}$, then we have*

$$\zeta(\{4l\}_n) = (-1)^{n+l}\frac{2^{1-2l}\pi^{4ln}i}{(2l(2n+1))!}\sum_{1\le k\le 2l-1}\sum_{i_k=0}^{1}(-1)^{i_1+\cdots+i_{2l-1}} \times \left(1+\sum_{k=1}^{2l-1}(-1)^{i_k}\omega_{4l}^k\right)^{2l(2n+1)}.$$

By (2.3) and (2.5), we have the following proposition.

PROPOSITION 2. *Let $\omega_l = \cos(2\pi/l)+i\sin(2\pi/l)$ be the l-th primitive roots of unity and $n, l \in \mathbb{N}$, then we have*

(1) *If $m \neq 2ln+n+l$, then*

$$\sum_{1\le k\le 2l}\sum_{i_k=0}^{1}(-1)^{i_1+\cdots+i_{2l}}\left(1+\sum_{k=1}^{2l}(-1)^{i_k}\omega_{2l+1}^k\right)^{2m+1} = 0.$$

(2) *If $m \neq 2ln+l$, then*

$$\sum_{1\le k\le 2l-1}\sum_{i_k=0}^{1}(-1)^{i_1+\cdots+i_{2l-1}}\left(1+\sum_{k=1}^{2l-1}(-1)^{i_k}\omega_{4l}^k\right)^{2m} = 0.$$

Now, we consider the multiple zeta values at even integers $s = 2l$ in (2.1) to relate the Bernoulli numbers and Genocchi numbers.

Put

$$g(z) = \sum_{n=1}^{\infty}\frac{B_{2n}(2\pi i)^{2n}}{2n(2n)!}z^{2n}.$$

and $f(z) = \log\frac{\sin \pi z}{\pi z}$. We have

$$\begin{aligned} f'(z) &= \frac{d}{dz}\log\left(\frac{\sin\pi z}{\pi z}\right) = \frac{d}{dz}\log\left(\frac{e^{\pi i z}-e^{-\pi i z}}{2\pi i z}\right) \\ &= \frac{2\pi i}{e^{2\pi i z}-1} + \pi i - \frac{1}{z} \\ &= \sum_{n=0}^{\infty}\frac{B_n(2\pi i)^n}{n!}z^{n-1} + \pi i + \frac{1}{z} \\ &= \sum_{n=2}^{\infty}\frac{B_n(2\pi i)^n}{n!}z^{n-1} = \sum_{n=1}^{\infty}\frac{B_{2n}(2\pi i)^{2n}}{(2n)!}z^{2n-1} \\ &= g'(z). \end{aligned}$$

Hence $f(z) = g(z)$ since $f(0) = g(0)$.

Let $s = 4l + 2$ in (2.1). We have

$$\sum_{n=0}^{\infty}(-1)^n\zeta(\{4l+2\}_n)z^{(4l+2)n}$$
$$= \frac{(-1)^l}{(2\pi z)^{2l}} \sum_{1\le k\le 2l} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l}} \frac{\sin((1+\sum_{k=1}^{2l}(-1)^{i_k}\omega_{2l+1}^k)\pi z)}{\pi z}$$
$$= \frac{(-1)^l}{(2\pi z)^{2l}} \sum_{1\le k\le 2l} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l}} \left(1+\sum_{k=1}^{2l}(-1)^{i_k}\omega_{2l+1}^k\right)$$
$$\times \exp\left[g\left\{\left(1+\sum_{k=1}^{2l}(-1)^{i_k}\omega_{2l+1}^k\right)z\right\}\right]$$
$$= \frac{(-1)^l}{(2\pi z)^{2l}} \sum_{1\le k\le 2l} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l}} \left(1+\sum_{k=1}^{2l}(-1)^{i_k}\omega_{2l+1}^k\right)$$
$$\times \sum_{m=0}^{\infty}\frac{1}{m!}\left[g\left\{\left(1+\sum_{k=1}^{2l}(-1)^{i_k}\omega_{2l+1}^k\right)z\right\}\right]^m.$$

By the multinomial theorem we obtain

$$\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{a_1+\cdots+a_{2ln+l+n}=m}\binom{m}{a_1,\cdots,a_{2ln+l+n}}$$
$$\times \prod_{j=1}^{2ln+l+n}\left[\frac{B_{2j}\left\{(1+\sum_{k=1}^{2l}(-1)^{i_k}\omega_{2l+1}^k)2\pi i z\right\}^{2j}}{2j(2j)!}\right]^{a_j}$$
$$= \sum_{2ln+l+n=0}^{\infty}\left\{\left(1+\sum_{k=1}^{2l}(-1)^{i_k}\omega_{2l+1}^k\right)2\pi i z\right\}^{4ln+2l+2n}$$
$$\times \sum_{(a_1,\cdots,a_{2ln+l+n})\in S(a;2ln+l+n)}\prod_{j=1}^{2ln+l+n}\frac{1}{a_j!}\left(\frac{B_{2j}}{2j(2j)!}\right)^{a_j}.$$

Comparing the coefficients of $z^{(4l+2)n}$ in the above formulae and using the Proposition 2, the proof of Theorem 1 is complete.

Put

$$\tilde{g}(z) = -\sum_{n=1}^{\infty} \frac{G_{2n}(2\pi i)^{2n}}{4n(2n)!} z^{2n},$$

and $\tilde{f}(z) = \log\cos\pi z$. We have

$$\begin{aligned}
\tilde{f}'(z) &= \frac{d}{dz}\log\cos\pi z = \frac{d}{dz}\log\left(\frac{e^{\pi iz}+e^{-\pi iz}}{2}\right) \\
&= \pi i - \frac{2\pi i}{e^{2\pi iz}+1} \\
&= -\frac{1}{2}\sum_{n=1}^{\infty} \frac{G_{2n}(2\pi i)^{2n}}{(2n)!} z^{2n-1} \\
&= \tilde{g}'(z).
\end{aligned}$$

Hence $\tilde{f}(z) = \tilde{g}(z)$ since $\tilde{f}(0) = \tilde{g}(0)$.

Let $s = 4l$ in (2.1). We have

$$\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^n\zeta(\{4l\}_n)z^{4ln} \\
&= \frac{2i}{(2\pi z)^{2l}} \sum_{1\le k\le 2l-1}\sum_{i_k=0}^{1}(-1)^{i_1+\cdots+i_{2l-1}} \cos\left\{\left(1+\sum_{k=1}^{2l-1}(-1)^{i_k}\omega_{4l}^k\right)\pi z\right\} \\
&= \frac{2i}{(2\pi z)^{2l}} \sum_{1\le k\le 2l-1}\sum_{i_k=0}^{1}(-1)^{i_1+\cdots+i_{2l-1}} \\
&\quad \times \exp\left[\tilde{g}\left\{\left(1+\sum_{k=1}^{2l-1}(-1)^{i_k}\omega_{4l}^k\right)z\right\}\right] \\
&= \frac{2i}{(2\pi z)^{2l}} \sum_{1\le k\le 2l-1}\sum_{i_k=0}^{1}(-1)^{i_1+\cdots+i_{2l-1}} \\
&\quad \times \sum_{m=0}^{\infty}\frac{1}{m!}\left[\tilde{g}\left\{\left(1+\sum_{k=1}^{2l-1}(-1)^{i_k}\omega_{4l}^k\right)z\right\}\right]^m.
\end{aligned}$$

By the multinomial theorem we obtain

$$\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{a_1+\cdots+a_{2ln+l}=m}\binom{m}{a_1,\cdots,a_{2ln+l}}$$
$$\times\prod_{j=1}^{2ln+l}\left[-\frac{G_{2j}\left\{(1+\sum_{k=1}^{2l-1}(-1)^{i_k}\omega_{4l}^{k})2\pi iz\right\}^{2j}}{4j(2j)!}\right]^{a_j}$$
$$=\sum_{2ln+l=0}^{\infty}\left\{\left(1+\sum_{k=1}^{2l-1}(-1)^{i_k}\omega_{4l}^{k}\right)2\pi iz\right\}^{4ln+2l}$$
$$\times\sum_{(a_1,\cdots,a_{2ln+l})\in S(a;2ln+l)}\prod_{j=1}^{2ln+l}\frac{1}{a_j!}\left(-\frac{G_{2j}}{4j(2j)!}\right)^{a_j}.$$

Comparing the coefficients of $z^{(4l+2)n}$ in the above formulae and using the Proposition 2, the proof of Theorem 2 is complete.

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