

ISHIKAWA ITERATIVE PROCESS WITH ERRORS FOR STRONGLY PSEUDO-CONTRACTIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, we study the Ishikawa iterative sequences with errors for Lipschitzian strongly pseudo-contractive operator in arbitrary real Banach spaces. Our results improve the results obtained previously by C. E. Chidume [3] and Zeng [9].

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and E^* be the dual space of E . We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x, f \rangle = \|x\| \cdot \|f\|, \|f\| = \|x\|\}.$$

Where $\langle x, f \rangle$ denotes the value of the continuous linear function $f \in E^*$ at $x \in E$. It is well known that if E^* is strictly convex then J is single-valued and if X^* is uniformly convex then J is uniformly continuous on bounded subsets of X .

An operator $T : D(T) \subset E \rightarrow E$ is said to be accretive, if the inequality

$$(1) \quad \|x - y\| \leq \|x - y + s(Tx - Ty)\|$$

holds for every $x, y \in D(T)$ and for all $s > 0$.

The accretive operators were introduced independently by Browder [1] and Kato [5] in 1967. An early fundamental result in the theory of

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accretive operators, due to Browder, states that the initial value problem

$$\begin{aligned}\frac{du}{dt} + Au &= 0 \\ u(0) &= u_0\end{aligned}$$

is solvable if A is locally Lipschitzian and accretive.

Closely related to the class of accretive maps is the class of pseudo-contractive operators. An operator T with domain $D(T)$ and range $R(T)$ in E is said to be a strong pseudo-contraction, if there exists $t > 1$ such that for all $x, y \in D(T)$ and $r > 0$, the following inequality holds:

$$(2) \quad \|x - y\| \leq \|(1+r)(x-y) - rt(Tx - Ty)\|.$$

If $t = 1$ in inequality (2) then T is called pseudo-contractive.

As a consequence of result of Kato [5], T is pseudo-contractive if and only if for each $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$(3) \quad \langle (I-T)x - (I-T)y, j(x-y) \rangle \geq 0.$$

Furthermore, T is strongly pseudo-contractive if and only if there exists $k > 0$ such that

$$(4) \quad \langle (I-T)x - (I-T)y, j(x-y) \rangle \geq k\|x-y\|^2.$$

Chidume [3] proved that if E is a real uniformly smooth Banach space, K is a nonempty closed convex bounded subset of E and $T : K \rightarrow K$ is a strongly pseudo-contraction, with a fixed point x^* in K , then both the Mann and the Ishikawa iteration schemes converge strongly to x^* , for an arbitrary initial point $x_0 \in K$. Zeng [9] and Li [6] consider an iterative process for Lipschitzian strongly pseudo-contractive operator in arbitrary real Banach spaces. In [3], Chidume proved the following theorem:

THEOREM 1.1. [3] *Suppose E is a real uniformly smooth Banach space and K is a bounded closed convex and nonempty subset of E . Suppose $T : E \rightarrow E$ is a strongly pseudo-contractive map such that $Tx^* = x^*$ for some $x^* \in K$. Let $\{\alpha_n\}, \{\beta_n\}$ be real satisfying the following conditions:*

- (i) $0 \leq \alpha_n, \beta_n \leq 1$ for all $n \geq 0$;
- (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$; $\lim_{n \rightarrow \infty} \beta_n = 0$;
- (iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then, for arbitrary $x_0 \in K$, the sequence $\{x_n\}$ defined iteratively by

$$(5) \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n,$$

$$(6) \quad y_n = (1 - \beta_n)x_n + \beta_n T x_n, n \geq 0,$$

converges strongly to x^* . Moreover, x^* is unique.

Our objective in this paper is to consider an iterative sequences with errors for Lipschitzian strongly pseudo-contractive operator in arbitrary real Banach spaces. Our results improve and extend the results of Chidume [3] and Zeng [9].

In order to prove the main results of the paper, we need the following lemma.

LEMMA 1.1. [7] *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be nonnegative sequence satisfying*

$$(7) \quad a_{n+1} \leq (1 - t_n)a_n + b_n + c_n.$$

With $\{t_n : n = 0, 1, 2, \dots\} \subset [0, 1], \sum_{n=1}^\infty t_n = \infty, b_n = o(t_n)$ and $\sum_{n=1}^\infty c_n < \infty$ then $\lim_{n \rightarrow \infty} a_n = 0$.

2.Main results

Now, we state and prove the following theorems.

THEOREM 2.1. *Suppose E is an arbitrary real Banach space and $T : E \rightarrow E$ be a Lipschitzian strongly pseudo-contractive map such that $Tx^* = x^*$ for some $x^* \in E$. Suppose $\{u_n\}, \{v_n\}$ be sequences in E and $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0, 1]$ such that*

- (1) $\sum_{n=1}^\infty \|u_n\| < \infty, \sum_{n=1}^\infty \|v_n\| < \infty$;
- (2) $\sum_{n=1}^\infty \alpha_n = \infty$;
- (3) $\sup \beta_n < \frac{\lambda k}{2L(L+1)}, \sup \alpha_n < \frac{(1-\lambda)k}{2(L^3+3L^2+2)}$ for some $\lambda \in (0, 1)$.

Then for any $x_0 \in E$, the Ishikawa iteration sequence $\{x_n\}$ with errors defined by

$$(8) \quad y_n = (1 - \beta_n)x_n + \beta_nTx_n + v_n$$

$$(9) \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n + u_n$$

converges strongly to x^* . Moreover, x^* is unique.

PROOF. Since $T : E \rightarrow E$ is strongly pseudo-contractive, we have $(I - T)$ is strongly accretive, so for any $x, y \in E$.

$$\langle (I - T)x - (I - T)y, j(x - y) \rangle \geq k\|x - y\|^2$$

where $k = (t - 1)/t$ and $t \in (1, \infty)$.

Thus

$$\langle ((1 - k)I - T)x - ((1 - k)I - T)y, j(x - y) \rangle \geq 0$$

and so it follows from Lemma 1.1 of Kato [5] that

$$\|x - y\| \leq \|x - y + r[((1 - k)I - T)x - ((1 - k)I - T)y]\|$$

for all $x, y \in E$ and $r > 0$.

From $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n + u_n$ we obtain

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - \alpha_n T y_n - u_n \\ &= (1 + \alpha_n)x_{n+1} + \alpha_n[(I - T)x_{n+1} - kx_{n+1}] \\ &\quad - (1 - k)\alpha_n x_n + (2 - k)\alpha_n^2(x_n - T y_n) \\ &\quad + \alpha_n(Tx_{n+1} - T y_n) - [(2 - k)\alpha_n + 1]u_n. \end{aligned}$$

It is easy to see that

$$x^* = (1 + \alpha_n)x^* + \alpha_n[((1 - k)I - T)x^*] - (1 - k)\alpha_n x^*$$

so that

$$\begin{aligned} x_n - x^* &= (1 + \alpha_n)(x_{n+1} - x^*) + \alpha_n[((1 - k)I - T)x_{n+1} \\ &\quad - ((1 - k)I - T)x^*] - (1 - k)\alpha_n(x_n - x^*) \\ &\quad + (2 - k)\alpha_n^2(x_n - T y_n) \\ &\quad + \alpha_n(Tx_{n+1} - T y_n) - [(2 - k)\alpha_n + 1]u_n. \end{aligned}$$

Hence

$$\begin{aligned} \|x_n - x^*\| &\geq (1 + \alpha_n)\|(x_{n+1} - x^*) + \frac{\alpha_n}{1 + \alpha_n}[((1 - k)I - T)x_{n+1} \\ &\quad - ((1 - k)I - T)x^*]\| \\ &\quad - (1 - k)\alpha_n\|x_n - x^*\| - (2 - k)\alpha_n^2\|x_n - T y_n\| \\ &\quad - \alpha_n\|Tx_{n+1} - T y_n\| - [(2 - k)\alpha_n + 1]\|u_n\| \\ &\geq (1 + \alpha_n)\|x_{n+1} - x^*\| \\ &\quad - (1 - k)\alpha_n\|x_n - x^*\| - (2 - k)\alpha_n^2\|x_n - T y_n\| \\ &\quad - \alpha_n\|Tx_{n+1} - T y_n\| - [(2 - k)\alpha_n + 1]\|u_n\| \end{aligned}$$

so that

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \left[\frac{1 + (1 - k)\alpha_n}{1 + \alpha_n} \right] \|x_n - x^*\| \\ &\quad + \alpha_n^2 \|x_n - T y_n\| \\ &\quad + \alpha_n \|Tx_{n+1} - T y_n\| + (2\alpha_n + 1) \|u_n\| \end{aligned}$$

$$\begin{aligned}
&\leq \left(1 - \frac{k\alpha_n}{1 + \alpha_n}\right)\|x_n - x^*\| \\
&\quad + \alpha_n^2\|x_n - Ty_n\| \\
&\quad + \alpha_n\|Tx_{n+1} - Ty_n\| + (2\alpha_n + 1)\|u_n\| \\
&\leq \left(1 - \frac{k\alpha_n}{2}\right)\|x_n - x^*\| \\
&\quad + \alpha_n^2\|x_n - Ty_n\| \\
&\quad + \alpha_n\|Tx_{n+1} - Ty_n\| + 3\|u_n\|.
\end{aligned}$$

Since T is Lipschitzian operator and L is Lipschitz bound, we have

$$\begin{aligned}
\|y_n - x^*\| &= \|(1 - \beta_n)(x_n - x^*) + \beta_n(Tx_n - x^*) + v_n\| \\
&\leq [1 + \beta_n(L - 1)]\|x_n - x^*\| + \|v_n\| \\
&\leq L\|x_n - x^*\| + \|v_n\| \\
\|x_n - Ty_n\| &\leq \|x_n - x^*\| + L\|y_n - x^*\| \\
&\leq (1 + L^2)\|x_n - x^*\| + L\|v_n\| \\
\|Tx_{n+1} - Ty_n\| &\leq L\|(1 - \alpha_n)(x_n - y_n) + \alpha_n(Ty_n - y_n) + u_n\| \\
&\leq L(1 - \alpha_n)[\beta_n(1 + L)\|x_n - x^*\| + \|v_n\|] \\
&\quad + [L\alpha_n(1 + L)[L\|x_n - x^*\| + \|v_n\|] + L\|u_n\| \\
&\leq [L(1 + L)\beta_n + (1 + L)L^2\alpha_n]\|x_n - x^*\| \\
&\quad + L(1 + L)\|v_n\| + L\|u_n\|
\end{aligned}$$

so there exist $M_1 > 0$ and $M_2 > 0$ such that

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\leq \left(1 - \frac{k\alpha_n}{2}\right)\|x_n - x^*\| \\
&\quad + \beta_n L(1 + L)\alpha_n\|x_n - x^*\| + (L^3 + 3L^2 + 2)\alpha_n^2\|x_n - x^*\| \\
&\quad + M_1\|u_n\| + M_2\|v_n\|.
\end{aligned}$$

Since $\text{Sup } \beta_n < \frac{\lambda k}{2L(L+1)}$, $\text{Sup } \alpha_n < \frac{(1-\lambda)k}{2(L^3+3L^2+2)}$ for some $\lambda \in (0, 1)$, there exists $\tau > 0$ and $N > 0$ such that for all $n > N$, we have

$$\frac{k}{2} - L(1 + L)\beta_n - (L^3 + 3L^2 + 2)\alpha_n > \tau > 0.$$

Thus

$$\|x_{n+1} - x^*\| \leq (1 - \tau\alpha_n)\|x_n - x^*\| + M_1\|u_n\| + M_2\|v_n\|.$$

Set

$$\begin{aligned}t_n &= \tau\alpha_n \\b_n &= 0 \\c_n &= M_1\|u_n\| + M_2\|v_n\| \\a_n &= \|x_n - x^*\|.\end{aligned}$$

Then we have

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n.$$

According to above argument, it is easy seen that

$$\sum_{n=1}^{\infty} t_n = \infty, \quad b_n = o(t_n), \quad \sum_{n=1}^{\infty} c_n < \infty$$

and so, by Lemma 1, we have $\lim a_n = \lim \|x_n - x^*\| = 0$. Uniqueness follows as in [2]. The proof of the theorem is complete. \square

REMARK 2.1. Our theorem 2.1 generalized the theorem of Chidume [3] from uniformly smooth Banach space to arbitrary Banach space and from Ishikawa iteration to Ishikawa iteration with errors. In addition, Our results extend, generalize and improve the corresponding results obtained by Zeng [9] and Li [6].

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