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Determining the Efficient Solutions for Bicriteria Programming Problems with Random Variables in Both the Objective Functions and the Constraints

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Abstract

This paper suggests an efficient approach for stochastic bicriteria programming problem (SBCPP) with random variables in both the objective functions and in the right-hand side of the constraints.

The suggested approach uses the statistical inference through two different techniques: In one of them , the SBCPP is transformed into an equivalent deterministic bicriteria programming problem (DBCPP), then the nonnegative weighted sum approach will be used to transform the bicriteria programming problem into a single objective programming problem, and the other technique, the nonnegative weighted sum approach is used to transform the SBCPP to an equivalent stochastic single objective programming problem, then apply the same procedure to convert stochastic single objective programming problem into its equivalent deterministic single objective programming problem (DSOPP). In both techniques the resulting problem can be solved as a nonlinear programming problem to get the efficient solutions. Finally, a comparison between the two different techniques is discussed, and illustrated example is given to demonstrate the actual application of these techniques.

1. Introduction

In most of the real life problems in which the decision maker (DM) would like to optimize different objectives at the same time, and the values of the parameters are uncertain to enable the DM to take the decision. If we consider the unknown parameters as random variables, the resulting problem is known as stochastic multicriterion programming problems (SMCPP).

Stochastic programming, as an optimization method based on the probability theory, has been developing in several ways (e.g., two stage problem by Dantiziq [4], chance constrained programming by Charnes and Cooper [1]. Especially, for multiobjective stochastic linear programming problems, Stancu-Minasian [6] considered the minimum risk approach, while Leclereq [9] and Teghem Jr. et al. [7] proposed interactive methods.

Research on the theoretical development of stochastic programming is going on for the last fifty years. The chance-constrained programming was first developed by Charnes and Cooper [1], and then, some researchers like Sengupta [8], Contini [3], Teghem et al. [7] and many others have established some theoretical results in the field of stochastic programming, Stancu-Minasian and Wets [5] have presented a review paper on stochastic programming with single objective function.

In this paper, we consider a stochastic bicriteria programming problem (SBCPP) with random variables in both the objective functions and the right-hand side of the constraints. In this process of the solution of the stochastic problem, several mathematical and statistical methods have been used.

Therefore, efficient solutions for SBCPP can be generated through two different techniques, in both techniques SBCPP will be transformed into an equivalent deterministic model using expected value and/or expected value-standard deviation criterion, and the nonnegative weighted sum approach, which is the one of the most familiar/popular approaches, used to convert BCPP into a single objective programming problem. A comparison of the efficiency between the two techniques will be discussed.

2. Bicriteria Programming problems formulation

Let us consider the following Bicriteria Programming Problem with random variables in the objective functions and the right-hand side of the constraints as:

$$\begin{aligned} &\min_{x\in X} \; F(x) = \{f_1(x), f_2(x)\} = \left\{ \sum_{j=1}^{n} c_{1j}x_j, \sum_{j=1}^{n} c_{2j}x_j \right\} \\ &\text{Subject to:} \\ &\qquad AX \geq b \\ &\qquad X \geq 0 \end{aligned} \tag{1}$$

Where, X is an n-dimensional decision variable column vector, and A is an $m \times n$ coefficient matrix, $c_k, k = 1,2$ are n-dimensional random variables row vectors. b is an m- dimensional random variables column vector, then , $AX \geq b$ can be stated as $\Sigma a_{ij}x_j \geq b_i, \quad i = 1,2,,...,m$ belong to normal distribution with mean $E(b_i) = \mu(b_i)$ and variance $Var(b_i) = \sigma^2(b_i)$, i.e $b_i \sim N(\mu(b_i), \sigma^2(b_i))$.

Since problem (1) contains random variables, definitions and solution methods for ordinary mathematical programming problems can not be applied directly. Therefore, we consider the expected value and/or expected value-standard deviation criterion in order to solve SBCPP, along with replacing the constraints by chance-constrained conditions with satisfying a certain probability, level of significance, $\alpha_i, i = 1,2,\ldots,m$ the problem will be converted into an equivalent deterministic BCPP.

3. Expected-Value Criterion

The expected value models are the most widely spread used stochastic programming and have been applied in the area of several scientifically fields.

Let us consider the expected value criterion in order to solve the SBCPP.

$$\underset{x\in X}{Min} \quad E(F(x)) = \{E(f_1(x), \ E(f_2(x)\}$$

The nonnegative weighted sum approach is used to transform the DBCPP into a single objective programming problem as follows :

$$\underset{x\in X}{Min} \quad E(F(x)) = \left\{\lambda E(\sum_{j=1}^{n} c_{1j}x_j) + (1-\lambda)E(\sum_{j=1}^{n} c_{2j}x_j)\right\}$$

We assume that $E(c_{1j}) = \mu(c_{1j})$ and $E(c_{2j}) = \mu(c_{2j})$, $j = 1,2,\ldots,n$,therefore,

$$\begin{aligned} E(\sum_{j=1}^{n} c_{1j}x_j) &= E(c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n) \\ &= x_1E(c_{11}) + x_2E(c_{12}) + \ldots + x_nE(c_{1n}) \\ &= x_1\mu(c_{11}) + x_2\mu(c_{12}) + \ldots + x_n\mu(c_{1n}) \\ &= \sum_{j=1}^{n} x_j\mu(c_{1j}) \end{aligned}$$

and

$$E(\sum_{j=1}^{n} c_{2j}x_j) == \sum_{j=1}^{n} x_j\mu(c_{2j})$$

Now, we obtain the following deterministic problem:

$$\begin{aligned} \underset{x\in X}{Min} \quad E(F(x)) &= \left\{\lambda(\sum_{j=1}^{n} \mu(c_{1j})x_j) + (1-\lambda)(\sum_{j=1}^{n} \mu(c_{2j})x_j)\right\} \\ &= \left\{\lambda(\sum_{j=1}^{n} x_j(\mu(c_{1j}) - \mu(c_{2j}))) + \sum_{j=1}^{n} \mu(c_{2j})x_j\right\} \end{aligned}$$

A chance-constrained programming with stochastic coefficients may be written as follows:

$$\underset{x \in X}{Min} \quad E(F(x))$$

subject to

$$P\left[\sum_{j=1}^{n} a_{ij} x_j \geq b_i\right] \geq 1 - \alpha_i, \quad i = 1,2,\ldots,m$$

$$x_j \geq 0 \;, \; j = 1,2,\ldots,n$$

where, b_i, $i = 1,2,\ldots,m$, as mentioned above belong to normal distribution with mean $E(b_i) = \mu(b_i)$ and variance $Var(b_i) = \sigma^2(b_i)$, then,

$$P\left[\frac{\sum_{j=1}^{n} a_{ij} x_j - \mu(b_i)}{\sigma(b_i)} \geq \frac{b_i - \mu(b_i)}{\sigma(b_i)}\right] \geq 1 - \alpha_i, \quad i = 1,2,\ldots,m$$

Let $Z_i = \dfrac{b_i - \mu(b_i)}{\sigma(b_i)}$ be a random variables with standard normal distribution with mean zero and variance one, i.e. $Z_i \sim N(0,1)$, $i = 1,2,\ldots,m$.

$$P\left[Z_i \leq \frac{\sum_{j=1}^{n} a_{ij} x_j - \mu(b_i)}{\sigma(b_i)}\right] \geq 1 - \alpha_i, \quad i = 1,2,\ldots,m$$

Therefore, $\phi\left(\dfrac{\sum_{j=1}^{n} a_{ij} x_j - \mu(b_i)}{\sigma(b_i)}\right) \geq 1 - \alpha_i, \quad i = 1,2,\ldots,m$

Where, $\phi(.)$ represents the cumulative distribution function of the standard normal variable. Let E_i represents the value of the standard normal variable at which $\phi(E_i) = 1 - \alpha_i$, $i = 1,2,\ldots,m$.

Let $E_i \leq \dfrac{\sum_{j=1}^{n} a_{ij}x_j - \mu(b_i)}{\sigma(b_i)}$, $i = 1,2,\ldots,m$

Then, $\phi(E_i) \leq \phi\left(\dfrac{\sum_{j=1}^{n} a_{ij}x_j - \mu(b_i)}{\sigma(b_i)}\right)$, $i = 1,2,\ldots,m$

This inequality will be satisfied only if

$$E_i \leq \frac{\sum_{j=1}^{n} a_{ij}x_j - \mu(b_i)}{\sigma(b_i)} \ , \ i = 1,2,\ldots,m$$

or

$$\sum_{j=1}^{n} a_{ij}x_j \geq \mu(b_i) + E_i \cdot \sigma(b_i) \ , \ i = 1,2,\ldots,m$$

where $\sigma(b_i)$ are the standard deviations of b_i , $i = 1,2,\ldots,m$.

Hence, the equivalent deterministic problem of SBCPP can be stated as :

$$\underset{x \in X}{Min}\ F(x) = \left\{ \lambda\left(\sum_{j=1}^{n} x_j(\mu(c_{1j}) - \mu(c_{2j}))\right) + \sum_{j=1}^{n} \mu(c_{2j})x_j \right\}$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \geq \mu(b_i) + E_i \cdot \sigma(b_i) \ , \ i = 1,2,\ldots,m \qquad (5)$$

$$x_j \geq 0 \ , \ j = 1,2,\ldots,n$$

Note that, if we apply the nonnegative weighted sum approach before or after using expected value criterion, the resulting problem (5) will be the same.

4. Expected value-standard deviation criterion

here, let us consider the expected value-standard deviation criterion in order to solve the weighted SBCPP

$$Min \left\{ \lambda \sum c_{1j}x_j + (1-\lambda)\sum c_{2j}x_j \right\}$$

(P(λ)) subject to (6)

$$\sum a_{ij}x_j \geq b_i , \quad i = 1,2,...,m$$

$$x_j \geq 0 \ , \ j = 1,2,...,n$$

If we apply the expected value – standard deviation criterion to the above problem (6).

$$E(F(x)) = \left\{ \lambda(\sum_{j=1}^{n} \mu(c_{1j})x_j) + (1-\lambda)(\sum_{j=1}^{n} \mu(c_{2j})x_j) \right\} \text{ and,}$$

$$Var(F(x)) = Var\left(\lambda \sum c_{1j}x_j + (1-\lambda)\sum c_{2j}x_j \right)$$

this is equal to

$$Var\left(\lambda \sum_{j=1}^{n} c_{1j}x_j \right) + Var\left((1-\lambda) \sum_{j=1}^{n} c_{2j}x_j \right) + 2\lambda(1-\lambda)Cov\left(\sum_{j=1}^{n} c_{1j}x_j, \sum_{j=1}^{n} c_{2j}x_j \right) \quad (7)$$

where,

$$Var\left(\lambda \sum_{j=1}^{n} c_{1j}x_j \right) = Var\left(\lambda(c_{11}x_1 + c_{12}x_2 + ... + c_{1n}x_n) \right)$$

$$= \lambda^2 Var(c_{11}x_1 + c_{12}x_2 + ... + c_{1n}x_n)$$
$$= \lambda^2 \Big(x_1^2 Var(c_{11}) + x_2^2 Var(c_{12}) + ... + x_n^2 Var(c_{1n}) + 2x_1x_2 \operatorname{cov}(c_{11}, c_{12}) + ... + 2x_1x_n \operatorname{cov}(c_{11}, c_{1n}) + ... + 2x_2x_n \operatorname{cov}(c_{12}, c_{1n}) \Big)$$

Consequently,

$$Var\left(\lambda \sum_{j=1}^{n} c_{1j}x_j \right) = \lambda^2 \left(\sum_{j=1}^{n} x_j^2 Var(c_{1j}) + 2 \sum_{s=1}^{n} \sum_{\substack{k=1 \\ k<s}}^{n} x_k x_s \operatorname{cov}(c_{1k}, c_{1s}) \right)$$

again, we find that

$$Var\left((1-\lambda) \sum_{j=1}^{n} c_{2j}x_j \right) = (1-\lambda)^2 \left(\sum_{j=1}^{n} x_j^2 Var(c_{2j}) + 2 \sum_{s=1}^{n} \sum_{\substack{k=1 \\ k<s}}^{n} x_k x_s \operatorname{cov}(c_{2k}, c_{2s}) \right)$$

Also, the third part of equation (7) can be expanded as:

$$Cov\left(\sum_{j=1}^{n} c_{1j}x_j, \sum_{j=1}^{n} c_{2j}x_j\right) = Cov\left(c_{11}x_1 + c_{12}x_2 + \dots + c_{1n}x_n, c_{21}x_1 + c_{22}x_2 + \dots + c_{2n}x_n\right)$$

$$= \sum x_j^2 Cov(c_{1j}, c_{2j}) + \sum_{k \neq s}\sum x_k x_s Cov(c_{1k}, c_{2s}) \qquad (8)$$

for simplicity the result in (8) will take this form

$$= \sum_{s=1}^{n}\sum_{k=1}^{n} x_k x_s Cov(c_{1k}, c_{2s}) = X^T V_{12} X \qquad (9)$$

then,

$$Var(F(x)) = \lambda^2\left(X^T V_1 X\right) + (1-\lambda)^2\left(X^T V_2 X\right) + 2\lambda(1-\lambda)\left(X^T V_{12} X\right) \qquad (10)$$

Where, $X^T = (x_1, x_2, \dots, x_n)$,

$$V_1 = \begin{bmatrix} Var(c_{11}) & Cov(c_{11}, c_{12}) \dots\dots\dots\dots & Cov(c_{11}, c_{1n}) \\ Cov(c_{12}, c_{11}) & Var(c_{12}) \dots\dots\dots\dots & Cov(c_{12}, c_{1n}) \\ . \quad . \quad . & . & . \quad . \quad . \\ . \quad . \quad . & . & . \quad . \quad . \\ Cov(c_{1n}, c_{11}) & Cov(c_{1n}, c_{12}) \dots\dots\dots\dots & Var(c_{1n}) \end{bmatrix},$$

$$V_2 = \begin{bmatrix} Var(c_{21}) & Cov(c_{21}, c_{22}) \dots\dots\dots\dots & Cov(c_{21}, c_{2n}) \\ Cov(c_{22}, c_{21}) & Var(c_{22}) \dots\dots\dots\dots & Cov(c_{22}, c_{2n}) \\ . \quad . \quad . & . & . \quad . \quad . \\ . \quad . \quad . & . & . \quad . \quad . \\ Cov(c_{2n}, c_{21}) & Cov(c_{2n}, c_{22}) \dots\dots\dots\dots & Var(c_{2n}) \end{bmatrix} \text{ and}$$

$$V_{12} = \begin{bmatrix} Cov(c_{11},c_{21}) & Cov(c_{11},c_{22}) \ldots\ldots\ldots\ldots Cov(c_{11},c_{2n}) \\ Cov(c_{12},c_{11}) & Cov(c_{12},c_{22}) \ldots\ldots\ldots\ldots Cov(c_{12},c_{2n}) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ Cov(c_{1n},c_{21}) & Cov(c_{1n},c_{22}) \ldots\ldots\ldots\ldots Cov(c_{1n},c_{2n}) \end{bmatrix}$$

The SBCPP is transformed into an equivalent deterministic problem using expected value-standard deviation criterion as follows :

$$\text{Min} \left\{ \lambda\left(\sum_{j=1}^{n} x_j(\mu(c_{1j}) - \mu(c_{2j})) \right) + \sum_{j=1}^{n} \mu(c_{2j})x_j + \sqrt{Var(F(x))} \right\}$$

Subject to (11)

$$\sum_{j=1}^{n} a_{ij}x_j \geq \mu(b_i) + E_i \cdot \sigma(b_i), \quad i = 1,2,\ldots,m$$

$$x_j \geq 0 \ , \quad j = 1,2,\ldots,n$$

Therefore, the solution of SBCPP can be obtained by solving the above equivalent deterministic nonlinear programming problem.

If we apply the expected value-standard deviation criterion for SBCPP to transform it into an equivalent deterministic bicriteria programming problem, then, nonnegative weighted sum approach is used to we get a single objective programming problem as follows

$$Min\, F(x) = \left\{ \sum_{j=1}^{n} \mu(c_{1j})x_j + \sqrt{X^T V_1 X},\ \sum_{j=1}^{n} \mu(c_{2j})x_j + \sqrt{X^T V_2 X} \right\}$$

Let $0 \leq \lambda \leq 1$, then, a single objective programming problem will be

$$Min \left\{ \lambda\left(\sum_{j=1}^{n} \mu(c_{1j})x_j + \sqrt{X^T V_1 X} \right) + (1 - \lambda)\left(\sum_{j=1}^{n} \mu(c_{2j})x_j + \sqrt{X^T V_2 X} \right) \right\}$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \geq \mu(b_i) + E_i \cdot \sigma(b_i), \quad i = 1,2,\ldots,m \tag{12}$$

$$x_j \geq 0 \ , \quad j = 1,2,\ldots,n$$

5. Discussion

According to the resulting problems (11) & (12) we have noted that problem (12) is less complicated than problem (11), and we can say that in problem (12) the non-dominated solutions and efficient solution are obtained, while in problem (11) efficient solutions were obtained only, which is shown in the following example.

6. Illustrative Example

Let us consider the following stochastic bicriteria programming problem :

$$Min \{c_{11}x_1 + c_{12}x_2 , c_{21}x_1 + c_{22}x_2\}$$

subject to

$$x_1 + 2x_2 \geq b_1$$
$$2x_1 + 4x_2 \geq b_2$$
$$x_1, x_2 \geq 0$$

where, $c_{11}, c_{12}, c_{21}, c_{22}$ are random variables with expected value as follows:

$$E(c_{11}) = 3, E(c_{12}) = 6, E(c_{21}) = 2, E(c_{22}) = 4$$

and with positive definite covariance matrix :

$$V = \begin{bmatrix} 9 & 2 & 1 & 0 \\ 2 & 16 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$$

Also, b_i are normal random variables with means $E(b_1) = 6$, $E(b_2) = 7$, and variances $Var(b_1) = 9$, $Var(b_2) = 16$, at level of significance $\alpha = 0.05$, therefore, $\phi^{-1}(\alpha) = 1.96$.

Now, we apply the expected value-standard deviation approach, the subset of efficient solutions will be as shown in table (1) according to the given different weights used in the problem.

From this example, we observed that, in problem (12) as the weights increase the decision variable x_1 decreases while the second decision variable x_2 increases, as well as f_1, f_2 as shown in table (2).

If we assume that the covariances of the objective functions are zero, the set of expected value-standard deviation efficient solutions for the problem (11) and (12) is achievable.

Table (1)
Subset of efficient solutions

λ	x_1	x_2	$F(x)$
0.0	5.940000	2.970000	32.16043
0.1	5.377903	3.251049	33.30103
0.2	4.707170	3.586415	34.92375
0.3	4.118718	3.880641	36.93954
0.4	3.702857	4.088571	39.27491
0.5	3.449033	4.215485	41.86402
0.6	3.310820	4.284590	44.64514
0.7	3.245316	4.317342	47.56637
0.8	3.222766	4.328617	50.58843
0.9	3.224664	4.327668	53.68322
1.0	3.240000	4.320000	56.83113

Where ; $0 \le \lambda \le 1$

x_1 , x_2 : are the decision variables.

$F(x)$: the scalar objective function.

Table (2)
Subset of efficient and non-dominated solutions

λ	x_1	x_2	f_1	f_2	$F(x)$
0.0	5.940000	2.970000	58.64552	32.16043	32.16043
0.1	5.435243	3.222379	58.04701	32.19071	34.77633
0.2	5.002462	3.438769	57.62256	32.26442	37.33605
0.3	4.635617	3.622191	57.33076	32.36058	39.85164
0.4	4.326581	3.776710	57.13538	32.46480	42.33303
0.5	4.066377	3.906811	57.00763	32.56841	44.78802
0.6	3.846460	4.016770	56.92637	32.66691	47.22259
0.7	3.659407	4.110297	56.87674	32.75830	49.64120
0.8	3.499093	4.190453	56.84854	32.84203	52.04724
0.9	3.360596	4.259702	56.83490	32.91827	54.44324
1.0	3.240000	4.320000	56.83113	32.98752	56.83113

Where ; $0 \le \lambda \le 1$

x_1 , x_2 : are the decision variables.

f_1 : the first objective function.

f_2 : the second objective function.

$F(x)$: the scalar objective function.

7. Conclusion:

This paper introduces an efficient approach for treating stochastic bicriteria programming problem (SBCPP). Two different techniques are used to determine the efficient set for SBCPP. A comparison is done to demonstrate the validity of the two techniques.

The efficient solutions of SBCPP are the same optimal solutions for the non-negative weighted sum approach. Furthermore, we found that, efficient solutions set can be obtained whether the covariances of the objective functions are zero or not.

Finally, we hope that this work will be asset for treating with a real life problem.

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