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ON DIRECT SUMS IN BOUNDED BCK-ALGEBRAS

YISHENG HUANG

ABSTRACT. In this paper we consider the decompositions of subdirect sums and direct sums in bounded BCK-algebras. The main results are as follows. Given a bounded BCK-algebra X, if X can be decomposed as the subdirect sum $\overline{\bigoplus}_{i\in I} A_i$ of a nonzero ideal family $\{A_i \mid i \in I\}$ of X, then I is finite, every A_i is bounded, and X is embeddable in the direct sum $\bigoplus_{i\in I} A_i$; if X is with condition (S), then it can be decomposed as the subdirect sum $\overline{\bigoplus}_{i\in I} A_i$ if and only if it can be decomposed as the direct sum $\bigoplus_{i\in I} A_i$; if X can be decomposed as the direct sum $\bigoplus_{i\in I} A_i$, then it is isomorphic to the direct product $\prod_{i\in I} A_i$.

1. Introduction and preliminaries

K. Iséki and S. Tanaka in [6] considered the direct products in BCK-algebras. Z. M. Chen in [1] generalized this notion to BCI-algebras and introduced the notions of subdirect sums and direct sums (who called them the interior subdirect products and interior direct products respectively) in BCI-algebras, and he in [2] also clarified some facts of subdirect sums and direct sums. The theory of subdirect sums and direct sums is a kind of decomposition theories in BCI-algebras. Based on this theory, Z. M. Chen and Y. S. Huang in [3] showed that for some classes of BCI-algebras, the notions of subdirect sums and direct sums are the same. In this paper, we will consider subdirect sums and direct sums in bounded BCK-algebras and we will obtain some interesting results.

Throughout this paper we will freely use the symbols and terminologies of [4], [6] or [7]. As preliminaries we enumerate some notions and results concerned as follows. Given a BCI-algebra $(X; *, 0)$, the following identity holds:

$$(x * y) * z = (x * z) * y. \tag{1.1}$$

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And X with respect to the *BCI-ordering* $\leq$ forms a partially ordered set $(X; \leq)$ satisfying the following condition: for any $x, y, z \in X$,

$$(1.2) \qquad x \leq y \text{ implies } x * z \leq y * z,$$

where $x \leq y$ is defined by $x * y = 0$. It is known that if the zero element 0 of X is the least element of X with respect to $\leq$ (i.e., $0 * x = 0$ for any $x \in X$), then X is called a *BCK-algebra*; if the *greatest element* b of X exsits (i.e., there is $b \in X$ such that $x * b = 0$ for any $x \in X$), then X must be a BCK-algebra, called a *bounded BCK-algebra*. An *ideal* A of X means that it is a subset of X containing the zero element 0 of X such that $x \in A$ and $y * x \in A$ imply $y \in A$ for any $x, y \in X$. Every ideal A of X determines a *congruence* $\equiv$ on X where $x \equiv y \pmod{A}$ is defined by $x * y \in A$ and $y * x \in A$. An ideal of X is said *closed* if it is a subalgebra of X. It is known that every ideal of a BCK-algebra is closed. It is easy to verify that the following is true.

PROPOSITION 1.1. *Let X be a BCK-algebra and A, B be two ideals of X. Then $A \cap B = \{0\}$ if and only if $x * y = x$ for any $x \in A$ and $y \in B$.*

Given an ideal family $\{A_i \mid i \in I\}$ of a BCI-algebra X, we let $\sum_{i \in I} A_i$ denote the generated ideal $\langle \bigcup_{i \in I} A_i \rangle$ of X, and call it the *sum* of $\{A_i \mid i \in I\}$. It has been known (refer to [6, Theorem 3]) that $x \in \sum_{i \in I} A_i$ if and only if there is a finite number of elements $x_1, x_2, \ldots, x_n$ in $\bigcup_{i \in I} A_i$ such that

$$(\cdots((x * x_1) * x_2) * \cdots) * x_n = 0.$$

For convenience, we denote $\sum_{j \in I - \{i\}} A_j$ by A_i^*. Putting $x \in \sum_{i \in I} A_i$, if there exists $x_i \in A_i$ such that $x \equiv x_i \pmod{A_i^*}$, then x_i is called an *i-th component* of x. If $A_i \cap A_i^* = \{0\}$ for all $i \in I$, we call the sum $\sum_{i \in I} A_i$ the *subdirect sum* of $\{A_i \mid i \in I\}$, denoted by $\overline{\bigoplus}_{i \in I} A_i$. And A_i is called a *sub-summand* of $\overline{\bigoplus}_{i \in I} A_i$ (see [2, Definition 3]). The sum $\sum_{i \in I} A_i$ is called the *direct sum* of $\{A_i \mid i \in I\}$, denoted by $\bigoplus_{i \in I} A_i$, if it is the subdirect sum $\overline{\bigoplus}_{i \in I} A_i$ satisfying the following condition: for any finite subset K of the indexing set I and any element $x_i \in A_i$ $(i \in K)$, there exists $x \in \sum_{i \in I} A_i$ such that

$$\begin{cases} x \equiv x_i \pmod{A_i^*} & \text{if } i \in K, \\ x \equiv 0 \pmod{A_i^*} & \text{if } i \notin K. \end{cases}$$

In this case, every A_i is called a *summand* of $\bigoplus_{i \in I} A_i$ (see [2, Definition 4]).

THEOREM 1.2. *Let $\{A_i \mid i \in I\}$ be a closed ideal family of a BCI-algebra X and let $A = \sum_{i\in I} A_i$. If $A = \overline{\bigoplus}_{i\in I} A_i$, then the following hold:* for any x, $y \in A$,

(1) *there is one and only one i-th component x_i of x for any $i \in I$;*
(2) *there is a finite number of components $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ of x such that*
$$(\cdots((x * x_{i_1}) * x_{i_2}) * \cdots) * x_{i_{n-1}} = x_{i_n};$$
(3) *if x_i is the i-th component of x, and y_i of y, then $x_i * y_i$ is the only i-th component of $x * y$;*
(4) *$x = 0$ if and only if all components of x are zero;*
(5) *$x = y$ if and only if x and y have the same i-th components for any $i \in I$.*

PROOF. (1) See [2, Theorem 1].

(2) See [1, Proposition 2.6(iv)].

(3) Since x_i and y_i are respectively the i-th components of x and y, we have
$$x \equiv x_i \,(\mathrm{mod}\, A_i^*) \quad \text{and} \quad y \equiv y_i \,(\mathrm{mod}\, A_i^*)$$
where $A_i^* = \sum_{j\in I-\{i\}} A_j$. Then the substitution property of congruences implies
$$x * y \equiv x_i * y_i \,(\mathrm{mod}\, A_i^*).$$
Also, since A_i is a closed ideal of X and x_i, $y_i \in A_i$, we obtain $x_i * y_i \in A_i$. So, by (1) and the definition of components, $x_i * y_i$ is the only i-th component of $x * y$.

(4) It is obvious that 0 is an i-th component of itself, then (1) implies that all components of 0 are 0, the necessity holding. The sufficiency is a direct result of (2).

(5) Assume that x_i is the i-th component of x, and y_i of y, then $x_i * y_i$ is the i-th component of $x * y$ by (3). So, $x * y = 0$ if and only if $x_i * y_i = 0$ by (4). Likewise, $y * x = 0$ if and only if $y_i * x_i = 0$. Hence $x = y$ if and only if $x_i = y_i$ for every $i \in I$. □

A BCI-algebra X is called to be with *condition* (S) if the set
$$A(x, y) = \{t \in X \mid t * x \leq y\}$$
has the greatest element $x \circ y$ for all x, $y \in X$. It is known (refer to [5] and [7, §1.7]) that if X is with condition (S), then $\circ$ is a binary operation on X, and $(X; \circ, 0)$ forms a commutative monoid with 0 as the unit element, and the following conditions hold:

$$(x \circ y) * x \leq y, \tag{1.3}$$

$$x * (y \circ z) = (x * y) * z. \tag{1.4}$$

It is easily seen from (1.3) that every ideal of X is a submonoid of $(X;\circ,0)$.

Finally, let's recall the notion of direct products. Given a BCI-algebraic family $\{(X_i;\, *_i,\, 0_i) \mid i \in I\}$, if we denote $\prod_{i\in I} X_i$ for the set

$$\{f : I \to \textstyle\bigcup_{i\in I} X_i \mid f(i) \in X_i \text{ for any } i \in I\}$$

of mappings, then we have a BCI-algebra $(\prod_{i\in I} X_i;\, *,\, \theta)$, called the *direct product* of $\{X_i \mid i \in I\}$, where the operation $*$ on $\prod_{i\in I} X_i$ is defined by

$$f * g : I \to \textstyle\bigcup_{i\in I} X_i,\ i \mapsto f(i) *_i g(i),$$

and the zero element θ is the mapping $\theta : I \to \bigcup_{i\in I} X_i,\ i \mapsto 0_i$ (see [1, section I] or [6, section III]). Obviously, if every X_i is a BCK-algebra, so is $\prod_{i\in I} X_i$.

2. Subdirect sums and direct sums in bounded BCK-algebras

We begin our discussion with the indexing set I in the decomposition expression $X = \overline{\bigoplus}_{i\in I} A_i$.

THEOREM 2.1. *Let $\{A_i \mid i \in I\}$ be a nonzero ideal family of a BCK-algebra X, and let $X = \overline{\bigoplus}_{i\in I} A_i$. If X is bounded, then the indexing set I is finite and every sub-summand A_i of X is bounded.*

PROOF. As $X = \overline{\bigoplus}_{i\in I} A_i$, we have $A_i \cap A_i^* = \{0\}$ for all $i \in I$, where $A_i^* = \sum_{j\in I-\{i\}} A_j$. Also, since X is bounded, if we let b denote the greatest element of X, by Theorem 1.2(2), there is a finite subset K of I, say $K = \{1, 2, \ldots, n\}$, such that

$$(\cdots((b * b_1) * b_2) * \cdots) * b_n = 0, \tag{2.1}$$

where b_i is the i-th component of b for any $i \in K$.

For any $x \in X$, by b being the greatest element of X, we have $x \le b$. Repeatedly applying (1.2) and noticing (2.1), we obtain

$$(\cdots((x * b_1) * b_2) * \cdots) * b_n \le (\cdots((b * b_1) * b_2) * \cdots) * b_n = 0,$$

that is,

$$(\cdots((x * b_1) * b_2) * \cdots) * b_n = 0. \tag{2.2}$$

Since $b_i \in A_i \subseteq \bigcup_{k\in K} A_k$ for any $i \in K$, it follows from (2.2) that x is in the generated ideal $\sum_{k\in K} A_k$. So, $X \subseteq \sum_{k\in K} A_k$ and $X = \sum_{k\in K} A_k$.

Now, if $I \neq K$, then $I \supset K$. Putting $i_0 \in I - K$, we have $I - \{i_0\} \supseteq K$. Noticing $A_{i_0} \neq \{0\}$, we derive

$$A_{i_0} \cap A_{i_0}^* \supseteq A_{i_0} \cap \textstyle\sum_{k \in K} A_k = A_{i_0} \cap X = A_{i_0} \neq \{0\},$$

a contradiction with $A_{i_0} \cap A_{i_0}^* = \{0\}$. Therefore $I = K$, showing I is a finite set.

Next, put $i, j \in I$. If $i \neq j$, then $A_i \cap A_j \subseteq A_i \cap A_i^* = \{0\}$. So, $A_i \cap A_j = \{0\}$. Since the j-th component b_j of b is in A_j, Proposition 1.1 implies $x * b_j = x$ for any $x \in A_i$. From this, we have

$$x = (\cdots(((x * b_1) * \cdots) * b_{i-1}) * b_{i+1}) * \cdots) * b_n. \tag{2.3}$$

Right $*$ multiplying both sides of (2.3) by b_i and repeatedly applying (1.1), we obtain

$$x * b_i = (\cdots((x * b_1) * b_2) * \cdots) * b_n.$$

Hence $x * b_i = 0$ by (2.2), proving b_i is the greatest element of A_i. Therefore A_i is bounded. □

COROLLARY 2.2. *Let $\{A_i \mid i \in I\}$ be a nonzero ideal family of a BCK-algebra X and let $X = \overline{\bigoplus}_{i \in I} A_i$. Then the following assertions are true.*

(1) *If the indexing set I is infinite, then X is unbounded.*
(2) *If there exists $i \in I$ such that A_i is unbounded, then X is unbounded.*

The converse of Theorem 2.1 is false.

EXAMPLE 2.1. Let $X = \{0, 1, 2\}$. We define a binary operation $*$ on X as follows: $x * y = x$ if $x \neq y$, and $x * y = 0$ if $x = y$. Then X is a BCK-algebra (see [7], p.243). It is easy to verify that $A_1 = \{0, 1\}$ and $A_2 = \{0, 2\}$ are two bounded ideals of X such that $X = A_1 \overline{\oplus} A_2$. However, X is unbounded.

It is interesting that in Theorem 2.1, if X can be decomposed as the direct sum of $\{A_i \mid i \in I\}$, the converse is still true.

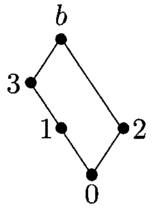
THEOREM 2.3. *Let $\{A_i \mid i \in I\}$ be a nonzero ideal family of a BCK-algebra X and let $X = \bigoplus_{i \in I} A_i$. Then X is bounded if and only if the indexing set I is finite and every summand A_i of X is bounded.*

PROOF. It suffices to prove the sufficiency. Assume that b_i is the greatest element of A_i. Since $X = \bigoplus_{i \in I} A_i$ and I is finite, by the definition of direct sums, there is $b \in X$ such that b_i is the i-th component of b. Also, according to Theorem 1.2(1), the i-th component x_i of any element $x \in X$ exists. Then Theorem 1.2(3) gives that $x_i * b_i$ is the i-th component of $x * b$. Note that b_i is the greatest element of A_i. It follows $x_i * b_i = 0$ for any $i \in I$. Hence Theorem 1.2(4) implies $x * b = 0$. We have proved that b is the greatest element of X. Therefore X is bounded. □

Let X be a bounded BCK-algebra and let $X = \overline{\bigoplus}_{i \in I} A_i$. Must the subdirect sum $\overline{\bigoplus}_{i \in I} A_i$ be the direct sum $\bigoplus_{i \in I} A_i$? It is a pity that the answer is negative.

EXAMPLE 2.2. The set $X = \{0, 1, 2, 3, b\}$ together with the operation $*$ on X defined by the following $*$ multiplication table forms a bounded BCK-algebra with b as the greatest element, whose Hasse diagram is described as follows (see [7], p.258).

$*$	0	1	2	3	b
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	3	3	0	0
b	b	b	3	2	0



It is easy to verify that $A_1 = \{0, 1, 3\}$ and $A_2 = \{0, 2\}$ are two ideals of X such that $X = A_1 \overline{\oplus} A_2$. But X can not be expressed as the direct sum of A_1 and A_2 (in fact, there does not exist any element $x \in X$ such that 1 and 2 are respectively the first and second components of x).

Nevertheless, if X is with condition (S), we have a nice result as follows.

THEOREM 2.4. *Let X be a bounded BCK-algebra, and $\{A_i \mid i \in I\}$ a nonzero ideal family of X. If X is with condition (S), then $X = \overline{\bigoplus}_{i \in I} A_i$ if and only if $X = \bigoplus_{i \in I} A_i$.*

PROOF. As every direct sum is a subdirect sum, we only need to show the necessity. Assume that $X = \overline{\bigoplus}_{i \in I} A_i$. Because X is bounded, the indexing set I is finite by Theorem 2.1. To fix our ideas, we suppose that $I = \{1, 2, \ldots, n\}$. For any $i \in I$ and any $x_i \in A_i$, putting

$$x = x_1 \circ x_2 \circ \cdots \circ x_n \in X \quad \text{and} \quad x_i^* = x_1 \circ \cdots \circ x_{i-1} \circ x_{i+1} \circ \cdots \circ x_n,$$

we have $x = x_i \circ x_i^*$ by the commutative law of $\circ$. Denoting $A_i^* = \sum_{j \in I - \{i\}} A_j$, we obtain $x_i^* \in A_i^*$ by A_i^* being a submonoid of $(X; \circ, 0)$.

Then (1.3) gives

$$x * x_i = (x_i \circ x_i^*) * x_i \leq x_i^* \in A_i^*,$$

that is, $x * x_i \in A_i^*$. Also, by (1.4), the following holds:

$$x_i * x = x_i * (x_i \circ x_i^*) = (x_i * x_i) * x_i^* = 0 * x_i^* = 0 \in A_i^*,$$

that is, $x_i * x \in A_i^*$. Hence $x \equiv x_i \pmod{A_i^*}$. Therefore $X = \bigoplus_{i \in I} A_i$. □

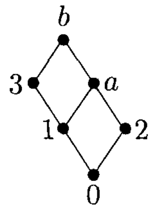
For the last example, whether we can make an extension $\overline{X}$ of X or not, such that $\overline{X}$ is a bounded BCK-algebra with b as the greatest element, satisfying $\overline{X} = A_1 \oplus A_2$. It is interesting that the answer is definite.

EXAMPLE 2.3. Let X be as in Example 2.2 and let $a \notin X$. Denote $\overline{X} = X \cup \{a\}$. If we extensively define

$$0 * a = 0,\ 1 * a = 0,\ 2 * a = 0,\ 3 * a = 3,\ b * a = 3,\ a * a = 0,$$
$$a * 0 = a,\ a * 1 = 2,\ a * 2 = 1,\ a * 3 = 2,\ a * b = 0,$$

then we have a binary operation $*$ on $\overline{X}$ whose $*$ multiplication table and Hasse diagram are as follows:

$*$	0	1	2	3	a	b
0	0	0	0	0	0	0
1	1	0	1	0	0	0
2	2	2	0	2	0	0
3	3	3	3	0	3	0
a	a	2	1	2	0	0
b	b	b	3	2	3	0



In order to verify that $(\overline{X}; *, 0)$ is a BCK-algebra, let's turn to consider the direct product $A_1 \times A_2$ where A_1 and A_2 are as in Example 2.2. For any $f, g \in A_1 \times A_2$ (i.e., f and g are any mappings from the indexing set $\{1, 2\}$ to the union set $A_1 \cup A_2$), if $f \neq g$, then $f(1) \neq g(1)$ or $f(2) \neq g(2)$. And the converse is still true. So we are able to denote f by $f_{(x_1, x_2)}$ in which $x_1 = f(1) \in A_1$ and $x_2 = f(2) \in A_2$. Now, it is not difficult to verify that the mapping $\varphi : \overline{X} \to A_1 \times A_2$, sending

$$0 \mapsto f_{(0,0)},\ 1 \mapsto f_{(1,0)},\ 2 \mapsto f_{(0,2)},\ 3 \mapsto f_{(3,0)},\ a \mapsto f_{(1,2)},\ b \mapsto f_{(3,2)}$$

is an isomorphism. Hence $\overline{X}$ is a BCK-algebra. It is easy to see that $\overline{X}$ is bounded with b as the greatest element, satisfying $\overline{X} = A_1 \oplus A_2$. Obviously, X is a subalgebra of $\overline{X}$.

As the general case of Example 2.3, we have the following result.

THEOREM 2.5. *Let X be a bounded BCK-algebra and $\{A_i \mid i \in I\}$ be a nonzero ideal family of X. If $X = \overline{\bigoplus}_{i\in I} A_i$, then X is embeddable in the direct sum $\overline{X} = \bigoplus_{i\in I} A_i$ such that X and $\overline{X}$ have the same greatest element.*

PROOF. Since $X = \overline{\bigoplus}_{i=I} A_i$, the i-th component x_i of any element $x \in X$ is unique by Theorem 1.2(1). Then

$$f_x : I \to \bigcup_{i\in I} A_i,\ i \mapsto x_i$$

is a mapping and $f_x \in \prod_{i\in I} A_i$. So we can define a mapping

$$\varphi : X \to \prod_{i\in I} A_i,\ x \mapsto f_x.$$

For any x, $y \in X$, if x_i and y_i are respectively the i-th components of x and y, then $x_i * y_i$ is the i-th component of $x * y$ by Theorem 1.2(3). So,

$$f_{x*y}(i) = x_i * y_i = f_x(i) * f_y(i),$$

that is, $\varphi(x * y) = \varphi(x) * \varphi(y)$. Hence φ is a homomorphism. Also, if $\varphi(x) = \varphi(y)$, then $f_x = f_y$, and so $f_x(i) = f_y(i)$, i.e., $x_i = y_i$, in other words, the i-th components of x and y are the same. Hence Theorem 1.2(5) gives $x = y$, and φ is injective. Moreover, if b is the greatest element of X, according to the second half part of the proof in Theorem 2.1, the i-th component b_i of b is the greatest element of A_i. Then for any $f \in \prod_{i\in I} A_i$, by $f(i) \in A_i$, we have $f(i) \leq b_i$, i.e., $f(i) * b_i = 0$ for any $i \in I$. Hence the definition of the operation $*$ on $\prod_{i\in I} A_i$ gives $f * f_b = \theta$. Therefore f_b is the greatest element of $\prod_{i\in I} A_i$. We have shown that X is embeddable in $\prod_{i\in I} A_i$ and f_b is the greatest element of $\prod_{i\in I} A_i$. Now, for any $j \in I$, if we denote B_j for the set

$$\{f \in \prod_{i\in I} A_i \mid f(i) = 0 \text{ whenever } i \neq j\},$$

then B_j is an ideal of $\prod_{i\in I} A_i$. Since I is finite (by Theorem 2.1), it is easy to verify that the sum $\sum_{j\in I} B_j$ is a direct sum and $\prod_{i\in I} A_i = \bigoplus_{j\in I} B_j$. Obviously, B_j is isomorphic to A_j. Also, if $i \neq j$, since $A_i \cap A_j = \{0\}$, there are no common elements of A_i and A_j except the zero element 0. Thus, in the viewpoint of isomorphisms, we are allowed in regarding B_j as A_j and f_b as b. Therefore $\bigoplus_{j\in I} A_j$ is well-defined and $\bigoplus_{j\in I} A_j$ is bounded with b as the greatest element. We now see that so long as we let $\overline{X} = \bigoplus_{j\in I} A_j$, i.e., $\overline{X} = \bigoplus_{i\in I} A_i$, we obtain the required conclusion: X is embeddable in $\overline{X}$ such that X and $\overline{X}$ have the same greatest element b. □

We remark that from the first half part of the proof in Theorem 2.5, we have a general result as follows.

PROPOSITION 2.6. *Let X be a BCI-algebra and $\{A_i \mid i \in I\}$ be a closed ideal family of X. If $X = \overline{\bigoplus}_{i\in I} A_i$, then X is embeddable in the direct product $\prod_{i\in I} A_i$.*

Finally, let's consider the relation between direct sums and direct products in bounded BCK-algebras.

THEOREM 2.7. *Let X be a bounded BCK-algebra and $\{A_i \mid i \in I\}$ be a nonzero ideal family of X. If $X = \bigoplus_{i\in I} A_i$, then X is isomorphic to $\prod_{i\in I} A_i$.*

PROOF. Assume that $\varphi : X \to \prod_{i\in I} A_i$, $x \mapsto f_x$, is as in Theorem 2.5 where $f_x(i)$ is the i-th component of x for all $i \in I$, then φ is an injective homomorphism. Also, for any $f \in \prod_{i\in I} A_i$, we have $f(i) \in A_i$ for all $i \in I$. Since $X = \bigoplus_{i\in I} A_i$, the indexing set I is finite by Theorem 2.1. Then the definition of direct sums implies that there exists $x \in X$ such that $f(i)$ is exactly the i-th component of x. So, $f_x(i) = f(i)$ and $f_x = f$. Hence $\varphi(x) = f_x = f$, proving φ is surjective. Therefore φ is an isomorphism and X is isomorphic to $\prod_{i\in I} A_i$. □

References

[1] Z. M. Chen, *Direct product theory of well BCI-algebras*, J. of Fujian Teachers Univ. **3** (1987), no. 2, 17–28.

[2] ———, *Direct sums of well BCI-algebras*, J. of Fujian Teachers Univ. **9** (1993), no. 2, 13–16.

[3] Z. M. Chen and Y. S. Huang, *Additive decomposition of BCI-algebras with condition* (S), Math. Japonica **41** (1995), no. 2, 303–310.

[4] K. Iséki, *On BCI-algebras*, Math. Seminar Notes **8** (1980), 125–130.

[5] ———, *On BCI-algebras with condition* (S), Math. Seminar Notes **8** (1980), 171–172.

[6] K. Iséki and S. Tanaka, *Ideal theory of BCK-algebras*, Math. Japonica **21** (1976), 351–366.

[7] J. Meng and Y. B. Jun, *BCK-algebras*, Kyung Moon Sa Co. Seoul, Korea, 1994.

Department of Mathematics
Sanming College
Sanming, Fujian 365004, P.R.China
E-mail: smcaihy@126.com