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ON OPERATOR INTERPOLATION PROBLEMS

YOUNG SOO JO*, JOO HO KANG AND KI SOOK KIM

ABSTRACT. In this paper we obtained the following : Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let X and Y be operators acting on $\mathcal{H}$. If the range of X is dense in $\mathcal{H}$, then the following are equivalent:

(1) there exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$,

(2) $\sup\left\{\dfrac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} : f \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty.$

Moreover, if condition (2) holds, we may choose the operator A such that $\|A\| = K$.

1. Introduction

On the process of solving operator equation $AX = Y$ for two given X and Y in the algebra $\mathcal{B}(\mathcal{H})$, the class of all bounded operators acting on a Hilbert space $\mathcal{H}$, many mathematicians investigated the problem on their field of studies. What is the condition for an operator A satisfying $AX = Y$ to be a member of $\mathcal{A}$ which is a specified subalgebra of $\mathcal{B}(\mathcal{H})$?

Douglas [2] used the range inclusion property of operators to show necessary and sufficient conditions for the existence of an operator A satisfying $AX = Y$. In [7], authors investigated the problem on nest algebra. Hopenwasser [3] studied on CSL and Moore [9] showed the condition is a necessary and sufficient condition for the existence of an interpolating operator in Alg$\mathcal{L}$ when $\mathcal{L}$ is a CSL and X has dense range. In [5], authors has found the other necessary and sufficient condition for interpolating operator when $\mathcal{L}$ is a subspace lattice and X has dense range. In this paper, we are going to show that the condition given in [9] holds for a subspace lattice $\mathcal{L}$.

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Let $\mathcal{H}$ be a Hilbert space. A *subspace lattice* $\mathcal{L}$ is a strongly closed lattice of orthogonal projections in $\mathcal{B}(\mathcal{H})$ containing the trivial projections 0 and I. The symbol Alg$\mathcal{L}$ denotes the algebra of bounded operators on $\mathcal{H}$ that leave invariant every projection in $\mathcal{L}$; Alg$\mathcal{L}$ is a weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$. A lattice $\mathcal{L}$ is a *commutative subspace lattice*, or CSL, if the projections in $\mathcal{L}$ all commute; in this case, Alg$\mathcal{L}$ is called a *CSL algebra.*

We want to introduce the simplest case of the operator interpolation problem that relaxes all restrictions on A, requiring it simply to be a bounded operator. In this case, the existence of A is nicely characterized by the well-known factorization theorem of Douglas.

THEOREM A. ([2], [4]) *Let X and Y be bounded operators acting on a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(1) *range* $Y^* \subseteq$ *range* X^*,

(2) $Y^*Y \leq \lambda^2 X^*X$ *for some* $\lambda \geq 0$,

(3) *there exists a bounded operator A on $\mathcal{H}$ so that $AX = Y$.*

Moreover, if (1), (2), *and* (3) *are valid, then there exists a unique operator A so that*

(a) $\|A\|^2 = \inf\{\mu : Y^*Y \leq \mu X^*X\}$,

(b) *ker* Y^* = *ker* A^* *and*

(c) *range* $A^* \subseteq$ *range* X^-.

2. Operator interpolation problems in $\mathcal{B}(\mathcal{H})$

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ a subset of $\mathcal{H}$. Then $\overline{\mathcal{M}}$ means the closure of $\mathcal{M}$ and $\mathcal{M}^{\perp}$ is the orthogonal complement of $\mathcal{M}$.

We use the convention $\frac{0}{0} = 0$, when necessary.

We will introduce theorems that are obtained easily.

THEOREM 2.1. *Let X and Y be bounded operators acting on $\mathcal{H}$. Then the following are equivalent:*

(1) *there exists* $\lambda(\geq 0)$ *such that* $Y^*Y \leq \lambda^2 X^*X$,

(2) *there exists* $\lambda(\geq 0)$ *such that* $\|Yf\| \leq \lambda\|Xf\|$ *for every* $f \in \mathcal{H}$,

(3) $\sup\left\{\dfrac{\|Yf\|}{\|Xf\|} : f \in \mathcal{H}\right\} < \infty.$

We can get the following theorem by Theorems A and 2.1.

THEOREM 2.2. *Let X and Y be bounded operators acting on $\mathcal{H}$. Then the following are equivalent:*

(1) *range* $Y^* \subseteq$ *range* X^*,

(2) *there exists* $\lambda(\geq 0)$ *such that* $Y^*Y \leq \lambda^2 X^*X$,
(3) *there exists an operator* A *in* $\mathcal{B}(\mathcal{H})$ *such that* $AX = Y$,
(4) *there exists* $\lambda(\geq 0)$ *such that* $\|Yf\| \leq \lambda \|Xf\|$ *for every* $f \in \mathcal{H}$,
(5) $\sup\left\{ \dfrac{\|Yf\|}{\|Xf\|} \ : \ f \in \mathcal{H} \right\} < \infty.$

THEOREM 2.3. *Let* $X_1, X_2, \ldots, X_n$ *and* $Y_1, Y_2, \ldots, Y_n$ *be bounded operators acting on* $\mathcal{H}$. *Then the following are equivalent:*

(1) *there exists an operator* A *in* $\mathcal{B}(\mathcal{H})$ *such that* $AX_i = Y_i$ *for* $i = 1, 2, \ldots, n$,

(2) $\sup\left\{ \dfrac{\|\sum_{i=1}^n Y_i f_i\|}{\|\sum_{i=1}^n X_i f_i\|} \ : \ f_i \in \mathcal{H} \right\} < \infty.$

Proof. If $\sup\left\{ \dfrac{\|\sum_{i=1}^n Y_i f_i\|}{\|\sum_{i=1}^n X_i f_i\|} : f_i \in \mathcal{H} \right\} < \infty$, then we may assume that $\sup\left\{ \dfrac{\|\sum_{i=1}^n Y_i f_i\|}{\|\sum_{i=1}^n X_i f_i\|} : f_i \in \mathcal{H} \right\} = 1$. Then

$$\|\sum_{i=1}^n Y_i f_i\| \leq \|\sum_{i=1}^n X_i f_i\|$$

for all f_i in $\mathcal{H}$. Let $\mathcal{M} = \{\sum_{i=1}^n X_i f_i \ : \ f_i \in \mathcal{H}\}$. Then $\mathcal{M}$ is a linear manifold. We define a function $A : \mathcal{M} \longrightarrow \mathcal{H}$ by $A(\sum_{i=1}^n X_i f_i) = \sum_{i=1}^n Y_i f_i$ for f_i in $\mathcal{H}$. Then A is well-defined.

For, if $\sum_{i=1}^n f_i f_i = \sum_{i=1}^n X_i g_i$, then $\sum_{i=1}^n X_i f_i - (\sum_{i=1}^n X_i g_i) = 0$. So, $\|\sum_{i=1}^n X_i f_i - (\sum_{i=1}^n X_i g_i)\| = \|\sum_{i=1}^n X_i (f_i - g_i)\| = 0$. Since $\|\sum_{i=1}^n Y_i (f_i - g_i)\| \leq \|\sum_{i=1}^n X_i (f_i - g_i)\| = 0$, $\sum_{i=1}^n X_i f_i = \sum_{i=1}^n X_i g_i$.

And A is continuous. Extend A to $\overline{\mathcal{M}}$ by continuity and define $Ah = 0$ for $h \in \overline{\mathcal{M}}^{\perp}$. Then, A is a bounded operator on $\mathcal{H}$ clearly. Moreover, $AX_i = Y_i$ for $i = 1, 2, \ldots, n$.

The converse is proved easily. □

We can extend the above fact for infinitely many operators easily.

THEOREM 2.4. *Let* X_i *and* Y_i *be bounded operators acting on* $\mathcal{H}$ *for each* i *in* $\mathbb{N}$. *Then the following are equivalent:*

(1) *there exists an operator* A *in* $\mathcal{B}(\mathcal{H})$ *such that* $AX_i = Y_i$ *for* $i = 1, 2, \ldots$,

(2) $\sup\left\{ \dfrac{\|\sum_{i=1}^n Y_i f_i\|}{\|\sum_{i=1}^n X_i f_i\|} \ : \ f_i \in \mathcal{H} \text{ and } n \in \mathbb{N} \right\} < \infty.$

Theorems 2.3 and 2.4 show that if we get necessary and sufficient conditions which are equivalent to the second case of the above Theorems, then it gives conditions for the existence of an operator A such that $AX_i = Y_i$ for any bounded operators X_i and Y_i. So we want to find conditions that are equivalent to the second case.

THEOREM 2.5. *Let* $X_1, \dots, X_n$ *and* $Y_1, \dots, Y_n$ *be bounded operators acting on* $\mathcal{H}$. *Assume that* $Re < X_i f, X_j g > \geq 0$ *for* $i < j$ *and all* f, g *in* $\mathcal{H}$. *If there exists* $\lambda (\geq 0)$ *such that* $Y_i^* Y_i \leq \lambda^2 X_i^* X_i$ *for each* $i = 1, 2, \dots, n$, *then* $\sup \left\{ \frac{\| \sum_{i=1}^n Y_i f_i \|}{\| \sum_{i=1}^n X_i f_i \|} : f_i \in \mathcal{H} \right\} < \infty$.

Proof. If $Y_i^* Y_i \leq \lambda^2 X_i^* X_i$ for each $i = 1, 2, \dots, n$, then $\frac{\|Y_i f\|}{\|X_i f\|} \leq \lambda$ for each $i = 1, 2, \dots, n$. Since $Re <X_i f, X_j g> \geq 0$ for $i < j$, $\|X_k f_k\| \leq \| \sum_{i=1}^n X_i f_i \|$ for each $k = 1, 2, \dots, n$. For,

$$\begin{aligned} & \| \sum_{i=1}^n X_i f_i \|^2 \\ = & < X_k f_k, X_k f_k > + < \sum_{i \neq k} X_i f_i, X_k f_k > \\ + & < X_k f_k, \sum_{i \neq k} X_i f_i > + < \sum_{i \neq k} X_i f_i, \sum_{i \neq k} X_i f_i > \\ = & \|X_k f_k\|^2 + \| \sum_{i \neq k} X_i f_i \|^2 + \sum_{i<j} 2Re < X_i f_i, X_j f_j > \geq \|X_k f_k\|^2 . \end{aligned}$$

Hence

$$n\lambda \geq \sum_{i=1}^n \frac{\|Y_i f_i\|}{\|X_i f_i\|} \geq \sum_{i=1}^n \frac{\|Y_i f_i\|}{\| \sum_{i=1}^n X_i f_i \|} = \frac{\sum_{i=1}^n \|Y_i f_i\|}{\| \sum_{i=1}^n X_i f_i \|} \geq \frac{\| \sum_{i=1}^n Y_i f_i \|}{\| \sum_{i=1}^n X_i f_i \|}.$$

So, $\sup \left\{ \frac{\| \sum_{i=1}^n Y_i f_i \|}{\| \sum_{i=1}^n X_i f_i \|} : f_i \in \mathcal{H} \right\} < \infty$. □

From Theorem 2.2, we have the following corollary.

COROLLARY 2.6. *Let* $X_1, \dots, X_n$ *and* $Y_1, \dots, Y_n$ *be bounded operators acting on* $\mathcal{H}$. *Assume that* $Re < X_i f, X_j g > \geq 0$ *for* $i < j$ *and all* f, g *in* $\mathcal{H}$. *If range* $Y_i^* \subseteq$ *range* X_i^* *for each* $i = 1, 2, \dots, n$, *then* $\sup \left\{ \frac{\| \sum_{i=1}^n Y_i f_i \|}{\| \sum_{i=1}^n X_i f_i \|} : f_i \in \mathcal{H} \right\} < \infty$.

THEOREM 2.7. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be bounded operators acting on $\mathcal{H}$. If $\sup\left\{\frac{\|\sum_{i=1}^n Y_i f_i\|}{\|\sum_{i=1}^n X_i f_i\|} : f_i \in \mathcal{H}\right\} < \infty$, then there exists $\lambda(\geq 0)$ such that $Y_i^* Y_i \leq \lambda^2 X_i^* X_i$ for each $i = 1, 2, \ldots, n$.*

Proof. If $\sup\left\{\frac{\|\sum_{i=1}^n Y_i f_i\|}{\|\sum_{i=1}^n X_i f_i\|} : f_i \in \mathcal{H}\right\} < \infty$, then

$$\sup\left\{\frac{\|Y_i f\|}{\|X_i f\|} : f \in \mathcal{H}\right\} < \infty$$

for each $i = 1, 2, \ldots, n$. So, by the Theorem 2.2, there exists $\lambda_i(\geq 0)$ such that $Y_i^* Y_i \leq \lambda_i^2 X_i^* X_i$ for each $i = 1, 2, \ldots, n$. Hence we can take $\lambda = max\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that $Y_i^* Y_i \leq \lambda^2 X_i^* X_i$ for each $i = 1, 2, \ldots, n$.□

From the above theorems, we have the following.

THEOREM 2.8. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be bounded operators acting on $\mathcal{H}$. If $Re < X_i f, X_j g > \geq 0$ for $i < j$ and all f, g in $\mathcal{H}$, then the following are equivalent:*

(1) *range $Y_i^* \subseteq$ range X_i^* for each $i = 1, 2, \ldots, n$,*

(2) *there exists $\lambda(\geq 0)$ such that $Y_i^* Y_i \leq \lambda^2 X_i^* X_i$ for each $i = 1, 2, \ldots, n$,*

(3) *there exists an operator A in $\mathcal{B}(\mathcal{H})$ such that $AX_i = Y_i$ for $i = 1, 2, \ldots, n$,*

(4) *there exists $\lambda(\geq 0)$ such that $\|Y_i f\| \leq \lambda \|X_i f\|$ for $i = 1, 2, \ldots, n$,*

(5) $\sup\left\{\frac{\|\sum_{i=1}^n Y_i f_i\|}{\|\sum_{i=1}^n X_i f_i\|} : f_i \in \mathcal{H}\right\} < \infty$.

If we look through the proof of the above theorems, then we know that it is not extended for the countable bounded operators by the same type. But we have a sufficient condition as the following theorem and corollary.

THEOREM 2.9. *Let X_i and Y_i be bounded operators acting on $\mathcal{H}$ for $i = 1, 2, \ldots$. If $Re < X_i f, X_j g > \geq 0$ for $i < j$ and f, g in $\mathcal{H}$ and there exists a sequence $\{\lambda_n\}$ of non-negative numbers such that $\sum_{i=1}^{\infty} \lambda_i < \infty$ and $Y_i^* Y_i \leq \lambda_i^2 X_i^* X_i$ for $i \in \mathbb{N}$, then*

$$\sup\left\{\frac{\|\sum_{i=1}^n Y_i f_i\|}{\|\sum_{i=1}^n X_i f_i\|} : f_i \in \mathcal{H} \text{ and } n \in \mathbb{N}\right\} < \infty.$$

Proof. If $Re < X_i f, X_j g > \geq 0$ for $i < j$ and $f,\ g$ in $\mathcal{H}$, then for each $n \in \mathbb{N}$, $\|X_k f_k\| \leq \|\sum_{i=1}^{n} X_i f_i\|$ for every $k \leq n$ with the same proof as Theorem 2.5. Hence for each $n \in \mathbb{N}$,

$$\begin{aligned} \infty > \sum_{i=1}^{\infty} \lambda_i &\geq \sum_{i=1}^{n} \frac{\|Y_i f_i\|}{\|X_i f_i\|} \\ &\geq \sum_{i=1}^{n} \frac{\|Y_i f_i\|}{\|\sum_{i=1}^{n} X_i f_i\|} \geq \frac{\sum_{i=1}^{n} \|Y_i f_i\|}{\|\sum_{i=1}^{n} X_i f_i\|} \\ &\geq \frac{\|\sum_{i=1}^{n} Y_i f_i\|}{\|\sum_{i=1}^{n} X_i f_i\|}. \end{aligned}$$

So, $\sup \left\{ \frac{\|\sum_{i=1}^{n} Y_i f_i\|}{\|\sum_{i=1}^{n} X_i f_i\|} : f_i \in \mathcal{H} \text{ and } n \in \mathbb{N} \right\} < \infty.$ □

COROLLARY 2.10. *Let X_i and Y_i be bounded operators acting on $\mathcal{H}$ for $i = 1, 2, \ldots$. Assume that $Re < X_i f, X_j g > \geq 0$ for $i < j$ and $f,\ g$ in $\mathcal{H}$. If there exists a sequence $\{\lambda_n\}$ of non-negative numbers such that $\sum_{i=1}^{\infty} \lambda_i < \infty$ and $Y_i^* Y_i \leq \lambda_i^2 X_i^* X_i$ for $i \in \mathbb{N}$, then there exists a bounded operator A in $\mathcal{B}(\mathcal{H})$ such that $AX_i = Y_i$ for any $i \in \mathbb{N}$.*

3. Interpolation problems in Alg$\mathcal{L}$

In [9], the authors showed a necessary and sufficient condition that there exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$ when $\mathcal{L}$ is a commutative subspace lattice. We state the theorem.

THEOREM B [9]. *Let $\mathcal{L}$ be a commutative subspace lattice in $\mathcal{B}(\mathcal{H})$ and let X be an operator with dense range, and let Y be a bounded operator acting on $\mathcal{H}$. The following conditions are equivalent:*
(1) *there exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$,*
(2) $\rho(X, Y; \mathcal{L}) =^{def} \sup \left\{ \frac{\|E^{\perp} Y f\|}{\|E^{\perp} X f\|} : E \in \mathcal{L} \text{ and } f \in \mathcal{H} \right\} < \infty.$
Furthermore, if condition (2) holds, then there is an operator $A \in$Alg$\mathcal{L}$ such that $AX = Y$ and whose norm is equal to $\rho(X, Y; \mathcal{L})$.

In the following theorem, we obtain the above result for non-commutative subspace lattice case.

THEOREM 3.1. *Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$ and let X and Y be in $\mathcal{B}(\mathcal{H})$. If the range of X is dense in $\mathcal{H}$, then the following are equivalent:*

(1) *there exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$,*

(2) $\sup\left\{\dfrac{\|E^\perp Yf\|}{\|E^\perp Xf\|} : f \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty.$

Moreover, if condition (2) holds, we may choose the operator A such that $\|A\| = K$.

Proof. Assume that $\sup\left\{\dfrac{\|E^\perp Yf\|}{\|E^\perp Xf\|} : f \in \mathcal{H}, E \in \mathcal{L}\right\} = K < \infty$. Then for each E in $\mathcal{L}$, there exists an operator A_E in $\mathcal{B}(\mathcal{H})$ such that $A_E(E^\perp X) = E^\perp Y$ and $\|A_E\| \leq K$ by Theorem 2.2. In particular, if $E = 0$, then we have an operator A_0 in $\mathcal{B}(\mathcal{H})$ such that $A_0X = Y$ and $\|A_0\| \leq K$. And for each $E \in \mathcal{L}$, $A_E(E^\perp X) = E^\perp Y = E^\perp(A_0X)$. Since the range of X is dense in $\mathcal{H}$, $A_E E^\perp = E^\perp A_0$ for each E in $\mathcal{L}$. Hence for each E in $\mathcal{L}$,

$$E^\perp A_0 E^\perp = A_E E^\perp E^\perp = A_E E^\perp = E^\perp A_0 .$$

So A_0 is in Alg$\mathcal{L}$. Since $A_0X = Y$ and $E^\perp A_0 E^\perp = E^\perp A_0$,

$$\|E^\perp Yf\| = \|E^\perp A_0 Xf\| = \|E^\perp A_0 E^\perp Xf\| \leq \|A_0\|\|E^\perp Xf\|.$$

So, $K \leq \|A_0\|$. Therefore, $\|A_0\| = K$.

The proof of the converse is obvious. □

THEOREM 3.2. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be bounded operators acting on $\mathcal{H}$. If the range of X_k is dense in $\mathcal{H}$ for some k in $\{1, 2, \ldots, n\}$, then the following are equivalent:*

(1) *there exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \ldots, n$,*

(2) $\sup\left\{\dfrac{\|E^\perp(\sum_{i=1}^n Y_i f_i)\|}{\|E^\perp(\sum_{i=1}^n X_i f_i)\|} : f_i \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty.$

Proof. If $\sup\left\{\dfrac{\|E^\perp(\sum_{i=1}^n Y_i f_i)\|}{\|E^\perp(\sum_{i=1}^n X_i f_i)\|} : f_i \in \mathcal{H},\ E \in \mathcal{L}\right\} = K$, then

$$\frac{\|E^\perp(\sum_{i=1}^n Y_i f_i)\|}{\|E^\perp(\sum_{i=1}^n X_i f_i)\|} \leq K$$

for each $E \in \mathcal{L}$ and all $f_i \in \mathcal{H}$. So for each $E \in \mathcal{L}$, there exists an operator A_E in $\mathcal{B}(\mathcal{H})$ such that $A_E(E^\perp X_i) = E^\perp Y_i$ for each $i = 1, 2, \ldots, n$ by Theorem 2.3. If $E = 0$, then $A_0 X_i = Y_i$ for $i = 1, 2, \ldots, n$. Hence

$$A_E(E^\perp X_i) = E^\perp Y_i = E^\perp A_0 X_i \text{ for each } i = 1, 2, \ldots, n.$$

Since $\overline{\text{range } X_k} = \mathcal{H}$ for some k in $\{1, 2, \ldots, n\}$, $A_E E^\perp = E^\perp A_0$ for each $E \in \mathcal{L}$. Also,

$$E^\perp A_0 E^\perp = A_E E^\perp E^\perp = A_E E^\perp = E^\perp A_0 .$$

So A_0 is in Alg$\mathcal{L}$.

We omit the proof of the converse since it can be proved easily. □

We can generalize Theorem 3.2 to the countable case easily.

THEOREM 3.3. *Let X_i and Y_i be bounded operators acting on $\mathcal{H}$. If the range of X_k is dense in $\mathcal{H}$ for some $k \in \mathbb{N}$, then the following are equivalent:*

(1) *there exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \ldots,$*

(2) $\sup \left\{ \dfrac{\|E^\perp(\sum_{i=1}^n Y_i f_i)\|}{\|E^\perp(\sum_{i=1}^n X_i f_i)\|} : f_i \in \mathcal{H},\ E \in \mathcal{L},\ n \in \mathbb{N} \right\} = K < \infty.$

Proof of the following theorem can be obtained by the same way as the proof of Theorem 2.5.

THEOREM 3.4. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be bounded operators acting on a Hilbert space $\mathcal{H}$. Assume that Re $<E^\perp X_i f, E^\perp X_j g> \geq 0$ for each $E \in \mathcal{L}$, $i < j$ and all f, g in $\mathcal{H}$. If there exists $\lambda (\geq 0)$ such that $(E^\perp Y_i)^*(E^\perp Y_i) \leq \lambda^2 (E^\perp X_i)^*(E^\perp X_i)$ for each $E \in \mathcal{L}$ and $i = 1, 2, \ldots, n$, then* $\sup \left\{ \dfrac{\|E^\perp(\sum_{i=1}^n Y_i f_i)\|}{\|E^\perp(\sum_{i=1}^n X_i f_i)\|} : f_i \in \mathcal{H}, E \in \mathcal{L} \right\} < \infty.$

THEOREM 3.5. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be bounded operators acting on a Hilbert space $\mathcal{H}$. If*

$$\sup \left\{ \frac{\|E^\perp(\sum_{i=1}^n Y_i f_i)\|}{\|E^\perp(\sum_{i=1}^n X_i f_i)\|} : f_i \in \mathcal{H}, E \in \mathcal{L} \right\} < \infty,$$

then there exists $\lambda (\geq 0)$ such that

$$(E^\perp Y_i)^*(E^\perp Y_i) \leq \lambda^2 (E^\perp X_i)^*(E^\perp X_i)$$

for each $E \in \mathcal{L}$ and $i = 1, 2, \ldots, n$.

Proof. If $\sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H}, E \in \mathcal{L}\right\} = \lambda < \infty$ for some nonnegative real number λ, then

$$\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\| \leq \lambda \|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|$$

for each $E \in \mathcal{L}$ and all f_i in $\mathcal{H}$. If $f_i = 0$ for $i \neq k$, then $\|E^{\perp} Y_k f_k\| \leq \lambda \|E^{\perp} X_k f_k\|$ for $k = 1, 2, \ldots, n$. So

$$(E^{\perp} Y_i)^*(E^{\perp} Y_i) \leq \lambda^2 (E^{\perp} X_i)^*(E^{\perp} X_i)$$

for each $E \in \mathcal{L}$ and $i = 1, 2, \ldots, n$. □

We have the following theorem from the above theorems.

THEOREM 3.6. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be bounded operators acting on a Hilbert space $\mathcal{H}$. If Re $< E^{\perp} X_i f, E^{\perp} X_j g > \geq 0$ for each $E \in \mathcal{L}$, $i < j$ and all f, g in $\mathcal{H}$ and if the range of X_k is dense in $\mathcal{H}$ for some k, then the following are equivalent:*

(1) *range $(E^{\perp} Y_i)^* \subseteq$ range $(E^{\perp} X_i)^*$ for each $E \in \mathcal{L}$ and $i = 1, 2, \ldots, n$,*

(2) *there exists $\lambda (\geq 0)$ such that $(E^{\perp} Y_i)^*(E^{\perp} Y_i) \leq \lambda^2 (E^{\perp} X_i)^*(E^{\perp} X_i)$ for each $E \in \mathcal{L}$ and $i = 1, 2, \ldots, n$,*

(3) *there exists $\lambda (\geq 0)$ such that $\|E^{\perp} Y_i f\| \leq \lambda \|E^{\perp} X_i f\|$ for each $E \in \mathcal{L}$ and $i = 1, 2, \ldots, n$,*

(4) $\sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H}, E \in \mathcal{L}\right\} < \infty$,

(5) *there exists an operator A in $\mathcal{B}(\mathcal{H})$ such that $AE^{\perp} X_i = E^{\perp} Y_i$ for each $E \in \mathcal{L}$ and $i = 1, 2, \ldots, n$,*

(6) *there exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for each $i = 1, 2, \ldots, n$.*

From Theorem 2.9, we can get the following theorem.

THEOREM 3.7. *Let X_i and Y_i be bounded operators on a Hilbert space $\mathcal{H}$ for $i = 1, 2, \ldots$. If Re $< E^{\perp} X_i f, E^{\perp} X_j g > \geq 0$ for each $E \in \mathcal{L}$, $i < j$ and all f, g in $\mathcal{H}$ and if there exists a sequence $\{\lambda_n\}$ of non-negative numbers such that $\sum_{i=1}^{\infty} \lambda_i < \infty$ and $(E^{\perp} Y_i)^*(E^{\perp} Y_i) \leq \lambda_i^2 (E^{\perp} X_i)^*(E^{\perp} X_i)$ for each $E \in \mathcal{L}$ and $i \in \mathbb{N}$, then*

$$\sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H}, E \in \mathcal{L}, n \in \mathbb{N}\right\} < \infty.$$

Proof. Let $Re < E^{\perp}X_i f, E^{\perp}X_j g > \geq 0$ for each $E \in \mathcal{L}$, $i < j$ and all f, g in $\mathcal{H}$ and let $n \in \mathbb{N}$. Then $\|E^{\perp}X_k f_k\| \leq \|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|$ for every $k \leq n$. Indeed,

$$\begin{aligned}
&\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|^2 \\
&= < E^{\perp}X_k f_k, E^{\perp}X_k f_k > + < E^{\perp}(\sum_{i \neq k} X_i f_i), E^{\perp}X_k f_k > \\
&\quad + < E^{\perp}X_k f_k, E^{\perp}(\sum_{i \neq k} X_i f_i) > + < E^{\perp}(\sum_{i \neq k} X_i f_i), E^{\perp}(\sum_{i \neq k} X_i f_i) > \\
&= \|E^{\perp}X_k f_k\|^2 + \|E^{\perp}(\sum_{i \neq k} X_i f_i)\|^2 + \sum_{i<j} 2Re < E^{\perp}X_i f_i, E^{\perp}X_j f_j > \\
&\geq \|E^{\perp}X_k f_k\|^2 \text{ for } i, j = 1, 2, \ldots, n.
\end{aligned}$$

Hence for each $n \in \mathbb{N}$,

$$\begin{aligned}
\infty > \sum_{i=1}^{\infty} \lambda_i &\geq \sum_{i=1}^{n} \frac{\|E^{\perp}Y_i f_i\|}{\|E^{\perp}X_i f_i\|} \\
&\geq \sum_{i=1}^{n} \frac{\|E^{\perp}Y_i f_i\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} \geq \frac{\sum_{i=1}^{n} \|E^{\perp}Y_i f_i\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} \\
&\geq \frac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|}.
\end{aligned}$$

So, $\sup\left\{ \frac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H}, E \in \mathcal{L}, n \in \mathbb{N} \right\} < \infty.$ □

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Young Soo Jo
Department of Mathematics
Keimyung University
Daegu 704-701, Korea
E-mail: ysjo@kmu.ac.kr

Joo Ho Kang and Ki Sook Kim
Department of Mathematics
Daegu University
Daegu 712-714, Korea
E-mail: jhkang@daegu.ac.kr