J. KSIAM **Vol.9**, No.1, 9-16, 2005

THE RANDOM SIGNALS SATISFYING THE PROPERTIES OF THE GAUSSIAN WHITE NOISE

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ABSTRACT. The random signals defined as sums of the single frequency sinusoidal signals with random amplitudes and random phases or equivalently sums of functions obtained by adding a Sine and a Cosine function with random amplitudes, are used in the double randomization method for the Monte Carlo solution of the turbulent systems. We show that these random signals can be used for studying the properties of the Johnson noise by proving that constant multiples of these signals with uniformly distributed frequencies in a fixed frequency band satisfy the properties of the Gaussian white noise.

1. INTRODUCTION

The Monte Carlo methods[1,7] have been used in many studies to simulate systems with a turbulent fluid component which is modeled by an Ornstein-Uhlenbeck equation. The simulation methods are employed because it is difficult to conceive of a deterministic mechanism for generating a velocity field with disordered fluctuations over a wide range of scales. The velocity field used in the randomization method [2,6] is most often given by

$$u(x) = \frac{1}{\sqrt{M}} \sum_{m=1}^{M} \sum_{j=1}^{M} \sqrt{2E}(\xi_j^{(m)} Cos(2\pi k_j^{(m)} x) + \eta_j^{(m)} Sin(2\pi k_j^{(m)} x)) \tag{1}$$

In this paper, we study an Ornstein-Uhlenbeck equation of the form

$$RI(t) + V(t) - L\frac{dI(t)}{dt} = 0 \tag{2}$$

where T is the temperature, R is the resistance, $I(t)$ is the electric current, L is the inductance, and $\Gamma(t)$ is the Gaussian white noise [3,4]. Our interests are in the the

Key words and phrases. Gaussian white noise; Johnson noise; Double randomization method; Markov process; Random signal; Ornstein-Uhlenbeck equation;

Johnson noise which is the fluctuating electromotive force $V(t) = (2kTR)^{\frac{1}{2}}\Gamma(t)$ satisfying the equation (2) in an arbitrary electric circuit.

It is not an easy task to measure $V(t)$ in (2) directly since the amplitude of $V(t)$ is extremely small, the signal is highly oscillating, and some channel noises must get picked up in any experimental setup for such measurements. Hence, a simulation technique is necessary in order to study the properties of the Johnson noise. In the following, we give an explicit proof that the random signals of the form

$$\Gamma(t) = c\sum_{j=M_0}^{M} a_j Sin(2\pi f_j t + b_j) \tag{3}$$

satisfies the properties of the Gaussian white noise, where a_j's, b_j's are uniformly distributed random numbers in $[0,1]$ and in $[0,2\pi]$ respectively, and $f_j = f_L + j$ where f_L is a nonnegative integer.

In our earlier work [5], we have shown that the random signal (3) satisfies the properties of the Gaussian white noise when the frequencies f_j's are also from a uniformly distributed random integers in a fixed frequency band, e.g. in $[2^{20}, 1.5\times 2^{20}]$. Note that the random signal in (3) can be identified the same as the function in (1). In the following, we prove that the frequencies do not have to be random so that the random signal (3) satisfies the properties of the Gaussian white noise. Note that if (3) is substituted in $V(t) = (2kTR)^{\frac{1}{2}}\Gamma(t)$ and the resulting $V(t)$ is substituted in (2), then the differential equation can be solved. The resulting solution verifies the experimental observation that the electric current $I(t)$ in (2) behaves also like a Gaussian white noise signal when it is in a steady state after the period of energy dissipation, i.e. after the exponential decay period.

2. RANDOM SIGNAL WITH GAUSSIAN WHITE NOISE PROPERTIES

A random signal is a Gaussian white noise by definition if it is normally distributed with a zero mean and if the samples at two different times are uncorrelated. In practical applications, the signal is given as a finite set of discrete sampled points and hence its Fourier transform must be finite, i.e. only a finite number of frequencies are used. Note that if K is the number of sampled points, then there can be at most only K/2 frequencies with nonzero amplitudes based on the Nyquist theorem.

Thus, the Gaussian white noise that we are concerned with must have a frequency band. A typical example of the frequency band is $[5\times 2^{19}, 1.2\times 2^{20}]$ and the corresponding sampling time is $h = \frac{1}{3\times 2^{20}}$. Hence, we restrict our attention to random signals of the form $\Gamma(t) = c\sum_{j=1}^{M} a_j Sin(2\pi f_j t + b_j)$ where f_j's are integers in $[f_L, f_U]$ for some positive integers f_L and f_U with $f_j = f_L + j$. We show that if the random variables a_j and b_j's are uniformly distributed, then the signal generates a Gaussian white noise, i.e. $E[\Gamma(t)] = 0$ and $E[\Gamma(t)\Gamma(t')] = \delta(t - t')$. Note that we won't have to prove $E(\Gamma(t)\Gamma(t)) = 1$ as this is achieved by choosing a proper constant c. In the following, we let $\Omega = \{\Gamma(t_k)|k = 1, 2, ...\infty\}$ where $t_k = \frac{k}{N}$ and let X_k be a random sample of the N consecutive points from Ω. Then we have the following, a simple proof of which is found in [3].

Lemma 1. Let X_n be as defined above, then the random variable $v = \frac{1}{N}\sum_{k=1}^{N} X_n$ is normally distributed.

Lemma 2. Let $X = r sin\theta$ be a random variable with $r \in [0,1]$, $\theta \in [0, 2\pi]$ as random numbers. If r and θ are independent and if θ is uniformly distributed, then the expectation value of X is zero.

Proof. Let $r_n = n\Delta r$, $\theta_m = m\Delta\theta$, $X_{n,m} = r_n Sin\theta_m$ with $\Delta r = \frac{1}{N}$ and $\Delta\theta = \frac{2\pi}{N}$ and let $A_{n,m} = \{(r,\theta)|r_n < r \le r_n + \Delta r, \theta_m \le \theta < \theta_m + \Delta\theta\}$. Due to the independence of r and θ, we have $p((r,\theta) \in A_{n,m}) = p(r_n < r \le r_n + \Delta r) \times p(\theta_m \le \theta < \theta_m + \Delta\theta) = \phi(r_n)\Delta r \times \frac{\Delta\theta}{2\pi}$, where $\phi(r)$ is the probability density function for the random variable r. Thus, we have $E[X] = \lim_{N\to\infty}\sum_{n=1}^{N}\sum_{m=1}^{N} P((r,\theta) \in A_{n,m})X_{n,m} = \lim_{N\to\infty}\sum_{n=1}^{N}\sum_{m=1}^{N}\phi(r_n)r_n Sin\theta_m \frac{\Delta r\Delta\theta}{2\pi}$
$= \frac{1}{2\pi}\int_0^1 r\phi(r)\{\int_0^{2\pi} Sin\theta d\theta\}dr = 0$. Q.E.D.

Lemma 3. Let a_k and b_k are random numbers in $[0,1]$, $[0,2\pi]$ respectively. If $b_k' s$ are uniformly distributed, then for any $\epsilon > 0$, we have $\lim_{N\to\infty} p(|\frac{1}{N}\sum_{k=1}^{N} a_k Sin(b_k)| > \epsilon) = 0$.

Proof. Let $X = aSin(b)$ and $X_k = a_k Sin(b_k)$ for $k = 1, 2, \cdots, N$. Then by Lemma 2, we have $E(X) = 0$ and hence using the weak law of large numbers, we have $p(|\frac{1}{N}\sum_{k=1}^{N} X_k| > \epsilon) = 0$. Q.E.D.

Lemma 4. Let f be an integer and $t_k = kh$ with $h = \frac{1}{N}$. Then we have $\sum_{k=0}^{N} Sin(2\pi f t_k + b) = Sin(b)$ and $\sum_{k=0}^{N} Cos(2\pi f t_k + b) = Cos(b)$.

Proof. Let $z_0 = e^{ib}, w = e^{i2\pi f h}$, and $z_k = e^{i(2\pi f t_k + b)}$. Then we have $z_k = z_0 w^k$ and hence $\sum_{k=0}^{N} Sin(2\pi f t_k + b) = Im(\sum_{k=0}^{N} z_k) = Im(z_0\sum_{k=0}^{N} w^k) = Im(z_0\frac{1-w^{N+1}}{1-w})$. Now, note that $w^N = e^{i2\pi f h N} = e^{i2\pi f} = 1$ since $hN = 1$ and f is an integer. Hence the sum becomes $Im(z_0) = Sin(b)$. A similar proof the cosine part is omitted. Q.E.D.

Corollary 1. Let $N = 2^n$ for some positive integer $n \geq 3$ and let $h = \frac{1}{N}$, $t_k = kh$. If f is an integer in $[1, 2^{n-1})$, then we have $\sum_{k=1}^{N} Cos(4\pi f t_k) = 0$ and $\sum_{k=1}^{N} Sin(4\pi f t_k) = 0$.

Proof. By Lemma 4, we have $\sum_{k=0}^{N} Cos(4\pi f t_k) = 1$ and $\sum_{k=0}^{N} Sin(4\pi f t_k) = 0$ with b=0. Now, by moving the first terms of the sums in the left hand side to the right, we obtain the results.

Theorem 1. Let $\Gamma(t)$ be as defined above and let $t_k = \frac{k}{N}$. Then for any $\epsilon > 0$, we have

$$\lim_{N \to \infty} p(|\frac{1}{N} \sum_{k=0}^{N} \Gamma(t_k)| > \epsilon) = 0$$

and hence $E[\Gamma(t)] = 0$.

Proof. Note that $\sum_{k=0}^{N} \Gamma(t_k) = \sum_{k=0}^{N} \{\sum_{j=1}^{M} a_j Sin(2\pi f_j t_k + b_j)\} = \sum_{j=1}^{M} a_j \{\sum_{k=0}^{N} Sin(2\pi f_j t_k + b_j)\}$. By Lemma 4, we have $\sum_{k=0}^{N} Sin(2\pi f_j t_k + b) = Sin(b)$ and hence $\sum_{k=0}^{N} \Gamma(t_k) = \sum_{j=1}^{M} a_j Sin(b_j)$. Now, we apply Lemma 3 to obtain $\lim_{N \to \infty} p(|\frac{1}{N} \sum_{k=0}^{N} \Gamma(t_k)| > \epsilon) = 0$ and hence we get $E[\Gamma(t)] = 0$. Q.E.D.

Fig.1 shows an example random signal $\Gamma(t) = \sum_{j=1}^{2^{14}} Sin(2\pi f_j t + b_j)$ sampled at a rate of 2^{15} per second. One can see from the figure that the signal is oscillating around 0, i.e. $E(\Gamma(t)) \simeq 0$. A direct calculation shows in fact that $E(\Gamma(t)) = 0.9 \times 10^{-5}$. Next, we consider the relation $E(\Gamma(t)\Gamma(t')) = \delta(t - t')$, i.e. the correlation coefficient between v_N and w_N, where v_N and w_N are samples of size N taken at different times. Recall that the temperature is constant so that the squared length of the vectors v_N and w_N must be the same and hence if the correlation coefficient approaches zero then so will the cross product sum.

Theorem 2. Let $a_j \in [0, 1)$, $b_j \in [0, 2\pi)$ be random numbers, and $f_j = f_L + j$ be for some nonnegative integer f_L and $j = 1, 2, ..., M$. Assume that ${a_j}^2$'s and b_j's are uniformly distributed and let $\Gamma(t) = \frac{c}{\sqrt{M}} \sum_{j=1}^{M} a_j Sin(2\pi f_j t + b_j)$. If $t' = t + \tau$ for some $\tau > 0$, then we have

$$\lim_{M \to \infty} [\lim_{N \to \infty} p(\frac{1}{N} \sum_{k=1}^{N} \Gamma(t_k)\Gamma(t_k + \tau) > \epsilon)] = 0$$

and hence we have $E(\Gamma(t)\Gamma(t')) = 0$ whenever $t \neq t'$.

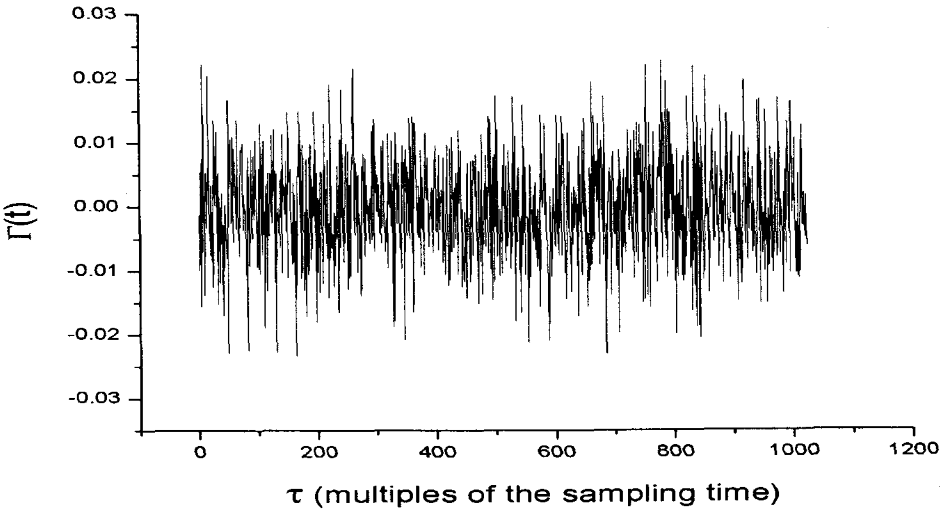

FIGURE 1. Sample Random Signal ($N = 2^{15}$)

Proof. Let $c_j = 2\pi f_j \tau + b_j$, then we have $\Gamma(t+\tau) = \frac{c}{\sqrt{M}} \sum_{j=1}^{N} a_j Sin(2\pi f_j t + c_j)$. By exchanging the order of sums, we have $\frac{1}{N}\sum_{k=1}^{N} \Gamma(t_k)\Gamma(t_k+\tau) = \frac{c^2}{M}\sum_{i,j=1}^{M} a_i a_j \{\frac{1}{N}\sum_{k=1}^{N} Sin(2\pi f_i t_k + b_i) Sin(2\pi f_j t_k + c_j)\}$. The product of the two sine functions can be expanded to obtain four terms; $(1) Cosb_i Cosc_j \frac{1}{N}\sum_{k=1}^{N} Sin(2\pi f_i t_k) Sin(2\pi f_j t_k)$,
(2) $Cosb_i Sinc_j \frac{1}{N}\sum_{k=1}^{N} Sin(2\pi f_i t_k) Cos(2\pi f_j t_k)$,
(3) $Sinb_i Cosc_j \frac{1}{N}\sum_{k=1}^{N} Cos(2\pi f_i t_k) Sin(2\pi f_j t_k)$,
(4) $Sinb_i Sinc_j \frac{1}{N}\sum_{k=1}^{N} Cos(2\pi f_i t_k) Cos(2\pi f_j t_k)$.
Note that as $N \to \infty$ the sum in (1) approaches $\int_0^1 \{Cos(2\pi(f_i - f_j)t) - Cos(2\pi(f_i + f_j)t)\}dt$ which is zero unless $i = j$. The same holds for all of the other three sums. Thus, the first term becomes
$\sum_{i=1}^{M} a_i^2 Cosb_i Cosc_i \frac{1}{N}\sum_{k=1}^{N} Sin^2(2\pi f_i t_k) = \sum_{i=1}^{M} a_i^2 Cosb_i Cosc_i \frac{1}{2N}\sum_{k=1}^{N}(1 - Cos(4\pi f_i t_k)$.
Now, the sum of the Cosine terms become zero by Corollary 1, and hence we have $\frac{1}{2}\sum_{i=1}^{M} a_i^2 Cosb_i Cosc_i$. Applying the same method, one can easily check that (2) and (3) become zero, and the fourth term with (4) becomes $\frac{1}{2}\sum_{i=1}^{M} a_i^2 Sinb_i Sinc_i$. Therefore, by adding the two nonzero terms and using the relation $c_i = b_i + 2\pi f_i \tau$, we obtain $\frac{c^2}{2M}\sum_{i=1}^{M} a_i^2 (Cosb_i Cosc_i + Sinb_i Sinc_i) = \frac{c^2}{2M}\sum_{i=1}^{M} a_i^2 Cos(2\pi f_i \tau)$. Finally, note that the set of points on the unit circle corresponding to $\{2\pi f_j \tau | j = 1, 2, ...M\}$ are uniformly distributed as M approaches to infinity. Hence, one can apply a similar argument as in Lemma 3 to prove that $lim_{M\to\infty} p(|\frac{1}{M}\sum_{i=1}^{M} a_i^2 Cos(2\pi f_i \tau)| > \epsilon) = 0$. Q.E.D.

To verify that the above theorem is correct, we performed various calculations with different values of N and M. Table 1 shows a summary of the results for the cases with $N = 2^{16}$ and with different values of M. In Table 1, the means and the standard deviations are for the 1,024 values of $\Gamma(t)\Gamma(t+\tau_j)$ with $\tau_j = j \times 2^{-16}$, j=1,2,...1,024. Table 2 shows how the correlation changes as N increases. Fig. 2 shows the computed values of $E(\Gamma(t)\Gamma(t+\tau))$ for different values of τ when $N = 2^{16}$. Two different values of M, i.e. $M = 2^{15}$ and $M = 2^{12}$ are used for the two curves in fig.2. One can see that $E(\Gamma(t)\Gamma(t+\tau))$ decreases to zero sharply as M increases.

Table 1. Sample Calculations of $\Gamma(t)\Gamma(t+\tau)$ - 1

N	M	Average	Standard Deviation
2^{16}	2^{16}	-3.625×10^{-6}	3.847×10^{-3}
2^{16}	2^{15}	-1.138×10^{-6}	4.510×10^{-3}
2^{16}	2^{14}	5.970×10^{-5}	2.430×10^{-2}
2^{16}	2^{13}	4.972×10^{-4}	4.313×10^{-2}
2^{16}	2^{12}	1.124×10^{-3}	7.492×10^{-2}

Table 2. Sample Calculations of $\Gamma(t)\Gamma(t+\tau)$ - 2

N	M	Average	Standard Deviation
2^{16}	2^{15}	-1.138×10^{-6}	4.510×10^{-3}
2^{15}	2^{14}	9.272×10^{-6}	6.564×10^{-3}
2^{14}	2^{13}	-9.710×10^{-5}	9.084×10^{-3}
2^{13}	2^{12}	-1.426×10^{-4}	1.336×10^{-2}
2^{12}	2^{11}	-6.282×10^{-4}	1.836×10^{-2}

Corollary 2. Let $a_j \in [0,1)$, $b_j \in [0,2\pi)$ be random numbers, and $f_j = f_L + j$ be integers for $j = 1, 2, ..., M$. Assume that ${a_j}^2$ and b_j are uniformly distributed random variables on the specified intervals. If $\Gamma(t) = c\sum_{j=1}^{M} a_j Cos(2\pi f_j t + b_j)$, then $\Gamma(t)$ generates a Markov process approximately for some constant c.

Proof. This follows from Theorem 1 and Theorem 2.

3. CONCLUSION

We gave an explicit proof that the random signals used in the double randomization method satisfy the properties of the Gaussian white noise. The random signals we used are low pass filtered so that they agree with the cases of actual measurements

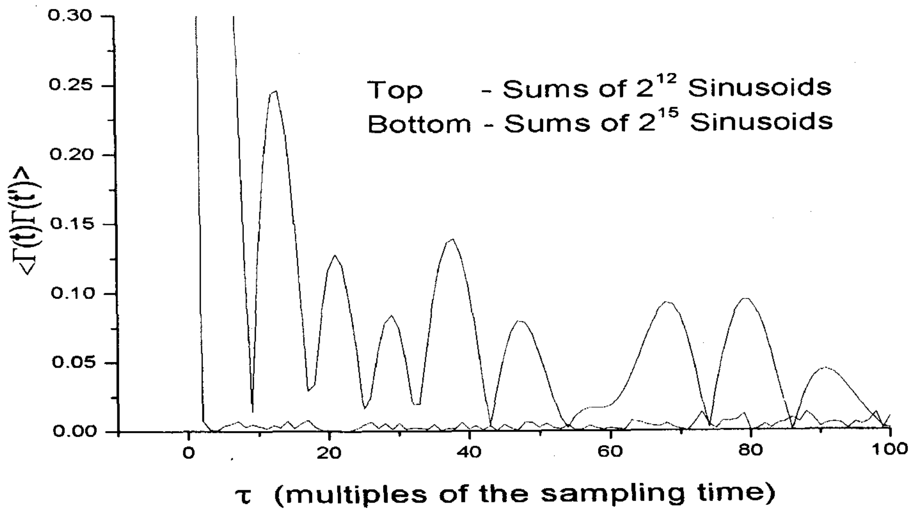

FIGURE 2. Sample Random Signal ($N = 2^{15}$)

where the low frequencies are filtered to reduce the channel noises. According to our proof, however, if the high frequency components are filtered, then the correlation of the signal starting from two different times may become different from zero. Since the random signals we studied satisfy the Gaussian white noise properties approximately, they can now be used to study the statistical properties of the Johnson noise for the Johnson noise thermometry.

REFERENCES

[1] Karl K. Sabelfeld, Monte Carlo methods in boundary value problems, Springer Series in Computational Physics, Springer-Verlag, Berlin, 1999.

[2] Peter R. Kramer, *A review of some Monte Carlo simulation methods for turbulent systems,* Monte Carlo Methods and Applications 7(3-4) (2001) 229-244.

[3] Daniel T. Gilespie, *The mathematics of Brownian motion and Johnson noise,* Am. J. Phys. 64(3) (1996) 225-240.

[4] Herman A. Tasman, Joseph Richter, *Unconventional methods for measuring high temperatures,* High Temperatures - High Pressure 11 (1979) 87-101.

[5] Byung Soo Moon, *A Gaussian white noise generator and its application to the fluctuation-Dissipation formula*J. Applied Math. & Computing 15(1-2) (2004) 363-375.

[6] Lorenz Bartosch, *Generation of Colored Noise* Int. J. Modern Phys. C, 12(6) (2001) 851-855.

[7] C. Jacobi and L. Reggiani, *The Monte Carlo method for the solution of charge transport in semiconductors with application to covalent materials,* Review of Modern Physics 55(3) (1983) 645-705.

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