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THE MASS FORMULA OF ORDERS OVER A DYADIC LOCAL FIELD

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ABSTRACT. In this paper, we study the arithmetic properties of orders in a quaternion algebra over a dyadic local field and we find the mass formula of orders.

1. Introduction

A primitive order in a quaternion algebra over a number field F is an order which contains the ring of integers in a quadratic extension field of F. Locally, there are two types of quaternion algebras over a local field k, i.e., a division algebra and a 2×2 matrix algebra. In these quaternion algebras over a local field, primitive orders can be classified into three types. Namely, an order in a quaternion division algebra which contains the ring of integers of a quadratic extension field of k is called primitive. In a 2×2 matrix algebra, there are two types of primitive orders. One is an order which contains $\mathcal{O} \times \mathcal{O}$ where $\mathcal{O}$ is the ring of integers in k and the other is an order which contains the ring of integers of a quadratic extension field of k.

Primitive orders in 2×2 matrix algebra which contain $\mathcal{O} \times \mathcal{O}$ where $\mathcal{O}$ is the ring of integers were studied by Hijikata [4]. Primitive orders in a division algebra, so called "special orders", were studied by Hijikata, Pizer and Shemanske [5]. The remaining type of primitive orders was studied by Brezinski only on a nondyadic local field [2].

In this paper, we will study the arithmetic properties of third type of orders and compute the Mass formula of the primitive orders in a 2×2 matrix algebra containing the ring of integers of a quadratic extension field of a dyadic local field k, which is the remaining type of primitive orders not studied by Brezinski.

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2. Preliminaries

Throughout this paper, we assume that k is a dyadic local field. In this section we summarize the arithmetic theory of dyadic local fields . Let $\mathcal{O} = \mathcal{O}_k$ denote the ring of integers in k, $P = P_k$ the maximal ideal of $\mathcal{O}$. By $\Delta(\alpha)$, we denote the discriminant of α.

$$\Delta(\alpha) = \mathrm{Tr}(\alpha)^2 - 4N(\alpha),$$

where Tr and N are the trace and norm of L over k where L is a quadratic extension field of k. If Γ is an $\mathcal{O}$ algebra of rank 2 contained in L, then $\Gamma = \mathcal{O} + \mathcal{O}x$ and the discriminant of Γ is

$$\Delta(\Gamma) = \Delta(x) \mod U^2,$$

where U is the set of all units in $\mathcal{O}$.

Let $\mathcal{O}^2 - 4\mathcal{O} = \{s^2 - 4n | s, n \in \mathcal{O}\}$. Then we consider the set of all possible discriminants, $(\mathcal{O}^2 - 4\mathcal{O})/U^2$.

DEFINITION 1. Let $\Delta_\sigma = ((\mathcal{O}^2 - 4\mathcal{O}) \cap \pi^\sigma U)/U^2$ for $\sigma = 0, 1, 2, \cdots$,

$$\Delta_0^* = \Delta_0 - \{1\}, \Delta_1^* = \Delta_1,$$
$$\Delta_\sigma^* = \Delta_\sigma - \pi^2 \Delta_{\sigma-2}.$$

Note that $\Delta_\sigma^* \neq \phi$ only if $\sigma = 2\rho, 0 \leq \rho \leq e$, or $\sigma = 2e+1$ where $e = \mathrm{ord}_k(2)$. Let

$$\Delta^* = \cup_{\sigma=0}^{\infty} \Delta_\sigma^* = \left(\cup_{\rho=0}^{e} \Delta_{2\rho}^*\right) \cup \Delta_{2e+1}^*.$$

Γ is a maximal order of a quadratic extension field of k if and only if $\Delta(\Gamma) \in \Delta^*$. If $e > 0$ and $1 \leq \rho \leq e$,

$$\Delta_{2\rho}^* = \pi^{2\rho}(U^2 + \pi^{2e-2\rho+1}U)/U^2.$$

There is a bijective correspondence between the elements of Δ^* and quadratic extension fields of k given by $\Delta(\Gamma) \to \Gamma \otimes \mathcal{O}$ for $\Delta(\Gamma)$ an element of Δ^*.

LEMMA 2.1. *Let U be the set of all units in $\mathcal{O}$ and $e > 0$. Then $U = U^2 + P \supset U^2 + P^2 \supset \cdots \supset U^2 + P^{2e+1} = U^2$ and*

$$(U^2 + P^\sigma)/(U^2 + P^{\sigma+1}) \simeq \begin{cases} 1 & \text{if } \sigma \text{ is even and} < 2e \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \sigma = 2e \\ \bar{k} & \text{if } \sigma \text{ is odd} \end{cases}$$

where $\bar{k} = \mathcal{O}/P$.

PROOF. See Proposition 1.4 in [5]. □

Thus we can classify all quadratic extension fields of a dyadic local field k as follows: Δ_0^* contains one point which corresponds to a unique unramified quadratic extension of k and

$$\Delta_{2e+1}^* = \pi^{2e+1}U/U^2$$

contains $2q^2$ points with $q = |\mathcal{O}/P|$.

Let L be a quadratic extension field of k and $x \to \bar{x}$ denote the conjugation of L/k. Further, let $\mathcal{O}_L$ be the ring of integers of L, $\mathcal{O}_L = \mathcal{O} + \mathcal{O}\alpha$ for some $\alpha \in L$. Then $\Delta(L) = \Delta(\mathcal{O}_L) = \Delta(\alpha)U^2$, while $\Delta(\alpha) = \mathrm{Tr}(\alpha)^2 - 4\mathrm{N}(\alpha) = (\alpha - \bar{\alpha})^2$. Whence, $\mathrm{ord}_k(\alpha - \bar{\alpha})^2 = \mathrm{ord}_k(\Delta(\alpha)) = \mathrm{ord}_k(\Delta(L))$.

DEFINITION 2. Let L be a quadratic extension of k.

$$t = t(L) = \mathrm{ord}_k(\Delta(L)) - 1.$$

REMARK. Note that if L is an unramified extension field of k, then $t = -1$. On the other hand, if L is a ramified extension field of a field k, then $t \geq 0$. Furthermore, if k is a dyadic local field, then $0 < t \leq 2e$ by 2.3 and 1.3 in [5].

LEMMA 2.2. *Let L be a quadratic extension of k. If $x \in \mathcal{O}_L$, then $ord_L(\Delta(x)) \geq ord_k(\Delta(L)) = t + 1$.*

PROOF. If $x = a + b\alpha \in \mathcal{O}_L = \mathcal{O} + \alpha\mathcal{O}$ with $a, b \in \mathcal{O}$, then $\mathrm{ord}_k(\Delta(x)) = \mathrm{ord}_k(b^2\Delta(\alpha)) \geq \mathrm{ord}_k(\Delta(L)) = t + 1$. □

3. Orders in quaternion algebra

Let A be a quaternion algebra which is split over a dyadic local field k (i.e. A is isomorphic to 2×2 matrix algebra over k). and let L be a quadratic extension field of k contained in A. Then there exists an element ξ in $A^\times$ such that $A = L + \xi L$ and $x\xi = \xi\bar{x}$ for all $x \in L$. To see this clearly, we can identify A with $\left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in L \right\}$ and L with $\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} | \alpha \in L \right\}$, where $-$ is the conjugation of L over k. Then ξ is identified with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence the norm and the trace of an element in A are defined as the determinant and the trace of corresponding element in

$\left\{\begin{pmatrix}\alpha & \bar{\beta}\\ \beta & \bar{\alpha}\end{pmatrix} \mid \alpha,\beta \in L\right\}$. Also, $N(\xi) = -1$ implies that $\bar{\xi} = -\xi$. Further, for an arbitrary $x \in L$, $\overline{\xi x} = \bar{x}\bar{\xi} = -\xi x$.

If α is integral of degree 2 over $\mathcal{O}$ satisfying $\alpha^2 - s\alpha + n = 0$. We denote the discriminant of α by $\Delta(\alpha) = s^2 - 4n$. Let $\mathcal{O}_L$ be the ring of integers in L, P_L the prime ideal of $\mathcal{O}_L$. Let π_L be the prime element of P_L. An order of a quaternion algebra A is a lattice in A which is also a subring containing the identity.

PROPOSITION 3.1. *Let the notation be as above. Let R be an order of A and L a quadratic extension field in A. Then R contains $\mathcal{O}_L$ if and only if*

$$R = \begin{cases} \mathcal{O}_L + \xi P_L^n & \text{if } L \text{ is an unramified extension field,} \\ \mathcal{O}_L + (1+\xi)P_L^{n-t-1} & \text{if } L \text{ is a ramified extension field , or} \\ \mathcal{O}_L + (1-\xi)P_L^{n-t-1} & \end{cases}$$

for some nonnegative integer n and $t = t(L)$.

PROOF. Suppose that R is an order of A which contains $\mathcal{O}_L$. Then $R = \mathcal{O}_L + y\mathcal{O}_L$ for some $y \in A$. $y \in R \subset A = L + \xi L$. Let $y = \alpha + \xi\beta$ for some $\alpha, \beta \in L$ and $\beta \neq 0$. If $x \in \mathcal{O}_L$, then $xy = x(\alpha+\xi\beta) = (x-\bar{x})\alpha + y\bar{x}$. So $(x-\bar{x})\alpha = xy - y\bar{x} \in R$ for any $x \in \mathcal{O}_L$. Since $\mathrm{ord}_L(x-\bar{x}) \geq t+1$ by Lemma 2.2, $\alpha \in P_L^{-t-1}$.

If $\alpha \in \mathcal{O}_L$, then $\beta \in \mathcal{O}_L$. For $N(y) = N(\alpha) - N(\beta) \in \mathcal{O}$. Let $n = \mathrm{ord}_L\beta$. Then $R = \mathcal{O}_L + \xi\beta\mathcal{O}_L = \mathcal{O}_L + \xi P_L^n$.

If $\alpha \notin \mathcal{O}_L$, then $\alpha \in P_L^{-t-1} - \mathcal{O}_L$. This is the case that L is ramified. Let $\alpha = \pi_L^{-s}u$ and $\beta = \pi_L^{-s}w$ for $1 \leq s \leq t+1$. From $N(y) = N(\alpha) - N(\beta) \in \mathcal{O}$, it is easy to see $N(u/w) \equiv 1 \mod P$. This implies that $u/w \equiv \pm 1 \mod P_L$. Thus R is of the form, $\mathcal{O}_L + (1+\xi)P_L^{n-t-1}$ or $\mathcal{O}_L + (1-\xi)P_L^{n-t-1}$. The other direction of the proof is trivial. □

COROLLARY 3.2. *Let the notations be as above and $e = \mathrm{ord}_k(2)$. If L is a ramified extension field of k,*

$$\begin{cases} \mathcal{O}_L + (1+\xi)P_L^{-t} = \mathcal{O}_L + (1-\xi)P_L^{-t} & \text{if } t = 2e, \\ \mathcal{O}_L + (1+\xi)P_L^{-t-1} = \mathcal{O}_L + (1-\xi)P_L^{-t-1} & \text{if } t < 2e. \end{cases}$$

PROOF. Let $\alpha + (1-\xi)\pi_L^{n-t-1}\beta \in \mathcal{O}_L + (1-\xi)P_L^{n-t-1}$. Then $\alpha + (1-\xi)\pi_L^{n-t-1}\beta \in \mathcal{O}_L + 2\pi_L^{n-t-1}\beta + (1+\xi)\pi_L^{n-t-1}\mathcal{O}_L$. If $t = 2e$, $2\pi^{n-t-1}\beta \in \mathcal{O}_L$ only if $n = 0$. $\mathcal{O}_L + (1-\xi)P_L^{n-t-1} = \mathcal{O}_L + (1+\xi)P_L^{n-t-1}$ for $n \geq 1$. If $t < 2e$, $2\pi^{n-t-1}\beta \in \mathcal{O}_L$ for any nonnegative integer n. □

Let $\pi(\pi_L)$ be a prime element in the ring of integers in k (L, respectively). Then if L is ramified, $\pi \equiv \pi_L^2 \mod \mathcal{O}_L^\times$ and if L is unramified, $\pi \equiv \pi_L \mod \mathcal{O}_L^\times$. We now need new notations of orders for the next step.

DEFINITION 3. Let L be a quadratic extension field of k and $\mathcal{O}_L$ its ring of integers. Then

(1) if L is unramified, $R_n(L) = \mathcal{O}_L + \xi\pi_L^n\mathcal{O}_L$ for $n \geq 0$,

(2) if L is ramified,

  (a) if $t = 2e$, $R_n(L) = \mathcal{O}_L + (1+\xi)\pi_L^{n-t-1}\mathcal{O}_L$ for $n \geq 0$, or $\overline{R_0(L)} = \mathcal{O}_L + (1-\xi)\pi_L^{-t-1}\mathcal{O}_L$,

  (b) if $t < 2e$, $R_n(L) = \mathcal{O}_L + (1+\xi)\pi_L^{n-t-1}\mathcal{O}_L$ for $n \geq 0$.

REMARK. If L is unramified, then the index n of $R_n(L)$ is always an even number.

LEMMA 3.3. *Let the notations be as above. Then*

(1) *if L is unramified,*

$$\cdots \subset R_{2n}(L) \subset R_{2n-2}(L) \cdots \subset R_0(L),$$

(2) *if L is ramified,* $\cdots \subset R_n(L) \subset R_{n-1}(L) \cdots \subset R_1(L) \subset \begin{cases} R_0(L) \\ \overline{R_0(L)}. \end{cases}$

PROOF. This is immediate from Definition 3. □

PROPOSITION 3.4. *Let L be a ramified quadratic extension field of k and $t = 2e$. Then $R_n(L) \approx \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ P^n & \mathcal{O} \end{pmatrix}$ for $n = 0, 1$.*

PROOF. By the proof of Corollary 3.2, $R_0(L) \cap \overline{R_0(L)} = R_1(L)$. By Hijikata's results ([4], 2.2 p.65), $R_1(L) \approx \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ P^n & \mathcal{O} \end{pmatrix}$ for some nonnegative integer n. Since $R_1(L)$ is the second largest order contained in the maximal order, $R_0(L) \approx \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}$, we conclude that $n = 1$. Clearly, $R_1(L) \approx \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ P & \mathcal{O} \end{pmatrix}$. □

THEOREM 3.5. *Let the notations be as above and $p = |\mathcal{O}/P|$. If L is unramified, then $|R_{2k}^\times / R_{2k+2}^\times| = p^2$ for $k \geq 1$ and $|R_0^\times / R_2^\times| = p^2 - p$.*

PROOF $k > 0$, $\alpha + \xi\beta \in R_{2k}^\times$ implies that $\alpha \in \mathcal{O}_L$ and $\beta \in P_L^k$. $N(\alpha + \xi\beta) = N(\alpha) - N(\beta) \in \mathcal{O}^\times$. That is, $\alpha \in \mathcal{O}_L^\times$. Now $R_{2k}^\times = (1 + \xi P_L^{2k})R_{2k+2}^\times$. $(1+\xi\beta_1)\overline{(1+\xi\beta_2)} = (1+\xi\beta_1)(1-\xi\beta_2) = 1 - \overline{\beta_1}\beta_2 +$

$\xi(\beta_1 - \beta_2) \in R^{\times}_{2k+2}$. Thus $(1+\xi\beta_1)R^{\times}_{2k+2} = (1+\xi\beta_2)R^{\times}_{2k+2}$ if and only if $\beta_1 \equiv \beta_2 \mod P_L^{k+1}$. $R^{\times}_{2k}/R^{\times}_{2k+2} \approx P_L^k/P_L^{k+1}$ by the map $\phi(1+\xi\pi^{-t-1}\beta) = \beta$. Since $|P_L^k/P_L^{k+1}| = |\mathcal{O}_L/P_L| = p^2$, $|R^{\times}_{2k}/R^{\times}_{2k+2}| = p^2$. Next, if $\alpha + \xi\beta \in R_0^{\times}$, then $N(\alpha) - N(\beta) \in \mathcal{O}^{\times}$. There are three cases to be classified.

(1) $\alpha \in \mathcal{O}_L^{\times}$ and $\beta \in P_L$.
(2) $\beta \in \mathcal{O}_L^{\times}$ and $\alpha \in P_L$.
(3) $\alpha \in \mathcal{O}_L^{\times}$ and $\beta \in \mathcal{O}_L^{\times}$ and $N(\alpha) - N(\beta) \notin P$.

Hence, $R_0^{\times} = R_2^{\times} \cup \xi R_2^{\times} \cup \{(1+\xi s)R_2^{\times} | s$ a representative of $\mathcal{O}_L/P_L$ with $N(s) \not\equiv 1 \mod P\}$. Let $\tilde{N}$ be a homomorphism from $(\mathcal{O}_L/P_L)^{\times} \to (\mathcal{O}/P)^{\times}$ induced by the norm of $\mathcal{O}_L$ to $\mathcal{O}$. Then $(\mathcal{O}_L/P_L)^{\times}/\ker(\tilde{N}) \simeq (\mathcal{O}/P)^{\times}$. Hence $|\ker(\tilde{N})| = p+1$. We now conclude that $|R_0^{\times}/R_2^{\times}| = 1 + 1 + (|(\mathcal{O}_L/P_L)^{\times}| - p - 1) = p^2 - p$. □

THEOREM 3.6. *If L is ramified, then*
(1) $|R_0^{\times}/R_1^{\times}| = p+1$.
(2) $|R_n^{\times}/R_{n+1}^{\times}| = p$ *for* $n \geq 1$.

PROOF. Suppose that $n \geq 1$. Then $\alpha + (1+\xi)\beta \in R_n^{\times}$ implies that $N(\alpha+\beta) - N(\beta) \in \mathcal{O}^{\times}$. That is, $N(\alpha) + \mathrm{Tr}(\alpha\overline{\beta}) \in \mathcal{O}^{\times}$, which implies $N(\alpha) \in \mathcal{O}^{\times}$ and $\alpha \in \mathcal{O}_L^{\times}$. Now let $R_n^{\times} = (\mathcal{O}_L^{\times} + (1+\xi)\pi_L^{-t-1}P_L^n) = (1 + (1+\xi)\pi_L^{-t-1}P_L^n)R^{\times}_{n+1}$. Then we can define a map ϕ from $R_n^{\times}/R^{\times}_{n+1}$ to P_L^n/P_L^{n+1} by $\phi(\{1+(1+\xi)\beta\}R^{\times}_{n+1}) \equiv \beta \mod P_L^{n+1}$. Since $(1+(1+\xi)\delta)^{-1} = (1 + \mathrm{Tr}(\delta) - (1+\xi)\delta)/N(1+(1+\xi)\delta)$,

$$\begin{aligned}\{1 + \mathrm{Tr}(\delta) &- (1+\xi)\delta\}\{1+(1+\xi)\beta\} \\ &= 1 + \mathrm{Tr}(\delta) + (1+\xi)\{-\delta - \mathrm{Tr}(\delta)\beta + (1+\mathrm{Tr}(\delta))\beta \\ &= 1 + \mathrm{Tr}(\delta) - (1+\xi)(\beta - \delta).\end{aligned}$$

Hence $\beta \equiv \delta \mod P_L$ implies that $\phi(1+(1+\xi)\delta R^{\times}_{n+1}) = \phi(1+(1+\xi)\beta R^{\times}_{n+1})$ if and only if $(1+(1+\xi)\delta)R^{\times}_{n+1} = (1+(1+\xi)\beta)R^{\times}_{n+1}$. ϕ is well defined. It is clear that $\tilde{\phi}$ is bijective. i.e. $R_n^{\times}/R^{\times}_{n+1} \approx P^n/P^{n+1}$. Hence (2) is proved.

By Proposition 3.4 and 2.2 in p.65 of [4] and direct computations, $|R_0^{\times}/R_1^{\times}| = (p^2-1)/(p-1) = p+1$. □

DEFINITION 4. Let M be an order of a quaternion algebra A over a number field F. Let $F_p = F \otimes \mathbb{Q}_p$ for a prime p and $\mathcal{O}_p$ is the ring of integers in F_p. M is said to be an order of level qp^n if $M \otimes \mathcal{O}_q$ is the maximal order of $A \otimes F_q$ and for some integer $n \geq 0$, $M \otimes \mathcal{O}_p = R_n(L)$ is an order of $A \otimes F_p \simeq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in F_p \right\}$ which contains the ring of integers of a quadratic extension field of F_p.

Our final goal is to compute the mass formula of an order of level $p^n q$. First of all, we need a definition of the mass formula. Let A be a quaternion algebra ramified at a prime q and ∞. Let M be an order of level N for some integer N and let $I_1, I_2, \cdots I_H$ be representatives of the left M-ideal classes.

DEFINITION 5. Let the notations be as above. The right order M_i of I_i is defined by

$$M_i = \{a \in A | I_i a \subset I_i\}.$$

DEFINITION 6. The Mass formula for M ideals where M is an order of A is given by

$$\text{Mass}(M) = 2 \sum_{i=1}^{H} \frac{1}{|U(M_i)|},$$

where $U(M_i) = M_i^\times$.

Finally we can compute the mass formula for orders, R_n.

THEOREM 3.7. *Let M be an order of level $p^n q$. Then*

$$Mass(M) = \frac{1}{12}(q-1)\delta$$

where $\delta = \begin{cases} (p^2 - p)p^{n-2} & \text{if } L \text{ is unramified} \\ (p+1)p^{n-1} & \text{if } L \text{ is ramified.} \end{cases}$

PROOF. Let M^0 be a maximal order in A containing M. Then as in Proposition 24 and Proposition 25 in [9] p.685,

$$\text{Mass}(M) = \text{Mass}(M^0)([U(M^0) : U(M)]).$$

By Eichler's result [3], $\text{Mass}(M^0) = \frac{1}{12}(q-1)$. Therefore we need to find $[U(M_0) : U(M)] = \prod_p [U(M_p^0) : U(M_p)]$. Since M_p^0 is a maximal order, by Definition 3, $M_p^0 = R_0(L)$ and $M_p = R_n(L)$. Hence, if L is unramified extension field of k, then

$$\begin{aligned}[U(M_p^0) : U(M_p)] &= [R_0^\times : R_2^\times] \cdots [R_{n-2}^\times : R_n^\times] \\ &= (p^2 - p)p^{n-2}.\end{aligned}$$

If L is ramified extension field of k, then

$$\begin{aligned}[U(M_p^0) : U(M_p)] &= [R_0^\times : R_1^\times] \cdots [R_{n-1}^\times : R_n^\times] \\ &= (p+1)p^{n-1}. \end{aligned}$$ □

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