

WORST CASES OF POLYNOMIAL ZEROS BAD

SEON-HONG KIM

ABSTRACT. In this paper, we provide an example of “worst cases” of bad pairs of polynomial zeros, namely sums of two polynomials with all pairs of zeros bad and all their zeros real.

1. Introduction

Given some (or all) information about two individual polynomials, particularly about their factorizations, what can be said about the factorization of their sum? For various results and examples about this, see [3], [4] and [5].

Let $P_A(x)$ and $P_B(x)$ be monic real polynomials with degree n having all zeros distinct and real. Then it is natural for us to be interested in the number of real zeros of $P_A(x) + P_B(x)$. It is an easy consequence of Fell [1] that, if all the zeros of $P_A(x)$ and $P_B(x)$ form good pairs, $P_A(x) + P_B(x)$ has all its zeros real. Here the definitions of good pairs and bad pairs were introduced in [2] as follows.

DEFINITION 1. If U is a finite multiset of complex numbers, write

$$P_U(x) = \prod_{\alpha \in U} (x - \alpha).$$

If U and V are sets of real numbers, with no repeated elements, and moreover

$$|U| = |V| = n, \quad U \cap V = \phi,$$

we may write

$$T := U \cup V = \{t_1, t_2, \dots, t_{2n}\}$$

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with $t_i < t_{i+1}$ for all i . Define

$$T_1 = \{\{t_1, t_2\}, \{t_3, t_4\}, \dots, \{t_{2n-1}, t_{2n}\}\}.$$

We say that a U -bad pair for T or for (P_U, P_V) is a pair of T_1 such that both elements belong to U ; let $N_U(U, V)$ denote the number of U -bad pairs. The number of bad pairs is defined by

$$N_U(U, V) + N_V(U, V).$$

Also a pair that is not bad is called a good pair.

The cases having bad pairs have been studied in [2] and [6]. In this paper, we consider a case of all pairs of polynomial zeros bad whose all adjacent two bad pairs have zeros of $P_A(x)$ and $P_B(x)$, respectively.

We can check that, by computational search, an arithmetic progression $F = \{1, 2, \dots, 2n\}$ of length ≤ 10 satisfies

CONDITION 2. *Whenever $F = A \cup B$, $A \cap B = \emptyset$, $|A| = |B| = n$, the number of bad pairs of F is equal to the number of nonreal zeros of*

$$(1) \quad P_A(x) + P_B(x) = \prod_{a_i \in A} (x - a_i) + \prod_{b_j \in B} (x - b_j).$$

So one might ask: is there a set $F = \{r_1, r_2, \dots, r_{2n}\}$ of real numbers in increasing order such that Condition 2 holds? But, an arithmetic progression F does not satisfy Condition 2. For example, there are ten polynomials (of $\binom{12}{6} = 924$ polynomials) for the arithmetic progression of length 12 that do not satisfy Condition 2. In fact, most sets do not seem to satisfy Condition 2. So it would be natural to ask a weaker question: if an arithmetic progression of length $2n$ and the number of bad pairs are given, can we find two monic polynomials $P_A(x)$ and $P_B(x)$ with the same degrees whose set of all zeros form an arithmetic progression such that the number of bad pairs is equal to the number of nonreal zeros? Author [2] considered the case that both the number of bad pairs and the number of nonreal zeros are two. Moreover, author [2] obtained the fundamental relation between the number of bad pairs and the number of nonreal zeros, and showed that the polynomial in x where the coefficient of x^k is the number of sequences having $2k$ bad pairs has all zeros real and negative.

We assume that F in Condition 2 is an arithmetic progression. Then we now consider cases of all pairs of polynomial zeros bad. For convenience, we denote \blacksquare a bad pair with both elements belonging to one of

A and B in (1), and \square a bad pair with both elements belonging to the other in (1). Then the cases of all pairs bad listed in increasing order of the form

$$(2) \quad \blacksquare \blacksquare \dots \blacksquare \square \square \dots \square$$

satisfy the conclusion of Condition 2. In fact, if x is a zero of polynomial with pairs of the form (2), then x lies on the vertical line with the point in the middle of (2). Moreover, gaps between the zeros in the upper half-plane strictly increase as one proceeds upward. For further study of this, see [4]. An example of (2) is the polynomial equation

$$\prod_{k=0}^n (x - k) + \prod_{k=n+1}^{2n+1} (x - k) = 0$$

which has all pairs bad and all its zeros nonreal on $\Re x = n + 1/2$. However author showed in [6] that, for an odd integer n and large positive integer m , the polynomial

$$(3) \quad \prod_{k=0}^n (x - k^{1+\frac{1}{m}}) + \prod_{k=n+1}^{2n+1} (x - k^{1+\frac{1}{m}}) = 0$$

(that seems to have all its zeros lying very near on $\Re x = n + 1/2$ and all nonreal) has two real zeros near $(n + 1)^{1+\frac{1}{m}} + \frac{1}{2}$ even though all pairs of the zeros of $\prod_{k=0}^n (x - k^{1+\frac{1}{m}})$ and $\prod_{k=n+1}^{2n+1} (x - k^{1+\frac{1}{m}})$ in (3) are still bad. Hence without restriction to arithmetic progressions, one might ask the existence of “worst cases”, namely sums of polynomials with all pairs bad and all zeros real. But it does not seem to be obvious how to construct such examples. Also, if exists, it does not seem to have bad pairs that are of the form in (2). In this paper, we provide such an example whose all bad pairs are listed in increasing order of the form

$$(4) \quad \blacksquare \square \blacksquare \square \dots \blacksquare \square \blacksquare \square$$

using same notation as before. We note that the zeros of polynomial in this example are related to the geometric progression. In fact, we show that

PROPOSITION 3. Let $n \geq 1$ be an odd integer and $a > 2$. For q large, the zeros of

$$P_n(x) = \prod_{k=0}^n (x - q^{ak}) + \left(\prod_{k=0}^{n-1} \left(x - \left(1 + \frac{(-1)^{k+1}}{q} \right) q^{ak} \right) \right) \left(x + \left(1 + \frac{1}{q} \right) q^{an} \right)$$

are all real and distinct.

We observe that all pairs

$$\begin{aligned} & \left(- \left(1 + \frac{1}{q} \right) q^{na}, 1 - \frac{1}{q} \right), (1, q^a), \left(\left(1 + \frac{1}{q} \right) q^a, \left(1 - \frac{1}{q} \right) q^{2a} \right), \\ & (q^{2a}, q^{3a}), \left(\left(1 + \frac{1}{q} \right) q^{3a}, \left(1 - \frac{1}{q} \right) q^{4a} \right), \dots, \\ & \left(\left(1 + \frac{1}{q} \right) q^{(n-2)a}, \left(1 - \frac{1}{q} \right) q^{(n-1)a} \right), (q^{(n-1)a}, q^{na}) \end{aligned}$$

are bad for the polynomial $P_n(x)$ defined in Proposition 3, and they are listed in increasing order of the form in (4).

2. Proof

The proof of Proposition 3 relies on

THEOREM 4. Let

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

be a polynomial of degree $n \geq 2$ with positive coefficients. If

$$a_i^2 - 4a_{i-1}a_{i+1} > 0, \quad 1 \leq i \leq n-1,$$

then all zeros of P_n are real and distinct.

For the proof of Theorem 4, see [7].

PROOF OF PROPOSITION 3. For an odd integer $n \geq 1$ and $a > 2$, let

$$A_n(x) = \prod_{k=0}^n (x - q^{ka})$$

and

$$B_n(x) = \left(\prod_{k=0}^{n-1} \left(x - \left(1 + \frac{(-1)^{k+1}}{q} \right) q^{ak} \right) \right) \left(x + \left(1 + \frac{1}{q} \right) q^{an} \right)$$

so that

$$P_n(x) = A_n(x) + B_n(x) = \sum_{k=0}^{n+1} a_k x^k.$$

For q large, we observe that the coefficient of x^k of $A_n(x)$ is

$$(-1)^k q^{\frac{1}{2}(n(n+1)-k(k-1))a} + (-1)^k q^{\frac{1}{2}(n(n+1)-k(k-1)-2)a} + \text{lower terms.}$$

Here we note that most lower terms above are powers of q multiplied by integers whose absolute values are greater than 1. This is the reason why we take q large. With the same reason, for q large, the coefficient of x^k of $B_n(x)$ is that, for k odd,

$$q^{\frac{1}{2}(n(n+1)-k(k-1))a} + q^{\frac{1}{2}(n(n+1)-k(k-1))a-1} + \text{lower terms}$$

and, for k even ≥ 2 ,

$$-q^{\frac{1}{2}(n(n+1)-k(k-1))a} + \frac{n+1-k}{2} q^{\frac{1}{2}(n(n+1)-k(k-1))a-2} + \text{lower terms.}$$

Hence adding the coefficients of x^k of $A_n(x)$ and $B_n(x)$ gives that, for k odd,

$$\begin{aligned} a_k &= q^{\frac{1}{2}(n(n+1)-k(k-1))a-1} - q^{\frac{1}{2}(n(n+1)-k(k-1)-2)a} + \text{lower terms} \\ &= q^{\frac{1}{2}(n(n+1)-k(k-1)-2)a-1} (q^a - q) + \text{lower terms} \end{aligned}$$

and, for k even ≥ 2 ,

$$\begin{aligned} a_k &= q^{\frac{1}{2}(n(n+1)-k(k-1)-2)a} \\ &\quad + \frac{n+1-k}{2} q^{\frac{1}{2}(n(n+1)-k(k-1))a-2} + \text{lower terms} \\ &= q^{\frac{1}{2}(n(n+1)-k(k-1)-2)a-2} \left(q^2 + \frac{n+1-k}{2} q^a \right) + \text{lower terms.} \end{aligned}$$

Also we have

$$a_0 = \frac{n+1}{2} q^{\frac{n(n+1)}{2}a-2} + \text{lower terms.}$$

Hence, for q large, we see that all coefficients of $P_n(x)$ are positive. On the other hand,

$$\begin{aligned} & a_1^2 - 4a_0a_2 \\ &= \left(q^{\frac{1}{2}(n(n+1))a-1} - q^{\frac{1}{2}(n(n+1)-2)a} \right)^2 \\ &\quad - 4 \left(\frac{n+1}{2} q^{\frac{n(n+1)}{2}a-2} \right) \left(q^{\frac{1}{2}(n(n+1)-4)a} + \frac{n-1}{2} q^{\frac{1}{2}(n(n+1)-2)a-2} \right) \\ &\quad + \text{lower terms} \\ &= q^{(n^2+n-2)a-4} \left(q^{2(a+1)} - 2q^{a+3} - (n^2-1)q^a + q^4 - 2(n+1)q^2 \right) \\ &\quad + \text{lower terms,} \end{aligned}$$

and we can compute that, for k odd ≥ 3 ,

$$\begin{aligned} & a_k^2 - 4a_{k-1}a_{k+1} \\ (5) \quad &= q^{(n^2+n-k^2+k-2)a-2} (q^a - q)^2 \\ &\quad - q^{(n^2+n-k^2+k-3)a-4} ((n-k)q^a + 2q^2)((n-k+2)q^a + 2q^2) \\ &\quad + \text{lower terms} \end{aligned}$$

and, for k even ≥ 2 ,

$$\begin{aligned} (6) \quad & a_k^2 - 4a_{k-1}a_{k+1} = q^{(n^2+n-k^2+k-2)a-4} \left(\frac{1}{2}(n-k+1)q^a + q^2 \right)^2 \\ &\quad - 4q^{(n^2+n-k^2+k-3)a-2} (q^a - q)^2 + \text{lower terms.} \end{aligned}$$

Since q is large, the signs of the right sides of (5) and (6), respectively, are completely determined by comparing the powers of q of first two summands except error terms. In (5), the power of the second summand from that of the first summand is

$$(n^2+n-k^2+k-2)a-2+2a - ((n^2+n-k^2+k-3)a-4)+2a = a+2 > 0,$$

and, in (6), it is

$$(n^2+n-k^2+k-2)a-4+2a - ((n^2+n-k^2+k-3)a-2+2a) = a-2 > 0.$$

This completes the proof of the proposition by Theorem 4. □

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Department of Mathematics
College of Natural Science
Chosun University
Gwangju 501-759, Korea
E-mail: shkim17@mail.chosun.ac.kr