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EULER SUMS EVALUATABLE FROM INTEGRALS

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Abstract. Ever since the time of Euler, the so-called Euler sums have been evaluated in many different ways. We give here a proof of the classical Euler sum by following Lewin's method. We also consider some related formulas involving Euler sums, which are evaluatable from some known definite integrals.

1. Introduction and preliminaries

The *Riemann Zeta function* $\zeta(s)$ is defined by

(1.1)
$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ \displaystyle\left(1-2^{1-s}\right)^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0;\ s \neq 1), \end{cases}$$

which can, except for a simple pole at $s = 1$ with its residue 1, be continued meromorphically to the whole complex s-plane. The *Hurwitz* (or generalized) *Zeta function* $\zeta(s, a)$ defined by

(1.2)
$$\zeta(s,a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}$$
$$(\Re(s) > 1;\ a \notin \mathbb{Z}_0^- := \{0, -1, -2, \dots\}),$$

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can, just as $\zeta(s)$, be continued meromorphically to the whole complex s-plane except for a simple pole at $s = 1$ with its residue 1 (see, for details, [23] and [24]). Clearly, we have

$$\zeta(s,1) = \zeta(s) = (2^s - 1)^{-1}\,\zeta\left(s, \tfrac{1}{2}\right) \quad \text{and} \quad \zeta(s,2) = \zeta(s) - 1. \tag{1.3}$$

The following identity was discovered by Euler in 1775 and has a long history (see, for example, [4, p. 252 et seq.]):

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n}{n^2} = \zeta(3), \tag{1.4}$$

where $H_n := H_n^{(1)}$ denotes the harmonic numbers and $H_n^{(s)}$ denotes the generalized harmonic numbers defined by

$$H_n^{(s)} := \sum_{k=1}^{n} \frac{1}{k^s} \qquad (n \in \mathbb{N} := \{1,\, 2,\, 3,\, \ldots\};\ s \in \mathbb{C}), \tag{1.5}$$

so that, obviously, we have

$$H_n^{(s)} = \zeta(s) - \zeta(s, n+1) \quad (\Re(s) > 1;\ n \in \mathbb{N}). \tag{1.6}$$

The identity (1.4) is a special case of the following more general sum due to Euler:

$$2\sum_{k=1}^{\infty} \frac{H_k}{k^n} = (n+2)\,\zeta(n+1) - \sum_{k=1}^{n-2} \zeta(n-k)\,\zeta(k+1) \quad (n \in \mathbb{N}\setminus\{1\}), \tag{1.7}$$

or, equivalently,

$$2\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^n} = n\,\zeta(n+1) - \sum_{k=1}^{n-2} \zeta(n-k)\,\zeta(k+1) \quad (n \in \mathbb{N}\setminus\{1\}), \tag{1.8}$$

where (and in what follows) an empty sum is understood to be nil.

Many different techniques have been used, in the vast mathematical literature, in order to evaluate harmonic sums of the types (1.4) and (1.8). For example, Borwein and Borwein [5] established the following interesting sums by applying Parseval's identity to a Fourier series and contour integrals to a generating function:

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n+1}\right)^2 = \frac{11}{4}\,\zeta(4), \tag{1.9}$$

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 = \frac{17}{4}\,\zeta(4), \tag{1.10}$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4}\,\zeta(4). \tag{1.11}$$

We give here a proof of the Euler sum (1.8) by following Lewin's method and consider some formulas related to Euler sums.

For the sake of ready reference in our present investigation, we recall here the definitions of the Polylogarithm functions $\mathrm{Li}_n(z)$ $(n \in \mathbb{N})$ and the Polygamma functions $\psi^{(n)}(z)$ $(n \in \mathbb{N})$ as follows:

$$\begin{aligned} \mathrm{Li}_n(z) : &= \sum_{k=1}^{\infty} \frac{z^k}{k^n} \quad (|z| \leq 1;\ n \in \mathbb{N} \setminus \{1\}) \\ &= \int_0^z \mathrm{Li}_{n-1}(t)\,\frac{dt}{t} \quad (n \in \mathbb{N} \setminus \{1,\, 2\}) \end{aligned} \tag{1.12}$$

and

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z) = \frac{d^n}{dz^n}\psi(z) \quad \left(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right), \tag{1.13}$$

where $\psi^{(0)}(z) := \psi(z)$ denotes the Psi (or Digamma) function defined by

$$\psi(z) := \frac{d}{dz}\,\log\,\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log\,\Gamma(z) = \int_1^z \psi(t)\,dt \tag{1.14}$$

in terms of the classical Gamma function. Furthermore, in terms of the Hurwitz Zeta function $\zeta(s, a)$ defined by (1.2), we have the relationship:

$$\psi^{(n)}(z) = (-1)^{n+1}\,n!\,\zeta(n+1, z) \quad (n \in \mathbb{N}). \tag{1.15}$$

2. Certain Euler sums evaluatable from definite integrals

Setting $x = 1$ in equation (7.65) in Lewin [13, p. 204] gives

$$\int_0^1 \log t \,\{\log(1-t)\}^2 \,\frac{dt}{t} = -\frac{\zeta(4)}{2}. \tag{2.1}$$

By using the Maclaurin series expansion of $\log(1-t)$, we obtain

$$\{\log(1-t)\}^2 = \sum_{k=2}^{\infty} \frac{2}{k} \left(\sum_{j=1}^{k-1} \frac{1}{j} \right) t^k \quad (-1 \leq t < 1). \tag{2.2}$$

We recall a known integral formula:

$$\int t^n \log(at)\, dt = t^{n+1} \left[\frac{\log(at)}{n+1} - \frac{1}{(n+1)^2} \right] \quad (n > 0), \tag{2.3}$$

which, upon setting $a = 1$ and $n = k - 1$, yields

$$\int t^{k-1} \log t \, dt = t^k \left[\frac{\log t}{k} - \frac{1}{k^2} \right] \quad (k > 1) \tag{2.4}$$

and

$$\int_0^1 t^{k-1} \log t \, dt = -\frac{1}{k^2} \quad (k \geq 1). \tag{2.5}$$

It follows from (2.2) and (2.5) that

$$\begin{aligned} \int_0^1 \log t \,\{\log(1-t)\}^2 \,\frac{dt}{t} &= 2 \sum_{k=2}^{\infty} \frac{1}{k} \left(\sum_{j=1}^{k-1} \frac{1}{j} \right) \int_0^1 t^{k-1} \log t \, dt \\ &= -\sum_{k=2}^{\infty} \frac{1}{k^3} \left(\sum_{j=1}^{k-1} \frac{1}{j} \right), \end{aligned}$$

which, in view of (2.1), gives a known Euler sum:

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^3} = \frac{1}{4}\, \zeta(4), \tag{2.6}$$

which is easily seen to be equivalent to (1.11).

Recall an integral formula (see [13, p. 310, equation (7)]):

$$\begin{aligned}(2.7)\qquad &\int_0^x \frac{\{\log(1+t)\}^2}{t}\,dt \\ &= \log x\,\{\log(1+x)\}^2 - \frac{2}{3}\,\{\log(1+x)\}^3 \\ &\quad - 2\,\log(1+x)\,\mathrm{Li}_2\left(\frac{1}{1+x}\right) - 2\,\mathrm{Li}_3\left(\frac{1}{1+x}\right) + 2\,\mathrm{Li}_3(1).\end{aligned}$$

By applying (2.2) to (2.7), we obtain
(2.8)

$$\int_0^x \frac{\{\log(1+t)\}^2}{t}\,dt = 2\sum_{k=2}^{\infty} \frac{(-1)^k}{k^2}\left(\sum_{j=1}^{k-1}\frac{1}{j}\right) x^k \quad (-1 < x \leq 1).$$

Setting $x = 1$ in (2.7) and (2.8), and considering the following known identities:

$$(2.9)\qquad \mathrm{Li}_2\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\,\zeta(2) - \tfrac{1}{2}\,(\log 2)^2$$

and

$$(2.10)\qquad \mathrm{Li}_3\left(\tfrac{1}{2}\right) = \tfrac{7}{8}\,\zeta(3) - \tfrac{1}{2}\,\zeta(2)\,\log 2 + \tfrac{1}{6}\,(\log 2)^3,$$

we get

$$(2.11)\qquad \int_0^1 \frac{\{\log(1+t)\}^2}{t}\,dt = \tfrac{1}{4}\,\zeta(3)$$

and

$$(2.12)\qquad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\,H_k}{(k+1)^2} = \tfrac{1}{8}\,\zeta(3).$$

Recall another integral formula similar to (2.7) (see [13, p. 310, equation (6)]):

$$\begin{aligned}(2.13)\qquad &\int_0^x \frac{\{\log(1-t)\}^2}{t}\,dt \\ &= \log x\,\{\log(1-x)\}^2 \\ &\quad + 2\,\log(1-x)\,\mathrm{Li}_2(1-x) - 2\,\mathrm{Li}_3(1-x) + 2\,\mathrm{Li}_3(1).\end{aligned}$$

Employing (2.13), similarly as in getting (2.12) and (2.13), we have the formula (1.4) and

$$(2.14)\qquad \sum_{k=1}^{\infty} \frac{H_k}{2^k\,(k+1)^2} = \tfrac{1}{4}\,\zeta(3) - \tfrac{1}{3}\,(\log 2)^3.$$

3. Further analysis of Lewin's method

We begin by recalling the Eulerian first integral (see [13, 7.9.5]):
(3.1)
$$\int_0^1 (1-t)^{\lambda}\, t^{\mu-1}\, dt = \frac{\Gamma(1+\lambda)\,\Gamma(1+\mu)}{\mu\,\Gamma(1+\lambda+\mu)} \quad (\Re(\lambda) > -1;\ \Re(\mu) > 0).$$

We obtain, by multiple differentiation of (3.1) with respect to λ and μ,
(3.2)
$$\int_0^1 \{\log(1-t)\}^n\, (\log t)^m\, \frac{dt}{t} = D_\lambda^n\, D_\mu^m \left[\frac{\Gamma(1+\lambda)\,\Gamma(1+\mu)}{\mu\,\Gamma(1+\lambda+\mu)}\right]\Bigg|_{\lambda=\mu=0}.$$

If we put $y = \log\Gamma(1+\lambda) + \log\Gamma(1+\mu) - \log\Gamma(1+\lambda+\mu)$, with the aid of (1.15), we have

$$D_\mu^q\, D_\lambda^p\, y = (-1)^{p+q-1}\,(p+q-1)!\,\zeta(p+q, 1+\lambda+\mu)$$

and so

$$D_\lambda^p\, D_\mu^q\, y\big|_{\lambda=\mu=0}\,(-1)^{p+q-1}\,(p+q-1)!\,\zeta(p+q)$$

and

$$y = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \frac{D_\lambda^p\, D_\mu^q\, y\big|_{\lambda=\mu=0}}{p!\,q!}\,\lambda^p\,\mu^q.$$

We thus have

(3.3)
$$\begin{aligned}&\int_0^1 \{\log(1-t)\}^n\,(\log t)^m\,\frac{dt}{t}\\ &= D_\mu^m\, D_\lambda^n\left[\frac{1}{\mu}\exp\left(-\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\lambda^p\,\mu^q\,A_{p,q}\right)\right]\Bigg|_{\lambda=\mu=0},\end{aligned}$$

where, for convenience,

$$A_{p,q} := (-1)^{p+q}\,\frac{(p+q-1)!}{p!\,q!}\,\zeta(p+q).$$

If we put

$$\sum_{q=1}^{\infty} \mu^q\, A_{p,q} := B_p,$$

then (3.3) can be rewritten in the following form:

$$\int_0^1 \{\log(1-t)\}^n (\log t)^m \frac{dt}{t} \tag{3.4}$$
$$= D_\mu^m D_\lambda^n \left[\frac{1}{\mu} \exp\left(-\sum_{p=1}^{\infty} \lambda^p B_p\right)\right]\Bigg|_{\lambda=\mu=0}.$$

In particular, the special case of (3.4) when $n = 2$ yields

$$\int_0^1 (\log t)^m \{\log(1-t)\}^2 \frac{dt}{t}$$
$$= D_\mu^m D_\lambda^2 \left[\frac{1}{\mu} \exp\left(-\sum_{p=1}^{\infty} \lambda^p B_p\right)\right]\Bigg|_{\lambda=\mu=0} \tag{3.5}$$
$$= D_\mu^m \left[\frac{1}{\mu} \left\{(B_1)^2 - 2\,B_2\right\}\right]\Bigg|_{\mu=0}.$$

Note that

$$(B_1)^2 = \left(\sum_{q=1}^{\infty} A_{1,q}\,\mu^q\right)^2 = \sum_{q=2}^{\infty} C_q\,\mu^q,$$

where

$$C_q := \sum_{j=1}^{q-1} A_{1,j}\,A_{1,q-j}.$$

We find that

$$\frac{1}{\mu}\left\{(B_1)^2 - 2\,B_2\right\} = -2\,A_{2,1} + \sum_{q=1}^{\infty} \left(C_{q+1} - 2\,A_{2,q+1}\right) \mu^q$$

and

$$D_\mu^m \left[\frac{1}{\mu} \left\{(B_1)^2 - 2\,B_2\right\}\right]\Bigg|_{\mu=0} = m!\,\left(C_{m+1} - 2\,A_{2,m+1}\right) \quad (m \in \mathbb{N}_0).$$

We thus have

$$\int_0^1 (\log t)^m \{\log(1-t)\}^2 \frac{dt}{t}$$
$$= m!\,(-1)^m \left[(m+2)\,\zeta(m+3) - \sum_{j=1}^{m} \zeta(1+j)\,\zeta(m+2-j)\right],$$

which, upon considering (2.2) and

$$\int_0^1 t^{k-1}\,(\log t)^m\,dt = \frac{(-1)^m\,m!}{k^{m+1}}, \tag{3.6}$$

yields an evaluation of the following Euler sum:

$$\begin{aligned}&\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^{m+2}} \\ &= \tfrac{m+2}{2}\,\zeta(m+3) - \tfrac{1}{2}\sum_{j=1}^{m} \zeta(1+j)\,\zeta(m+2-j) \quad (m \in \mathbb{N}_0),\end{aligned} \tag{3.7}$$

which is easily seen to be equivalent to the classical Euler sum (1.8).

It is also seen that

$$\begin{aligned}&D_\lambda^n \left[\exp\left(-\sum_{p=1}^{n} \lambda^p\,B_p\right)\right]\Bigg|_{\lambda=0} \\ &= n! \sum_{\substack{k_1,\,k_2,\,\ldots,\,k_n \in \mathbb{N}_0 \\ k_1+2k_2+\cdots+nk_n=n}} (-1)^{k_1+k_2+\cdots+k_n}\,\frac{{B_1}^{k_1}\,{B_2}^{k_2}\,\cdots\,{B_n}^{k_n}}{k_1!\,k_2!\,\cdots\,k_n!},\end{aligned} \tag{3.8}$$

which, for $n=2$, immediately yields

$$D_\lambda^2 \left[\exp\left(-\sum_{p=1}^{2} \lambda^p\,B_p\right)\right]\Bigg|_{\lambda=0} = {B_1}^2 - 2\,B_2. \tag{3.9}$$

Observe

$$\begin{aligned}{B_i}^{k_i} &= \left(\sum_{q=1}^{m+1} A_{i,q}\,\mu^q\right)^{k_i} + \text{terms of degrees} \geq m+2 \text{ in } \mu \\ &= \sum_{j=0}^{(m+1)\,k_i} E_j^{(i)}\,\mu^j + \text{terms of degrees} \geq m+2 \text{ in } \mu,\end{aligned}$$

where, for $i = 1, 2, \ldots, n$,

$$E_j^{(i)} := \sum_{\substack{\ell_1,\,\ell_2,\,\ldots,\,\ell_{m+1} \in \mathbb{N}_0 \\ \ell_1+\ell_2+\cdots+\ell_{m+1}=k_i \\ \ell_1+2\,\ell_2+\cdots+(m+1)\,\ell_{m+1}=j}} \frac{k_i!\,{A_{i,1}}^{\ell_1}\,{A_{i,2}}^{\ell_2}\,\cdots\,{A_{i,m+1}}^{\ell_{m+1}}}{\ell_1!\,\ell_2!\,\cdots\,\ell_{m+1}!}. \tag{3.10}$$

We find that the term in μ^{m+1} of

$$\prod_{i=1}^{n} B_i^{\,k_i} = \prod_{i=1}^{n} \left(\sum_{j=0}^{(m+1)k_i} E_j^{(i)} \mu^j \right)$$

is

$$\left(\sum_{\substack{j_1, j_2, \ldots, j_n \in \mathbb{N}_0 \\ j_1+j_2+\cdots+j_n=m+1}} E_{j_1}^{(1)} E_{j_2}^{(2)} \cdots E_{j_n}^{(n)} \right) \mu^{m+1}$$

and so

$$D_\mu^m \left[\frac{1}{\mu} B_1^{\,k_1} B_2^{\,k_2} \cdots B_n^{\,k_n} \right] \Bigg|_{\mu=0} = m! \sum_{\substack{j_1, j_2, \ldots, j_n \in \mathbb{N}_0 \\ j_1+j_2+\cdots+j_n=m+1}} E_{j_1}^{(1)} E_{j_2}^{(2)} \cdots E_{j_n}^{(n)}. \tag{3.11}$$

It follows from (3.4), (3.8), and (3.11) that

$$\begin{aligned} &\int_0^1 \{\log(1-t)\}^n (\log t)^m \frac{dt}{t} \\ &= n!\, m! \sum_{\substack{k_1, k_2, \ldots, k_n \in \mathbb{N}_0 \\ k_1+2k_2+\cdots+nk_n=n}} \frac{(-1)^{k_1+k_2+\cdots+k_n}}{k_1!\, k_2! \cdots k_n!} \\ &\quad \times \left(\sum_{\substack{j_1, j_2, \ldots, j_n \in \mathbb{N}_0 \\ j_1+j_2+\cdots+j_n=m+1}} E_{j_1}^{(1)} E_{j_2}^{(2)} \cdots E_{j_n}^{(n)} \right), \end{aligned} \tag{3.12}$$

where $E_{j_i}^{(i)}$ $(i = 1, 2, \ldots, n)$ are given as in (3.10).

It is remarked that the integral in (3.12) is evaluated in terms of $\zeta(s)$ and can be changed in the following form:

$$\int_0^{\frac{\pi}{2}} (\log \sin \theta)^m (\log \cos \theta)^n \cot \theta \, d\theta,$$

by substituting $t = \sin^2 \theta$, as commented in Lewin [13, p. 222].

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