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GENERALIZED LIOUVILLE PROPERTY FOR SCHRÖDINGER OPERATOR ON GRAPHS

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ABSTRACT. We prove that the dimension of the space of positive (bounded, respectively) solutions for the Schrödinger operator whose potential q is nonnegative on a graph with q-regular ends is equal to the number of ends (q-nonparabolic ends, respectively).

1. Introduction

In this paper, we study the Liouville properties of the Schrödinger operator on a graph. Given an operator $\mathcal{A}$ on a graph and a class $\mathcal{S}$ of solutions of $\mathcal{A}$, by the Liouville property, we mean that the dimension of the space of solutions in $\mathcal{S}$ is at most one. Taking this point of view, given an operator $\mathcal{A}$ on a graph, it is natural to regard the finite dimensionality of the solution space in $\mathcal{S}$ as the generalized Liouville property of the pair $(\mathcal{A}, \mathcal{S})$. For example, Liouville property of the space of harmonic functions on a graph is well understood by the works of Kanai, Soardi, Lee and others. In [2], Kanai proved that the parabolicity of a graph is an invariant under a rough isometry between graphs. In [6], Soardi proved that rough isometries between graphs preserve the Liouville property of the space of bounded harmonic functions with finite Dirichlet sum on a graph. Later, Lee[4] proved that the dimension of the space of all bounded harmonic functions with finite Dirichlet sum on a graph is invariant under rough isometries between graphs.

In section 2, we obtain basic properties of solutions of the Schrödinger operator on a graph. In section 3, we introduce the parabolicity and regularity of ends related to the Schrödinger operator on a graph. In section 4, we prove the generalized Liouville property of the space of positive (bounded, respectively) solutions of the Schrödinger operator on a graph.

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2. Preliminaries

Let $G = (V, E)$ be an infinite graph with no self-loops, where V is the vertex set of G and E is the edge set of G. If vertices x and y are the endpoints of the same edge, we say that x and y are neighbors to each other and write $y \in N_x$ and $x \in N_y$. From now on, $|A|$ denotes the cardinality of the set A and in particular, the degree of x is denoted by $|N_x|$. The graph G is said to be of bounded degree if there exists a constant ν such that $|N_x| \leq \nu < \infty$ for all $x \in V$.

A sequence $\mathbf{x} = (x_0, x_1, \cdots, x_l)$ of vertices in G is called a path from x_0 to x_l with the length l if each x_k is an element of $N_{x_{k-1}}$. We say that a subset U of G is connected if for any two points $x, y \in U$, there exists a path in U joining x and y. For any $x, y \in V$, we define $d(x, y)$ to be the minimum of the lengths of paths from x to y. Then d defines a metric on G. For this metric d, define an l-neighborhood $N_l(x) = \{y \in G : d(x, y) \leq l\}$ for each $x \in G$ and $l \in \mathbf{N}$. For a subset U of G, define its boundary ∂U by $\partial U = \{x \in G : d(x, U) = 1\}$. Through this paper, we assume that each graph G is connected and has bounded degree, unless otherwise specified.

Let f be a real-valued function defined on V. Define the gradient of f by $|Df|(x) = (\sum_{y \in N_x} |f(y) - f(x)|^2)^{1/2}$ for each $x \in V$.

DEFINITION 1. Let U be a subset of a graph G with $\partial U \neq \emptyset$ and q be a nonnegative real valued function on G. A real valued function h defined on a set $\overline{U} (= U \cup \partial U)$ is called q-harmonic (q-superharmonic, resp.) in U if

$$\sum_{x \in G} \Big(\sum_{y \in N_x} (h(y) - h(x))(\eta(y) - \eta(x)) + q(x)h(x)\eta(x) \Big) = 0 \;\; (\geq 0, \text{ resp.})$$

for any finitely supported (nonnegative, resp.) function η in U. The function h is called q-subharmonic if $-h$ is q-superharmonic in U.

PROPOSITION 2 (Local Harnack Inequality). *Let h be a nonnegative q-superharmonic function in a subset U of a graph G. Then for each $x \in U$, we have*

$$\max_{y \in N_x} h(y) \leq (1 + |N_x| + q(x))h(x).$$

PROOF. Fix a point $x \in U$. Assume that $h(y_0) = \max_{y \in N_x} h(y)$ for some $y_0 \in N_x$. Then, obviously, $h(y_0) \geq h(x)$. Choose a real valued function w on G such that $w(x) = 1$ and $w(y) = 0$ for all $y \in G \setminus \{x\}$.

Then we have

$$
\begin{aligned}
0 &\leq \sum_{z\in U}\Big(\sum_{y\in N_z}(h(y)-h(z))(w(y)-w(z))+2q(z)h(z)w(z)\Big)\\
&= 2\sum_{y\in N_x}(h(x)-h(y))+2q(x)h(x)\\
&= -2\sum_{\{y\in N_x: h(y)>h(x)\}}(h(y)-h(x))\\
&\quad +2\sum_{\{y\in N_x: h(y)<h(x)\}}(h(x)-h(y))+2q(x)h(x).
\end{aligned}
$$

Hence we have $h(y_0)\leq(1+|N_x|+q(x))h(x)$. □

PROPOSITION 3 (Maximum Principle). *Let U be a connected subset of a graph G. Let h be a nonnegative q-harmonic function in U. If h is not identically zero on $U\cup\partial U$, then h cannot attain an interior maximum. Furthermore, h is strictly positive in U.*

PROOF. First, suppose that $h(x)=0$ at a point $x\in U$. Then by the local Harnack inequality, we have $h(y)=0$ for all $y\in N_x$. Repeating this process, we have $h\equiv 0$ on $U\cup\partial U$. This is a contradiction to the assumption.

Next, assume that h attains an interior maximum $M>0$ at a point $x\in U$. Then $M-h$ is a nonnegative q-superharmonic in U and takes zero at the point x. By the above argument, we get a contradiction. □

3. q-parabolicity and q-regularity of ends

Fix a point o in a graph G. We denote by $\sharp(l)$ the number of unbounded components of $G\setminus N_l(o)$ for each $l\in\mathbf{N}$. Then $\sharp(l)$ is nondecreasing in l. If $\lim_{l\to\infty}\sharp(l)=k$, where k may be infinity, then we say that the number of ends of G is k. If k is finite, then we can choose $l_0\in\mathbf{N}$ such that $\sharp(l)=k$ for all $l\geq l_0$. In this case, there exist mutually disjoint unbounded components $E_1,E_2,\cdots,E_k$ of $G\setminus N_{l_0}(o)$ and we call each E_i an end of G.

Let q be a nonnegative real valued function on G. We now classify ends of G by the q-parabolicity: We say that an end E of G is q-nonparabolic if for some integer $l_1\geq l_0$, there exists a q-harmonic function u_E, called a q-harmonic measure, on $E \setminus N_{l_1}(o)$ such that

$$\begin{cases} u_E = 0 \quad \text{on} \quad \partial N_{l_1}(o) \cap E; \\ \sup_{E \setminus N_{l_1}(o)} u_E = 1. \end{cases}$$

Otherwise, E is called q-parabolic. We now construct a q-harmonic function L_E, called the Liouville function of E, on a q-nonparabolic end E. Let $\{L_{E,l}\}_{l>l_1}$ be a sequence of q-harmonic function on $(N_l(o) \setminus N_{l_1}(o)) \cap E$ such that

$$\begin{cases} L_{E,l} = 1 \quad \text{on} \quad \partial N_l(o) \cap E; \\ L_{E,l} = 1 \quad \text{on} \quad \partial N_{l_1}(o) \cap E. \end{cases}$$

Then by the comparison principle, $u_E \leq L_{E,r} \leq 1$ on $(N_l(o) \setminus N_{l_1}(o)) \cap E$. Since $\{L_{E,l}\}$ is monotone decreasing, its limit function L_E is q-harmonic on E such that

$$\begin{cases} u_E \leq L_E \leq 1 \quad \text{on} \quad E; \\ \sup_E L_E = 1; \\ \inf_E (L_E - u_E) = 0. \end{cases}$$

DEFINITION 4. We say that an end E of a graph G is q-regular if there exist a constant $C < \infty$ such that for any nonnegative q-harmonic function f on E and sufficiently large integer L,

$$\sup_{\partial N_L(o) \cap E} f \leq C \inf_{\partial N_L(o) \cap E} f.$$

If E is a q-regular end of G and f is a nonnegative q-harmonic function on E, then, by combining the q-regularity and the maximum principle, we can control the wild variance of f at infinity of E as follows:

$$\begin{cases} \text{If} \ \liminf_{x \to \infty, x \in E} f(x) = 0, \quad \text{then} \ \lim_{x \to \infty, x \in E} f(x) = 0. \\ \text{If} \ \limsup_{x \to \infty, x \in E} f(x) = \infty, \quad \text{then} \ \lim_{x \to \infty, x \in E} f(x) = \infty. \end{cases}$$

4. Main results

Let us begin with constructing some q-harmonic functions, which generate all bounded or positive q-harmonic functions on a graph G. From now on, we assume that each end of G is q-regular, unless otherwise specified. If we assume that E is q-nonparabolic, then there exists a constant $C_f < \infty$ such that

$$\limsup_{x \to \infty, x \in E} f(x) \leq C_f.$$

Thus, by the maximum principle, we can construct a bounded q-harmonic function f_E on G such that for any other q-nonparabolic end E',

$$\lim_{x\to\infty, x\in E}(f_E - L_E)(x) = 0 \quad \text{and} \quad \lim_{x\to\infty, x\in E'} f_E(x) = 0. \tag{1}$$

If e is a q-parabolic end, then, by the local Harnack inequality, we can find a q-harmonic function v_e on $e \setminus N_{l_0}(o)$ such that

$$\begin{cases} v_e \geq 0 & \text{on} \quad e \setminus N_{l_0}(o); \\ v_e = 0 & \text{on} \quad \partial N_{l_0}(o) \cap e; \\ \lim_{x\to\infty, x\in e} v_e(x) = \infty. \end{cases}$$

If we further assume that G has at least one q-nonparabolic end, by the strong maximum principle, one can construct a nonnegative q-harmonic function h_e on G in such a way that h_e is nonnegative on $G \setminus e$ and for any q-nonparabolic end E,

$$\lim_{x\to\infty, x\in E} h_e(x) = 0 \quad \text{and} \quad \lim_{x\to\infty, x\in e} h_e(x) = \infty. \tag{2}$$

We have a characterization of the q-parabolicity as follows:

LEMMA 5. *Let e be a q-parabolic* (not necessarily q-regular) *end, and f be a nontrivial q-harmonic function bounded above on $e \setminus N_{l_0}(o)$ for some $l_0 \in \mathbf{N}$. Then $\sup_e f > \limsup_{x\to\infty, x\in e} f(x)$.*

PROOF. Suppose that $\limsup_{x\to\infty, x\in e} f(x) = \sup_e f = m$. Since f is nontrivial, for sufficiently small $\epsilon > 0$, there exists a proper subset Ω of $e \setminus N_{l_0}(o)$ such that $\Omega = \{x \in e \setminus N_{l_0}(o) : f(x) > m - \epsilon\}$. Put $v = \max\{(f - m + \epsilon)/\epsilon, 0\}$. Then v is a nonnegative q-subharmonic function on $e \setminus N_{l_0}(o)$ such that

$$v \equiv 0 \;\text{ on }\; e \setminus \Omega \quad \text{and} \quad \sup_{e\setminus N_{l_0}(o)} v = 1.$$

By the maximum principle, we can construct a q-harmonic function u_e on $e \setminus N_{l_0}(o)$ such that

$$u_e = 0 \;\text{ on }\; \partial N_{l_0}(o) \cap e \quad \text{and} \quad \sup_{e\setminus N_{l_0}(o)} u_e = 1.$$

This is a contradiction to the q-parabolicity of e. □

By Lemma 5 and the maximum principle, the Liouville theorem for the Schrödinger operator immediately follows:

THEOREM 6 (Liouville Theorem). *Let G be a graph with only q-parabolic* (not necessarily q-regular) *ends. Then every positive q-harmonic function and every bounded q-harmonic function on G must be zero.*

We are ready to describe the structure of the space of bounded (positive, respectively) q-harmonic functions on G:

THEOREM 7. *Let G be a graph of bounded degree and $E_1, E_2, \cdots, E_l$ be q-nonparabolic ends of G, each of which is q-regular. Then for each bounded q-harmonic function f on G, there exist real numbers $a_1, a_2, \cdots, a_l$ such that*

$$f = \sum_{i=1}^{l} a_i f_{E_i}, \tag{3}$$

where f_{E_i}'s are bounded q-harmonic functions satisfying (1).

Furthermore, the set $\{f_{E_i} : i = 1, 2, \cdots, l\}$ is linearly independent, therefore, $\dim \mathcal{HB}_q(G) = l$, where $\mathcal{HB}_q(G)$ denotes the space of all bounded q-harmonic functions on G.

PROOF. Let f be a bounded q-harmonic function on G. By Lemma 5, there exists a q-nonparabolic end E such that $\sup_E f = \sup_G f$. Without loss of generality, we may assume that $\sup_G f \geq 0$. By the maximum principle, we get $f \leq C_f L_E \leq C_f$ on E, where $C_f = \sup_G f$. Since $\sup_E f = C_f$, there is a sequence $\{x_i\}_{i\in\mathbf{N}}$ in E such that $\lim_{i\to\infty}(C_f L_E - f)(x_i) = 0$. By q-regularity, we have $\lim_{x\to\infty, x\in E}(C_f L_E - f)(x) = 0$, hence by (1), $\lim_{x\to\infty, x\in E}(C_f f_E - f)(x) = 0$. Repeating the above process, we can choose nonnegative real numbers $c_1, c_2, \cdots, c_l$ such that

$$\lim_{x\to\infty, x\in E}(f - \sum_{i=1}^{l} c_i f_{E_i})(x) = 0, \tag{4}$$

whenever E is a q-nonparabolic end satisfying $\limsup_{x\to\infty, x\in E} f \geq 0$. Thus by Lemma 5, $f - \sum_{i=1}^{l} c_i f_{E_i} \leq 0$ on G. Put $g = \sum_{i=1}^{l} c_i f_{E_i} - f$. Then g is a nonnegative bounded q-harmonic function on G. Applying the above argument to g again, there exist nonnegative real numbers $b_1, b_2, \cdots, b_l$ such that

$$\lim_{x\to\infty, x\in E}(g - \sum_{i=1}^{l} b_i f_{E_i})(x) = 0, \tag{5}$$

whenever E is an q-nonparabolic end satisfying $\liminf_{x\to\infty, x\in E} f \leq 0$. Combining (4) and (5), we have (3), where $a_i = c_i - b_i$ for each $i = 1, 2, \cdots, l$.

Clearly, the set $\{f_{E_i} : i = 1, 2, \cdots, l\}$ is linearly independent. □

THEOREM 8. *Let G be a graph of bounded degree with q-parabolic ends $e_1, e_2, \cdots, e_s$ and q-nonparabolic ends $E_1, E_2, \cdots, E_l$, each of which* *is q-regular. Then for each positive q-harmonic function f, there exist nonnegative real numbers $a_1, a_2, \cdots, a_l$ and $b_1, b_2, \cdots, b_s$ such that*

$$f = \sum_{i=1}^{l} a_i f_{E_i} + \sum_{j=1}^{s} b_j h_{e_j},$$

where f_{E_i}'s and h_{e_j}'s are q-harmonic functions satisfying (1) *and* (2), *respectively.*

Furthermore, the set $\{f_{E_i}, h_{e_j} : i = 1, 2, \cdots, l,\ j = 1, 2, \cdots, s\}$ is linearly independent, therefore, $\dim \mathcal{H}_q^+(G) = s + l$, *where $\mathcal{H}_q^+(G)$ denotes the space spanned by all positive q-harmonic functions on G.*

PROOF. Let f be a positive q-harmonic function on G. Then by the q-nonparabolicity, there exists a nonnegative constant $C_{E_i} < \infty$ for each $i = 1, 2, \cdots, l$ such that $\liminf_{x\to\infty, x\in E} f(x) = C_{E_i}$. We may assume that $C_{E_i} \geq 0$ for all $i = 1, 2, \cdots, l$. By the maximum principle, $f \geq C_{E_i} u_{E_i}$ on E_i. Hence there exist a nonnegative constant d_i and a sequence $\{x_n\}$ of points in E_i such that $f - d_i u_{E_i} > 0$ on E_i and $(f - d_i u_{E_i})(x_n) \to 0$ as $x_n \to \infty$. By the q-regularity, we have

$$\lim_{x\to\infty, x\in E_i} (f - d_i u_{E_i})(x) = 0.$$

Hence

$$\lim_{x\to\infty, x\in E_i} (f - d_i f_{E_i})(x) = 0.$$

Repeating this process, we can choose nonnegative constants $d_1, d_2, \cdots, d_l$ such that

$$\lim_{x\to\infty, x\in E_k} (f - \sum_{i=1}^{l} d_i f_{E_i})(x) = 0,$$

for each $k = 1, 2, \cdots, l$. If $f - \sum_{i=1}^{l} d_i f_{E_i}$ is also bounded on each q-parabolic end, then by Lemma 5, $f = \sum_{i=1}^{l} d_i f_{E_i}$ on M.

Assume that $h = f - \sum_{i=1}^{l} d_i f_{E_i}$ is unbounded on q-parabolic ends $e_1, e_2, \cdots, e_t$ where $1 \leq t \leq s$. We claim that for each $j = 1, 2, \cdots, t$, there exists a constant $0 < c_j < \infty$ such that $h - c_j h_{e_j}$ is bounded on e_j. Otherwise, there exists a q-parabolic end e_j for some $1 \leq j \leq t$ such that for any constant $0 < c < \infty$,

(6) $$h - c h_{e_j} \text{ is still unbounded on } e_j.$$

By the comparison principle, there exists a constant $0 < c_j < \infty$ such that

(7) $$h \geq c_j h_{e_j} \quad \text{or} \quad h \leq c_j h_{e_j} \quad \text{on } e_j.$$

First, assume that $c_j h_{e_j} - h \geq 0$ on e_i. Set $\underline{c}_i = \inf\{c_j : c_j h_{e_j} \geq h \text{ on } e_j\}$, then $0 < \underline{c}_j < \infty$ because $h > 0$. By (6), $\underline{c}_j h_{e_j} - h \geq 0$ is still unbounded on e_j. On the other hand, by (7) and the definition of the number $\underline{c}_j$, there exists a constant $0 < a_j < \infty$ such that

$$(\underline{c}_j - a_j) h_{e_j} \leq h.$$

Set $\overline{c}_j = \sup\{c_j : c_j h_{e_j} \leq h \text{ on } e_j\}$. If $h = \overline{c}_j h_{e_j}$, then we get the claim. Otherwise, by the strong maximum principle, we get

$$\underline{c}_j h_{e_j} - h > 0 \quad \text{and} \quad h - \overline{c}_j h_{e_j} > 0 \quad \text{on} \quad e_j, \tag{8}$$

and each of them is unbounded on e_j. There exists a constant $0 < b_j < \infty$ such that $\underline{c}_j h_{e_j} - h \geq b_j(h - \overline{c}_j h_{e_j})$ or $\underline{c}_j h_{e_j} - h \leq b_j(h - \overline{c}_j h_{e_j})$ on e_j, hence

$$\frac{\underline{c}_j + b_j \overline{c}_j}{b_j + 1} h_{e_j} \geq h \quad \text{or} \quad \frac{\underline{c}_j + b_j \overline{c}_j}{b_j + 1} h_{e_j} \leq h \quad \text{on} \quad e_j.$$

From the definition of $\underline{c}_j$ and $\overline{c}_j$, we have $\underline{c}_j = \overline{c}_j$. This implies that $\underline{c}_j h_{e_j} = h$ or $\overline{c}_j h_{e_j} = h$ on e_j. This is a contradiction to (8).

Consequently, for each $j = 1, 2, \cdots, t$, there exists a constant $0 < c_j < \infty$ such that $h - c_j h_j$ is bounded on e_j. For such constants $0 < c_j < \infty$, $h - \sum_{j=1}^{t} c_j h_{e_j}$ is bounded on G and

$$\lim_{x \to \infty, x \in E_i} (h - \sum_{j=1}^{t} c_j h_{e_j})(x) = 0$$

for all $i = 1, 2, \cdots, l$. By Lemma 5, we have $h = \sum_{j=1}^{t} c_j h_{e_j}$ on G, i.e.,

$$f = \sum_{i=1}^{l} d_i f_{E_i} + \sum_{j=1}^{t} c_j h_{e_j} \quad \text{on} \quad M.$$

It is easy to check that the set $\{f_{E_i}, h_{e_j} : i = 1, 2, \cdots, l,\ j = 1, 2, \cdots, s\}$ is linearly independent. □

The case of the space of harmonic functions on G is directly applicable to our result, since it is the case when $q = 0$. In particular, if each end of a graph satisfies the volume doubling condition, the Poincaré inequality, the Sobolev inequality, and the finite covering condition on the end, then it becomes regular for harmonic functions. (See [1] and [3].)

On the other hand, in [5], Lee pointed out that the dimension of the space of energy finite bounded solutions of the Schrödinger operator is preserved under rough isometries between graphs. Hence, our result can be extended to the case being rough isometric to the graphs satisfying the assumptions in Theorem 7 in the case of the energy finite bounded solutions of the Schrödinger operator.

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