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SIMPLE VALUATION IDEALS OF ORDER TWO IN 2-DIMENSIONAL REGULAR LOCAL RINGS

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Abstract. Let (R, m) be a 2-dimensional regular local ring with algebraically closed residue field R/m. Let K be the quotient field of R and v be a prime divisor of R, i.e., a valuation of K which is birationally dominating R and residually transcendental over R. Zariski showed that there are finitely many simple v-ideals $m = P_0 \supset P_1 \supset \cdots \supset P_t = P$ and all the other v-ideals are uniquely factored into a product of those simple ones. It then was also shown by Lipman that the predecessor of the smallest simple v-ideal P is either simple (P is free) or the product of two simple v-ideals (P is satellite), that the sequence of v-ideals between the maximal ideal and the smallest simple v-ideal P is saturated, and that the v-value of the maximal ideal is the m-adic order of P. Let $m = (x, y)$ and denote the v-value difference $|v(x) - v(y)|$ by n_v. In this paper, if the m-adic order of P is 2, we show that $o(P_i) = 1$ for $1 \leq i \leq \lceil \frac{b+1}{2} \rceil$ and $o(P_i) = 2$ for $\lceil \frac{b+3}{2} \rceil \leq i \leq t$, where $b = n_v$. We also show that $n_w = n_v$ when w is the prime divisor associated to a simple v-ideal $Q \supset P$ of order 2 and that $w(R) = v(R)$ as well.

1. Backgrounds

Let (R, m) be a 2-dimensional regular local ring with algebraically closed residue field R/m. Let K denote the quotient field of R. If v is a valuation of K dominating R with the valuation ring (V, n), then $\operatorname{tr.deg}_{R/m} V/n \leq 1$. If the residual transcendence degree is 0 (1, respectively), then v is called a 0-dimensional (1-dimensional, respectively) valuation. We call v a prime divisor of R if it is a 1-dimensional valuation. For a detailed background we refer to [3], [6], and [14].

Let v be a prime divisor of R and (V, n) be the associated valuation ring of v, i.e., $K \supset V \supset R$ and $n \cap R = m$. Since $v : K \to \mathbf{Z}$ then is a discrete rank one valuation, the image $v(V)$ is the set of nonnegative

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integers $\mathbf{N} \cup \{0\} = \{0, 1, 2, \ldots\}$ (cf. [1, Theorem 1], [12]). For an ideal I of R, $v(I) = \min\{v(a) | a \in I\}$ is a nonnegative integer and I is called a v-ideal if $IV \cap R = I$, i.e., if $I = \{r \in R | v(r) \geq v(I)\}$. The following sequence of contractions of the powers of the maximal ideals of V

$$n \cap R = m \supseteq n^2 \cap R \supseteq \cdots \supseteq n^i \cap R \supseteq \cdots\cdots$$

forms an infinite descending sequence of v-ideals in R :

$$m = I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_j \supset I_{j+1} \supset \cdots\cdots \tag{1}$$

For each j, $I_j = \{r \in R | v(r) \geq v(I_j)\}$ is the j^{th} largest v-ideal in R. For a consecutive pair $I_j \supset I_{j+1}$ of v-ideals, I_j is called the v-predecessor of I_{j+1} and I_{j+1} is called the v-successor of I_j.

The set of nonnegative integers $v(R) = \{v(r) | r \in R\} \subseteq \mathbf{N} \cup \{0\}$ is called the value semigroup of v on R which consists of the following nonnegative integers:

$$0 < r_0 < r_1 < r_2 < \cdots\cdots < r_j < r_{j+1} < \cdots\cdots , \tag{2}$$

where $r_j = v(I_j)$ for all $j \geq 0$. The value semigroup $v(R)$ is known to be symmetric [7, Theorem 1], i.e., there exists some integer z such that $a \in v(R)$ if and only if $z - a \notin v(R)$ for every integer $a \in \mathbf{Z}$. The conductor element of $v(R)$ is the smallest ingeger c such that $c-1 \notin v(R)$ but $c+j \in v(R)$ for all $j \geq 0$. The corresponding ideal C of v-value c is called the conductor ideal of v.

In [14], Zariski showed that there are only finitely many simple v-ideals $P_i's$ among infinite v-ideals $I_j's$ as follows:

$$P_0 \supset P_1 \supset P_2 \supset \cdots\cdots \supset P_t \tag{3}$$

and that any other v-ideal I_j can be uniquely factored into a product of simple v-ideals $I_j = \prod_{i=0}^{t} P_i^{a_i}$. It is clear that $m = P_0$ and let us denote the smallest simple v-ideal P_t by P. The number t of nonmaximal simple v-ideals is defined to be the rank of v, or the rank of P which is the smallest simple v-ideal. For such valuation v of K, there is a unique quadratic sequence of 2-dimensional regular local rings in K:

$$R = R_0 \subset R_1 \subset R_2 \subset \cdots\cdots \subset R_t = S \subset K \tag{4}$$

in which the transform of P_i in R_i becomes the maximal ideal m_i for each $0 \leq i \leq t$ and v is the m_t-adic order valuation. If v_i denotes the m_i-adic order valuation of K, then P_i is the smallest simple v_i-ideal in R for each i ([14, Theorem (F), p.392]). The conductor ideal C of v is also called the adjoint ideal of the smallest simple v-ideal P ([5]).

Combining notations of v-ideals in two sequences (1) and (3), we rewrite the sequence (1) with the conductor ideal C in it:

(5) $m = P_0 \supset P_1 \supset \cdots \supset C \supset \cdots \supset I_{s-1} \supset P_t = P = I_s \supset I_{s+1} \supset \cdots$.

It is known that the above sequence is saturated from m to P, i.e., $\lambda(I_j/I_{j+1}) = 1$ for $0 \leq j \leq s-1$ [8, Lipman, Theorem A.2], and hence $s = \lambda(R/P) - 1$ since k is algebraically closed. The length between any two consecutive v-ideals $I_j \supset I_{j+1}$ for $j \geq s$ can be measured in terms of the largest integer $\nu \in \mathbf{N}$ such that $I^\nu | I_j$ ([8, Theorem 3.1]).

For a simple v-ideal $J \supset P$ with the associated prime divisor w, the sequence of w-ideals containing J coincides with that of v-ideals [8, Lipman, Theorem A.2]. For two regular local rings $T \subset S$ in K, S is said to be proximate to T (denoted by $S \succ T$) if the $m(T)$-adic order valuation ring contains S ([6, (1.3)]). In the sequence (5), the v-predecessor I_{s-1} is the unique integrally closed ideal adjacent to P from above [6, Theorem 4.11], [9, Theorem 3.1]. It was also known that I_{s-1} is the product of simple v-ideals $P_i's$ associated to $R_i's$ to which R_t is proximate, and that there are at most two such quadratic transformations $R_i's$ [6, Theorem 4.11]. One of them is R_{t-1} since R_t is a first quadratic transformation of R_{t-1}. Hence we have either $I_{s-1} = P_{t-1}$ or $I_{s-1} = P_{t-1}P_i$ for some $0 \leq i \leq t-2$ when R/m is algebraically closed. The simple v-ideal P is said to be free for the former and satellite for the latter. Note that Lipman showed this result without the assumption R/m being algebraically closed [6]. We refer [2] for the proximity relations between valuation ideals for 0–dimensional valuation case.

For an ideal L of R, the (m-adic) order $o(L)$ of L is defined to be the integer r such that $L \subseteq m^r \backslash m^{r+1}$. Let us assume P is a simple integrally closed ideal associated to a prime divisor v, $o(P) = r \geq 1$ and $\operatorname{rank}(P) = t \geq 0$. Let us denote the number of simple v-ideals of order i by n_i for $1 \leq i \leq r = o(P)$ among t nonmaximal simple v-ideals in the following sequence:

$$P_0 \supset P_1 \supset P_2 \supset \cdots \supset P_{t-1} \supset P_t = P.$$

We are interested in finding the satellite simple v-ideals, i.e., simple v-ideal P_i whose v-predecessor is not simple.

Let $o(P) = 1$. If $t = 0$, then $P = m$ and hence $n_1 = t = 0$, and

$$m \supset m^2 \supset m^3 \supset m^4 \supset \cdots\cdots$$

is the sequence of all the v-ideals. If we further assume $o(P) = 1$ and $t > 0$, it is easy to see that $n_1 = t$, the nonmaximal simple v-ideals of order 1 are free, and they form the saturated sequence of all the v-ideals from m to P. The complete sequence of v-ideals was described in detail for $o(P) = 1$ case [10].

Let $o(P) = 2$. Any simple v-ideal P_i is of order one or two in the above sequence (3). If P is free, then $o(P_{t-1}) = 2$ and if P is satellite, then $o(P_{t-1}) = 1$ and therefore $o(P_i) = 1$ for all $i \leq t-2$ as well. Therefore, there exists some ℓ such that $o(P_\ell) = 1$ and $o(P_{\ell+1}) = 2$. In this paper we find such ℓ in terms of the v-value difference n_v of a regular system of parameters x, y when $o(P) = 2$. The results were stated without a proof and used to describe the complete sequence (5) of v-ideals in [11].

Throughout the paper, we assume $m = (x, y)$, $o(P) = 2$, $\text{rank}(P) = t \geq 2$, $v(y) = 2$, $v(x) = 2 + b$ for $b \geq 1$, i.e., $n_v = b$. We show that there are $n_1 = \lceil \frac{b+1}{2} \rceil$ simple nonmaximal v-ideals of order 1 and hence there are $n_2 = t - \lceil \frac{b+1}{2} \rceil$ simple v-ideals of order 2. It is also shown that $P_{\lceil \frac{b+3}{2} \rceil}$ is the only satellite simple v-ideal and that $C = P_{\lceil \frac{b-1}{2} \rceil}$ is the conductor ideal of v. The v-predecessor of $P_{\lceil \frac{b+3}{2} \rceil}$ is then obtained as $P_{\lceil \frac{b-1}{2} \rceil} \cdot P_{\lceil \frac{b+1}{2} \rceil} = C \cdot P_{\lceil \frac{b+1}{2} \rceil}$, i.e., $P_{\lceil \frac{b+3}{2} \rceil}$ is proximate to two previous simple v-ideals $C = P_{\lceil \frac{b-1}{2} \rceil}$ and $P_{\lceil \frac{b+1}{2} \rceil}$. For any other simple v-ideal Q of order 2 which is associated to the prime divisor w, we show that $n_w = n_v$ as well as $w(R) = v(R)$.

2. Simple valuation ideals of order two

Throughout this section, we assume that v is a prime divisor of a 2–dimensional regular local ring R, P is the associated simple integrally closed ideal of v, $o(P) = 2$, $\text{rank}(P) = t$ for $t \geq 2$. Let us assume that $m = (x, y)$ and denote $|v(x) - v(y)|$ by n_v. Note that $v(m) = o(P) = 2$ by reciprocity [6, Corollary (4.8)].

Let us assume $v(y) \leq v(x)$. Since P then is contracted from a first quadratic transformation $R_1 = R[\frac{m}{y}]_N$ for some maximal ideal N of $R[\frac{m}{y}]$ such that $m(V) \cap R[\frac{m}{y}] = N$. Therefore $v(x) > v(y)$ and $v(x) = 2 + n_v$ for some $n_v \geq 1$.

Let us denote n_v by b. In either $b = 2k$ even case for $k \geq 1$ or $b = 2k+1$ odd case for $k \geq 0$, we have $\lceil \frac{b-1}{2} \rceil = k$, $\lceil \frac{b+1}{2} \rceil = k+1$, $\lceil \frac{b+3}{2} \rceil = k+2$. With this invariant k for given prime divisor v, we describe the sequence of simple v-ideals from m to P.

Since $o(P) = 2$, $t = n_1 + n_2$, i.e., there are n_1 nonmaximal simple v-ideals of order 1 and the rest are the ones of order 2 in the sequence:

$$m = P_0 \supset P_1 \supset \cdots \supset P_{n_1} \supset P_{n_1+1} \supset \cdots \supset P_t = P.$$

We use the invariant n_v to determine these number n_1 and hence $n_2 = t - n_1$ as well.

THEOREM 2.1. *Let (R, m, k) be a 2-dimensional regular local ring with algebraically closed residue field k. Let P be a simple integrally closed ideal of R which is associated to the prime divisor v. Let $o(P) = 2$, $n_v = b$ and* rank$(P) = t$. *Let n_i be the number of nonmaximal simple v-ideals of order i for $i = 1, 2$. Then, $n_1 = \lceil \frac{b+1}{2} \rceil$ and $n_2 = t - \lceil \frac{b+1}{2} \rceil$.*

PROOF. Let us assume that $m = (x, y)$, $v(y) = 2$, and $v(x) = 2 + b$ for $b \geq 1$.

If $b = 1$, then m^2 is a v-ideal [8, Theorem 1.2]. Hence $P_1 = (x, y^2)$ is the only nonmaximal simple v-ideal of order 1 and $P_2 = (x^2, xy^2, y^3)$ is the simple v-ideal of order 2 and rank 2. Therefore, among simple v-ideals

$$m \supset P_1 \supset P_2 \supset \cdots\cdots \supset P_t = P$$

there exists only one nonmaximal simple v-ideal, i.e., $n_1 = 1 = \lceil \frac{b+1}{2} \rceil$ and therefore $n_2 = t - 1 = t - \lceil \frac{b+1}{2} \rceil$ for $t \geq 2$ and $b = 1$.

Assume $b \geq 2$.

Case 1: b is even($b = 2k$, $k \geq 1$).

In this case,

$$P_1 = (x, y^2) \supset P_2 = (x, y^3) \supset \cdots \supset P_k = (x, y^{k+1})$$

is the sequence of saturated v–ideals of v-values $4, 6, \ldots, 2k+2$ such that $v(P_k) = v(x) = v(y^{k+1}) = 2k + 2$ for $k \geq 1$. Since $\lambda(P_k/mP_k) = 2$, $v(mP_k) = 2k + 4$, and $P_k \supset I_{k+1} \supset mP_k$, we have that $I_{k+1} = (x - \alpha y^{k+1}, y^{k+2})$ is also simple for some $\alpha \neq 0 \in R/m$, i.e., $I_{k+1} = P_{k+1}$. Note that $I_{k+2} = mP_k$ since $v(P_{k+1}) = 2k + 3$, $v(mP_k) = 2k + 4$, and $I_{k+1} \supset I_{k+2}$ are adjacent. Therefore, $I_{k+2} = mP_k$ is the largest v-ideal of order 2 and hence $n_1 = k + 1 = \lceil \frac{b+1}{2} \rceil$ and $n_2 = t - \lceil \frac{b+1}{2} \rceil$.

Case 2: b is odd($b = 2k + 1$, $k \geq 1$).

In this case,

$$P_1 = (x, y^2) \supset P_2 = (x, y^3) \supset \cdots \supset P_k = (x, y^{k+1}) \supset P_{k+1} = (x, y^{k+2})$$

is the saturated sequence of v-ideals of v-values $4, 6, \ldots, 2k+2, 2k+3$. Therefore, $I_i = P_i$ for $1 \leq i \leq k + 1$. Since $\lambda(P_k/mP_k) = o(P_k) + 1 = 2$ (cf. [3, 4]) and $v(mP_k) = 2k + 4$, $I_{k+2} = mP_k$ is the v-ideal adjacent to

P_{k+1}, i.e., mP_k is the largest v-ideal of order 2. Since $\lambda(P_{k+1}/mP_{k+1}) = 2$ and $v(mP_{k+1}) = 2k+5$, $I_{k+3} = mP_{k+1}$ is the v-successor of $I_{k+2} = mP_k$. Therefore,

$$m \supset P_1 \supset \cdots \supset P_{k+1} \supset mP_k \supset mP_{k+1}$$

are all the v-ideals from m to mP_{k+1} and therefore $n_1 = k+1 = \lceil \frac{2k+2}{2} \rceil = \lceil \frac{b+1}{2} \rceil$ and $n_2 = t - \lceil \frac{b+1}{2} \rceil$.

In both cases, $o(P_i) = 2$ for $\lceil \frac{b+3}{2} \rceil \leq i \leq t$, i.e., $\lceil \frac{b+3}{2} \rceil$ is the largest simple v-ideal of order 2 and among the simple v-ideals from m to P,

$$m \supset P_1 \supset \cdots \supset P_{\lceil \frac{b-1}{2} \rceil} \supset P_{\lceil \frac{b+1}{2} \rceil} \supset P_{\lceil \frac{b+3}{2} \rceil} \supset \cdots \supset P_t = P$$

we see that $o(P_i) = 1$ for $i \leq \lceil \frac{b+1}{2} \rceil$ and $o(P_i) = 2$ for $i \geq \lceil \frac{b+3}{2} \rceil$. □

The conductor ideal of v(or the adjoint ideal of the associated simple v-ideal P) is the v-ideal C such that for any successive v-ideals $J \supset J'$ such that $C \supset J \supset J'$, $v(J') = v(J) + 1$ and it is known that $C = L : m$ for the largest v-ideal L of order $o(P)$ [5, Theorem 2.2]. Using this and Theorem 2.1, we now obtain the conductor ideal of v in our case.

COROLLARY 2.2. *Let P, v, $b = n_v$, t be as in Theorem 2.1. Then*

(i) *The largest v-ideal of order 2 is $mP_{\lceil \frac{b-1}{2} \rceil}$,*
(ii) *The conductor ideal of v is $C = P_{\lceil \frac{b-1}{2} \rceil}$,*
(iii) *P_i is satellite if and only if $i = \lceil \frac{b+3}{2} \rceil$,*
(iv) *$P_{\lceil \frac{b+3}{2} \rceil}$ is proximate to $P_{\lceil \frac{b-1}{2} \rceil}$ and $P_{\lceil \frac{b+1}{2} \rceil}$.*

PROOF. Note that $\lceil \frac{b+3}{2} \rceil = k+2$, $\lceil \frac{b+1}{2} \rceil = k+1$, $\lceil \frac{b-1}{2} \rceil = k$ for either $b = 2k$ even or $b = 2k+1$ odd.

(i)–(ii) Let $b = 2k$ for $k \geq 1$ or $b = 2k+1$ for $k \geq 0$. In either case, $P_k = (x, y^{k+1})$ by Theorem 2.1. Consider

$$P_{\lceil \frac{b-1}{2} \rceil} = P_k \supset P_{\lceil \frac{b+1}{2} \rceil} = P_{k+1} \supset mP_k \supset mP_{k+1} \supset P_{k+2}.$$

Note that $P_k = I_k$ and $P_{k+1} = I_{k+1}$ such that $v(P_k) = 2k+2$ and $v(P_{k+1}) = 2k+3$. Since $2 \in v(R)$, $v(I_{k+2}) = 2k+4$. Hence $mP_k \subseteq I_{k+2}$ since $v(mP_k) = 2 + (2k+2)$. However, $\mu(P_k) = o(P_k) + 1$ implies that mP_k is a v-ideal, too. Therefore, $mP_k = I_{k+2}$ is the largest v-ideal of order 2, hence $C = mP_k : m = P_k$, i.e., P_k is the conductor ideal of v by [5, Theorem 2.2].

(iii)–(iv) Since $o(P_{k+1}) = 1$ and $o(P_{k+2}) = 2$, they are not adjacent since P_{k+2} is simple. Therefore P_{k+2} is satellite and $o(P_j) = 2$ for all

$k+2 \leq j \leq t$, i.e., simple v-ideals of order 2 other than P_{k+2} are free. Since

$$m \supset P_1 \supset \cdots \supset P_k = C \supset P_{k+1}$$

is the set of all v-ideals of order 1 for either $b = 2k$ or $b = 2k+1$, they are all free. Since $\lambda(P_k/P_{k+1}) = 1$ and $\mu(P_k) = 2$, therefore $P_{k+1} \supset mP_k$ are adjacent. Since $v(P_{k+1}) = 2k+3$ and $v(mP_k) = 2k+4$, $P_{k+1} \supset I_{k+2} \supseteq mP_k$, where I_{k+2} is the v–successor of P_{k+1}. Therefore, $I_{k+2} = mP_k$. Similarly, $mP_k \supset I_{k+3} \supseteq mP_{k+1}$ since $v(mP_{k+1}) = 2 + (2k+3)$. But, $\lambda(mP_k/mP_{k+1}) = 1$ implies that $I_{k+3} = mP_{k+1}$. Since $v(P_1P_k) = 4 + 2k+2 = 2k+6$, $I_{k+4} \supseteq P_1P_k$. Since $v(P_1P_{k+1}) = 2k+7$, $I_{k+5} \supset P_1P_{k+1}$.

Now we claim that $\lambda(P_1P_k/P_1P_{k+1}) = 1$, i.e., $P_1P_k \supset P_1P_{k+1}$ are adjacent. For $1 \leq i \leq k$, let v_i be the prime divisor associated to P_i and consider two ideals $P_iP_k \supset P_iP_{k+1}$. By intersection multiplicity, we have $\lambda(P_iP_k/P_iP_{k+1}) = \lambda(P_k/P_{k+1}) + v_i(P_{k+1}) - v_i(P_k) = 1$ since P_{k+1} is not a v_i-ideal [8, Remark 2.2] while $P_k \supset P_{k+1}$ are adjacent. Therefore we have that $P_iP_k \supset P_iP_{k+1}$ are adjacent for $1 \leq i \leq k$. If $i = 1$, we have

$$I_{k+2} = mP_k \supset I_{k+3} = mP_{k+1} \supset I_{k+4} \supset P_1P_k \supset P_1P_{k+1}.$$

We then have $\lambda(mP_k/P_1P_k) = \lambda(m/P_1) + v_k(P_1) - v_k(m) = 1 + (2-1) = 2$ since P_k is a simple integrally closed ideal of order 1 and P_1 is also a v_k-ideal. Since $\lambda(I_i/I_{i+1}) = 1$ for any v-ideals containing P by [8, Theorem A.2], therefore we see that $I_{k+4} = P_1P_k$ and $I_{k+5} = P_1P_{k+1}$ by comparing the lengths.

We similarly can show that

$$P_{k+1} \supset mP_k \supset mP_{k+1} \supset P_1P_k \supset P_1P_{k+1} \supset \cdots\cdots \supset P_kP_k \supset P_kP_{k+1}$$

is a saturated sequence of v–ideals contained in P_{k+1}. Since this is saturated and none of them other than P_{k+1} are simple, we see that $P_kP_{k+1} \supset P_{k+2}$. Since $o(P_{k+2}) = 2$ and $o(P_{k+1}) = 1$, P_{k+2} is satellite, hence proximate to P_{k+1} and P_i for some $0 \leq i \leq k$, i.e., $P_{k+1}P_i \supset P_{k+2}$ are adjacent for some $0 \leq i \leq k$. Therefore, the v-predecessor of P_{k+2} is $P_{k+1}P_k$ from the containments as in the following sequence:

$$P_1P_{k+1} \supset P_2P_{k+1} \supset \cdots \supset P_{k-1}P_{k+1} \supset P_kP_{k+1} \supset P_{k+2}.$$

Since $o(P_t) = 2$, all the other simple v-ideals

$$P_{k+2} \supset P_{k+3} \supset \cdots\cdots \supset P_t$$

are saturated and hence P_i is free for $k+3 \leq i \leq t$ and for all $1 \leq i \leq k+1$ as well. Note that $t = \operatorname{rank}(P) \geq k+2 = \lceil \frac{b+3}{2} \rceil$ from the above construction. □

We showed that P_k is the conductor ideal of v, P_{k+1} is the smallest v-ideal of order 1, mP_k is the largest v-ideal of order 2, P_{k+2} is the only satellite simple v-ideal which is adjacent to $P_{k+1}P_k$, and the rank of P is at least $k+2$. Among the simple v-ideals of v, we also showed that $o(P_i) = 1$ for $1 \leq i \leq k+1$ and $o(P_i) = 2$ for $k+2 \leq i \leq t$. Let v_i denote the prime divisor associated to P_i for each $1 \leq i \leq t$.

If $i \leq k+1$, i.e., $o(P_i) = 1$, then the complete sequence of v_i-ideals was obtained in [10].

If $i \geq k+2$,

$$m \supset P_1 \supset \cdots\cdots \supset P_{k+1} \supset P_{k+2} \supset \cdots\cdots \supset P_i$$

is the sequence of all simple v_i-ideals as well. Furthermore, the sequence of all v_i-ideals from m to P_i coincides with the sequence of v-ideals from m to P_i by [8, Lipman, Theorem A.2]. It is known that if $J \supset I$ are adjacent simple integrally closed ideal associated to the prime divisors w and v respectively, then $o(J) = o(I)$ and $w(R) = v(R)$ [7, Theorem 2]. Now we further compare $w(x)$ to $v(x)$, $w(y)$ to $v(y)$, and b_w to n_v if w is the associated prime divisor of P_i for $k+2 \leq i < t$.

COROLLARY 2.3. *Let P, v, $b = n_v$, t be as in Theorem 2.1. Let $w = v_i$ be the prime divisor associated to the simple v-ideal P_i for $k+2 \leq i < t$. Then, $w(y) = v(y)$, $n_w = n_v$, and $w(R) = v(R)$.*

PROOF. By the previous theorem and corollary, we have $o(P_i) = 2$ for $k+2 \leq i \leq t$. Now let us denote $v_i = w$, $P_i = Q$ for $k+2 \leq i < t$ and $n_v = b$. Since $o(P) = 2$, we have $t \geq 2$. Since $P_{k+1} \supset \cdots \supset P_t$ are saturated simple v-ideals of order 2, we see that $w(R) = v(R)$ by using [7, Theorem 2] inductively.

Let $R = R_0 \subset R_1 \subset \cdots \subset R_i \subset \cdots \subset R_t$ be the quadratic sequence along v. Since $o(Q) = w(m) = 2$, $t \geq 2$, and $R_1 = R[\frac{x}{y}]_{(\frac{x}{y}, y)}$ is dominated by R_t, we have $w(y) = w(m) = 2 < w(x) = 2 + b_i$ for some $b_i > 0$, where $b_i = n_{v_i} = n_w$.

If $b = 1$, then m^2 is a v-ideal since $\lceil \frac{r}{b} \rceil = 2$ [8, Theorem 1.2]. Since $m^2 \supset Q$, m^2 is also a w-ideal [8, Theorem A.2]. Therefore, $\lceil \frac{r}{b_i} \rceil = \lceil \frac{2}{b_i} \rceil = 2$, hence $b_i = 1 = b$.

Assume $b \geq 2$. Then, m^2 is not a v-ideal and hence is not a w-ideal since $m^2 \supset Q \supset P$. Since Q is a w-ideal as well as a v-ideal, the sequence of v-ideals from m to Q in the following sequence

$$m \supset P_1 \supset \cdots\cdots \supset P_k \supset P_{k+1} \supset mP_k \supset \cdots$$
$$\supset P_{k+2} \supset \cdots \supset P_i = Q \supset \cdots \supset P_t = P$$

is also the sequence of w-ideals from m to Q [8, Theorem A.2].

Case 1: b is even($b = 2k$, $k \geq 1$).

In this case, we have the following containments:

$$P_{k-1} = (x, y^k) \supset P_k = (x, y^{k+1}) \supset P_{k+1} = (x - \alpha y^{k+1}, y^{k+2}) \supset mP_k \supset \cdots \supset P_i = Q$$

for some $\alpha \neq 0$ in R/m by Theorem 2.1. Since $y^k \in P_{k-1} \backslash P_k$ and $x \in P_k$ imply that $w(x) > w(y^k) = 2k$ and hence $w(P_k) = \min\{w(x), 2k+2\}$ is either $2k+1$ or $2k+2$. Suppose $w(x) = 2k+1$, i.e., $b_i = 2k-1$. Then $w(P_k) = 2k+1$ and $w(P_{k+1}) = 2k+2$ since $2 \in w(R)$. Then, $y^{k+1} \in P_{k+1}$ since $w(y^{k+1}) = 2k+2$, contradiction. Therefore, $w(x) \geq 2k+2$. Suppose $w(x) \geq 2k+3$. Then $w(P_k) = 2k+2$. Since $w(x) \geq 2k+3$ and P_{k+1} is the successive w-ideal of P_k, we see that $x \in P_{k+1}$, contradiction. Therefore, $w(x) = 2k+2 = b+2$ and hence $n_w = n_v$.

Case 2: b is odd($b = 2k+1$, $k \geq 1$).

In this case, we easily obtain $P_k = (x, y^{k+1})$ and $P_{k+1} = (x, y^{k+2})$. If $w(x) = 2k+2$, then $w(P_{k+1}) = w(P_k)$, contradiction. Therefore, $w(x) > 2k+2$ and hence $w(P_{k+1}) = \min\{w(x), 2k+4\}$ is either $2k+3$ or $2k+4$. Since mP_k is also a w-ideal of order 2, $x \notin mP_k$ implies that $w(x) < w(mP_k) = 2k+4$. Therefore, $w(x) = 2k+3 = 2+b$, hence $n_w = n_v$ as well. □

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