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Fundamental problems for an elastic plate weakened by a curvilinear hole

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Abstract: Muskhelishvili's complex variable method has been applied to derive exact and closed expressions for Gaursat's functions for the first and second fundamental problems of the infinite plate weakened by a curvilinear hole which is conformally mapped on the domain outside the unit circle by means of rational mapping function. The hole having three poles. The previous work of the authers in this domain is considered as special cases of this work.

1. Introduction:

problems dealing with isotropic homogeneous perforated infinite plate have been investigated by several authors. Some authors used laurant's theorem to express each complex potential as a power series, (see [1.2.3]), others used complex variable method of Cauchy integrals, (see [3.4]).

Muskelishvili, in his work [4] proved that, the first and second fundamental problems in the plane theory of elasticity are equivalent to finding two analytic functions $\phi_1(z)$ and $\psi_1(z)$ of one complex argument z= x+ iy, i $=\sqrt{-1}$, .

These functions must satisfy the boundary conditions:

$$k^{*}\phi_1(t) - t\overline{\phi_1^{1}(t)} - \overline{\Psi_1(t)} = f(t), \tag{1.1}$$

where k^{*}= -1 and f(t) is a given function of stresses for the first fundamental problem, while $K^{*} = x = \frac{\lambda + 3\mu}{\lambda + \mu} > 1$ and f(t) =2μ g(t), is a given function of displacements, for the

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second fundamental problems, λ, μ are called the Lame's constants, x is called Muskhelishvile's constant, and t denotes the affix of a point on the boundary.

In terms of z= cw (ζ). c > o, w'(ζ) does not vanish or become infinite for $|\zeta|>1$, the infinite region outside a closed contour conformal by mapped outside the unit circle γ. The two complex functions of potential ϕ_1 (z) and ψ_1(z) take the form (4).

$$\phi_1(z) = -\frac{X+iY}{2\pi(1+x)}\ln\zeta + c\Gamma\zeta + \phi(\zeta) \tag{1.2}$$

$$\psi_1(z) = \frac{(X+iY)}{2\pi(1+x)}\ln\zeta + c\Gamma^*\zeta + \psi(\zeta), \tag{1.3}$$

where X, Y are the components of the resultant vector of all exnternal forces acting on the boundary, and Γ, Γ^* are complex constants.

Generally, the two complex functions $\phi(\zeta)$, $\psi(\zeta)$, are single – valued analytic functions within the region outside the unit circle and $\phi(\infty) = \psi(\infty) = 0$. For the first fundamental problem, it will be assumed that $\Gamma=\overline{\Gamma}$ and X = Y = 0.

Muskhelishvili [4] used the transformation

$$z= c(\zeta + m\,\zeta^{-1}),\ c>0 \tag{1.4}$$

for solving the problem of stretching of an infinite plate weakened by an elliptic hole. England [1] considered an infinite plate weakened by a hypotrochoid hole, conformally mapped onto a unit circle $|\zeta| = 1$ by the transformation mapping

$$z(\zeta) = c(\zeta + m\,\zeta^{-n}),\quad c > 0\quad ,0 \le m > \frac{1}{n}, \tag{1.5}$$

where z'(ζ) does not vanish or become infinite outside the unit circle γ . and he solved the boundary value problem by suing Laurant's theorem.

In previous papers [5, 6, 7], the complex variable method has been applied to solve the first and second fundamental problems for the same domain of the infinite plate with general curvilinear hole conformally mapped on the domain outside a unit circle by using, respectively, the rational mapping functions:

$$z = c\,\frac{\zeta + m\zeta^{-1}}{1 - n\zeta^{-1}},\quad c > 0,\ |n| < 1 \qquad (1.6)$$

$$z = c\,\frac{\zeta + m_1\zeta^{-1} + m_2\zeta^{-2}}{1 - n\zeta^{-1}},\quad c >,\ |n| < 1 \qquad (1.7)$$

and

$$z = c\,\frac{\zeta + m_1\zeta^{-1} + m_2\zeta^{-2} + m_3\zeta^{-3}}{1 - n\zeta^{-1}},\quad c > 0,\ |n| < 1. \qquad (1.8)$$

In this paper, the complex variable method has been applied to solve the first and second fundamental problems for the same previous domain of the infinite plate with a general curvilinear hole C, with three pales, conformally mapped on the domain outside a unit circle γ by the rational mapping functions

$$z(\zeta) = c\,\frac{\zeta + m_1\zeta^{-1} + m_2\zeta^{-2} + m_3\zeta^{-3}}{(1 - n_1\zeta^{-1})(1 - n_2\zeta^{-1})(1 - n_3\zeta^{-1})}\quad c > 0,\ n_1 \neq n_2 \neq n_2) \qquad (1.9)$$

where $c > 0$, m's and n's are real parameters restricted such that $z'(\zeta)$ does not vanish or become infinite outside γ. The interesting cases when the shape of the hole is an ellipse, hypotrochoidal, a crescent or a cut having the shape of a circular arc are included as special ones. Many new cases, also can be derived from the work. All the previous works mentioned in this domain, are considered as special cases of this work. Holes corresponding to certain combinations of the parameters m's and n's are sketched (see figs 1,2,3,4). Some applications of the first and second fundamental problems for the plate with a curvilinear hole of Eq. (1.9) are investigated

2. Method of solution:

The rational mapping z = cw (ζ). c > 0, maps the boundary C of the given region occupied by the middle plane of the plate in the z–plane onto the unit circle λ in the ζ-plane. Curvilinear coordinates (ρ , θ) are thus introduced into the z – plane which are the maps of the polar coordinates in the ζ - plane as given by $\zeta = \rho e^{i\theta}, 0 \le \theta \le 2\pi$.

Substituting w(ζ) into (1.1), we have:

$$\phi_1((cw(\zeta)) - \frac{w(\zeta)}{w'(\zeta)}\overline{\phi_1'(cw(\zeta))} - \overline{\psi_1(cw(\zeta)} = f(cw(\zeta)). \qquad (2.1)$$

Using Eq. (1.9), the expressions $\frac{w(\zeta^{-1})}{w'(\zeta)}$ can be written in the form

$$\frac{w(\zeta^{-1})}{w'(\zeta)} = \alpha(\zeta^{-1}) + \beta(\zeta), \qquad (2.2)$$

where

$$\alpha(\zeta) = \sum_{k=1}^{3} \frac{h_k}{\zeta - n_k}.$$

$$h_k = \frac{\left[n_k^4 + \sum_{J=1}^{3} m_j n_k^{3-j}\right]\left(\prod_{J=1}^{3}(1 - n_j n_k)\right)\left(\prod_{\substack{k \ne j \\ J=1}}^{3}(n_k - n_j)\right)^{-1}}{\left[2 + \sum_{J=1}^{3}(3-j) m_j n_k^{1+j} - \sum_{J=1}^{3}\frac{1}{1 - n_j n_k} - \left(\sum_{j=1}^{3} m_j n_k^{1+j}\right)\sum_{j=1}^{3}\frac{1}{(1 - n_j n_k)}\right]} \qquad (2.3)$$

and β (ζ) is a regular function for | ζ | > 1

Using (1.2), (1.3) and (2.2), the boundary condition (2.1) can be written in the form.

$$k^*\phi(\sigma) - \alpha(\sigma)\overline{\phi'(\sigma)} - \overline{\psi_*(\sigma)} = f_*(\sigma) \qquad (2.4)$$

where $\sigma = e^{i\theta}$ denotes the value of ζ on the boundary of the unit circle γ, while.

$$\psi_*(\zeta) = \psi(\zeta) + \beta(\zeta)\phi'(\zeta).$$

$$f_*(\zeta) = F(\zeta) - CK^*\Gamma\zeta + \overline{C\Gamma^*}\zeta^{-1} + N(\zeta)\left(\alpha(\zeta) + \bar{\beta}(\zeta)\right)$$

$$N(\zeta) = C\bar{\Gamma} - \frac{X - iY}{2\pi(1 + x)}\zeta \cdot$$

and

$$F(\zeta) = f\left(cw(\zeta)\right). \quad (2.5)$$

Assume that the derivatives of F (σ) must satisfy the Holder condition, i.e.

$$\left|F(\sigma_1) - F(\sigma_2)\right| \le a_1 \mid \sigma_1 - \sigma_2 \mid^{\gamma}, \ \alpha_1 \text{ is constant, } 0 \le \gamma < 1 \quad (2.6)$$

our aim is to determine the two functions $\phi(\zeta)$ and $\psi(\zeta)$ for the various fundamental problems, from Eq. (2.4). For this, multiplying both sides of (2.4) by $\frac{1}{2\pi i}\cdot\frac{d\sigma}{\sigma - \zeta}$, then integrating the result around the unit circle γ and evaluating the integrals thus formulated by residue theorems, we have

$$k^*\phi(\zeta) + \frac{1}{2\pi_i}\int_{\gamma}\frac{\alpha(\sigma)\overline{\phi'(\sigma)}}{\sigma - \zeta}\,d\sigma = \overline{C\Gamma^*}\zeta^{-1} - A(\zeta) + \sum_{J=1}^{3}\frac{h_j N(n_j)}{\zeta - n_j} \quad (2.7).$$

where

$$A(\zeta) = -\frac{1}{2\pi i}\sum_{\upsilon=0}^{\infty}\zeta^{(-1-\upsilon)}\int \sigma^{\upsilon}F(\sigma)d\sigma, \ |\zeta| > 1 \quad (2.8)$$

Using (2.2), we have

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\alpha(\sigma)\overline{\phi'(\sigma)}}{\sigma - \zeta}d\sigma = c\sum_{J=1}^{3}\frac{h_j b_j}{n_j - \zeta} \quad (2.9)$$

where b's are complex constants to be determined.

Using (2.9) in (2.7), we get

$$-k^*\phi(\zeta) = A(\zeta) - c\overline{\Gamma^*}\zeta^{-1} + \sum_{J=1}^{3} \frac{h_j}{n_j - \zeta}\left(cb_j + N(n_j)\right) \tag{2.10}$$

Differentiating (2.10) with respect to ζ, and using the result in (2.9), we obtain

$$ck^* b_j + cn_j^2 \Gamma^* + d_j h_j \left(cb_j + \overline{N(n_j)}\right) = -\overline{A'(n_j)} \tag{2.11}$$

where

$d_{j,k} = n_j^2(1 - n_j n_k)^{-2}$, j, k = 1, 2, 3, $j \neq k$.

Hence, we have

$$b_j = \frac{k^* E_j - h_j d_{j,k} \overline{E_j}}{c(k^{*2} - h_j^2 d_{j,k}^2)} \tag{2.12}$$

where

$E_j = -\overline{A'(n_j)} - c\Gamma^* n_j^2 - h_j d_{j,k} \overline{N(n_j)}$.

Also, from (2.4), the function ψ(ζ) can be determined in the form

$$\psi(\zeta) = \frac{ck^*\overline{\Gamma}}{\zeta} - \frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta) + \sum_{J=1}^{3} \frac{h_j \zeta}{1 - n_j \zeta}\phi_*(n_j^{-1}) + B(\zeta) - B, \tag{2.13}$$

where

$\phi_*(\zeta) = \phi'(\zeta) + \overline{N(\zeta)}$

$B(\zeta) = \frac{1}{2\pi_j}\int \frac{\overline{F(\sigma)}}{\sigma - \zeta} d\sigma$

and:

$$B = \frac{1}{2\pi i}\int \overline{F(\sigma)}\frac{d\sigma}{\sigma}, \tag{2.14}$$

where the integrals are taken over the unit circle γ.

3. special cases

(i) For n's =0 , $0 \le m_j \le 1$, j = 1,2,3, we get the rational mapping function.

$$z = c(\zeta + m_1\zeta^{-1} + m_2\zeta^{-1} + m_3\zeta^{-1}). \quad (3.1).$$

The physical interest of mapping (3.1) comes from the following.

(1) A circle of radiusc : $m_j = 0$, $1 \le j \le 3$

(2) An ellipse $m_2 = m_3 = 0$

(3) A square with rounded corners with diagonals parallel to the x and y axis $m_1 = m_3 = 0$, m_2 = about 0.1. The same square with its sides parallel to the axis m_1=m_3= 0,
m_2 = about -0.1.

(4) An ovaloid $m_3 = 0$, m_1 = a bout 0.3 , m_2 = -0.05

(5) A triangle $m_1 = m_3 = 0$

(ii) For $m_2 = m_3 = 0$, $m_1 = -1$, the boundary C degenerate into a circular cut with three poles , and for m_1 takes values near -1 $m_2 = m_3 = 0$, the edge of the hole resembles the shape of a crescent. Many interesting cases for the reader can be derived and used according to the technology work.

4. Examples:

In this section, we will discuss, some examples for the first and second fundamental problems, when the external and applied forces takes different cases.

4.1: Curvilinear hole for an infinite plate subjected to a uniform tensile stress:

For $k^* = -1$, $\Gamma = \frac{P}{4}, \Gamma^* = -\frac{P}{2}e^{-2i\theta}, 0 \le \theta \le 2\pi,$

and X = Y = f = 0, we have an infinite plate stretched at infinity by the application of a uniform tensile stress of intensity P, making an angle θ with the $x-$axis. The plate weakened by a curvilinear hole C having finite poles which is free from stress.

Gaursat's functions (2.10), (2.13) take the form

$$\phi(\zeta) = \frac{cP}{2}\left[\zeta^{-1}\exp(2i\theta) + \sum_{j=1}^{3} \frac{h_j Q_j}{n_{j-\zeta}}\right], \tag{4.1}$$

$$\Psi(\zeta) = \frac{-cP}{4}\zeta^{-1} - \frac{w(\zeta^{-1})}{w^{'}(\zeta)}\phi_*(\zeta) + \sum_{j=1}^{3} \frac{h_j \phi_*(n_j^{-1})\zeta}{1-n_j\zeta} \tag{4.2}$$

where

$$Q_j = \frac{\frac{1}{2} - n_j^2 \cos 2\theta}{1-h_j d_j} + \frac{n_j^2 \sin 2\theta}{1+h_j d_j}$$

and

$$\phi_*(\zeta) = \phi^1(\zeta) + \frac{cp}{4} \tag{4.3}$$

4.2: Curvilinear hole having three poles the edge of which is subject to a uniform pressure

For $k^* = -1, X = Y = \Gamma = \Gamma^* = 0$ and f(t) = Pt,

where P is a real constant, the formulas (2.10), (2.13) become.

$$\phi(\zeta) = \sum_{k,j=1}^{3} \frac{(n_k^4 + m_j n_k^{3-j})(1-h_j d_{j,k})}{(n_k - \zeta)(1+h_j d_{j,k})} p \tag{4.4}$$

and

$$\Psi(\zeta) = -\frac{w(\zeta^{-1})}{w'(\zeta)}\phi'(\zeta) - cp\sum_{j=1}^{3}(n_j + \zeta^{-1}) + \sum_{j=1}^{3} \frac{h_j\zeta}{1-n_j\zeta}\phi'(n_j^{-1}). \tag{4.5}$$

Hence (4.4) and (4.5) give the solution of the first fundamental problem when the edge of the hole is subject to uniform pressure P . Putting in (4.4) and (4.5) –iT instead of P, we have the first fundamental problem when the edge of the hole is subject to a uniform tangential stresst T.

4.3: Uni- directional tension of an infinite plate with a rigid curvilinear centre.

For $k^* = x, \Gamma = \frac{P}{4}, \Gamma^* = -(\frac{P}{2})e^{-2i\theta}, X = Y = 0,$ F(t) = 2iPεt, we have the two complex function,

$$-\text{x}\phi(\zeta)=\frac{cP}{2}e^{2i\theta}\zeta^{-1}+2ic\mu\in\sum_{j,k=1}^{3}\frac{\text{x}(n_k^4+m_jn_k^{3-j})}{(n_k-\zeta)(\text{x}+h_jd_{j,k})}+\frac{cP}{2}\sum_{j,k=1}^{3}\frac{h_jQ_{j,k}^{(2)}}{n_j-\zeta} \tag{4.6}$$

$$\Psi(\zeta)=2ic\mu\in\sum_{j=1}^{3}n_j+c(\frac{\text{x}p}{4}+2\mu\in_i)\zeta^{-1}-\frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta)+\sum_{j=1}^{3}\frac{h_j\zeta}{1-n_j\zeta}\phi_*(n_j^{-1}) \tag{4.7}$$

where

$$Q_{j,k}^{(2)}=\frac{x+2n_j^2\cos 2\theta}{2(x+h_jd_{j,k})}-i\frac{n_j^2\sin 2\theta}{x-h_jd_{j,k}}.$$

and

$$\phi_*(\zeta)=\phi'(\zeta)+\frac{cP}{4}\cdot \tag{4.8}$$

Therefore, we have the case of uni- directional tension of an infinite plate with a rigid curvilinear center. The constant $\in$ can be determined from the condition that the resultant moment of the forces acting on the curvilinear center from the surrounding material must vanish , i. e

$$\text{M}=\text{Re}\left\{\int\left[\Psi(\zeta)-\frac{cP}{2}e^{-2i\theta}\zeta\right]w'(\zeta)d\zeta\right\}=0\cdot \tag{4.9}$$

Hence, we have

$$\varepsilon=\frac{P(1+\text{x})(\sum_{j=1}^{3}n_j+N)\sin 2\theta}{4\mu\left[1+\sum n_j+L\right]} \tag{4.10}$$

where

$$N=\sum_{j,k=1}^{3}\frac{h_jn_j^2n_k^2}{(1-n_jn_k)^2(\text{x}-h_jd_{j,k})},$$

and

$$L=\sum_{j,k=1}^{3}\frac{n_k^6+m_jn_k^{5-j}}{(1-n_jn_k)^2(\text{x}+h_jd_{j,k})} \tag{4.11}$$

Case 1: Bi – axial tension with $k^*=\text{x}$, X= Y = 0, $\Gamma=\overline{\Gamma}=\frac{P}{2}$, $\Gamma^*=0$ and f(t) = 2μ g(t) under the same condition of example (4.3), one obviously will have ε = 0 and the two complex functions are

$$\phi(\zeta)=\frac{cP}{2}\sum_{j,k=1}^{3}\frac{h_j}{(\zeta-n_j)(x+h_jd_{j,k})}, \tag{4.12}$$

$$\Psi(\zeta)=\frac{cxP}{2}\zeta^{-1}-\frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta)+\sum_{j=1}^{3}\frac{h_j\zeta}{1-n_j\zeta}\phi_*(n_j^{-1}) \tag{4.13}$$

where

$$\phi_*(\zeta)=\phi'+\frac{cP}{2} \tag{4.14}$$

Case 2: When the curvilinear centre is not allowad to rotate.

Under the condition of the preceding example (4.3) the rigid curvilinear kernel is restrained in its original position by a couple which is not sufficient to rotate the kernel , then $\varepsilon=0$.

Hence, the two complex functions are

$$-x\phi(\zeta)=\frac{cP}{2}e^{2i\theta}\zeta^{-1}+\frac{cP}{2}\sum_{j,k=1}^{3}\frac{h_jQ_{j,k}^{(2)}}{n_j-\zeta} \tag{4.15}$$

and

$$\psi(\zeta)=\frac{xcp}{4}\zeta^{-1}-\frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta)+\sum_{J=1}^{3}\frac{h_j\zeta}{1-n_j\zeta}\phi_*(n_j^{-1}) \tag{4.16}$$

where $Q_{j,k}^{(2)}$ and $\phi_r(\zeta)$ are g iven by (4.9).

The resultant momemt is given by

$$M=\frac{cP\pi(1+x)}{x}\left[\sum_{J=1}^{3}n_j+\sum_{J=1}^{3}\frac{h_jn_j^2n_k}{(1-n_jn_k)^2(x-h_jd_{j,k})}\right]\sin 2\theta. \tag{4.17}$$

Case 3: When a couple with a given moment acts on the curvilinear hole:

We assume that the stresses vanish at infinity, under the same conditions of example (4.3), the complex functions take the following form

$$\phi(\zeta)=2ic\mu\in\sum_{j,k=1}^{3}\frac{(n_k^4+m_jn_k^{3-j})}{(\zeta-n_k)(x+h_jd_{j,k})} \tag{4.18}$$

$$\psi(\zeta)=2ic\mu\in\left(\sum_{J=1}^{3}n_j+\zeta^{-1}\right)-\frac{w(\zeta^{-1})}{w'(\zeta)}\phi^1(\zeta)+\sum_{J=1}^{3}\frac{h_j\zeta}{1-n_j\zeta}\phi'(n_j^{-1}) \tag{4.19}$$

where

$$\varepsilon = \frac{M\text{x}}{2\pi c\mu\left[1+\sum_{j=1}^{3} n_j + L\right]}, \tag{4.20}$$

and L is given by (4.11)

4.4: The force acts on the centre of the curvilinear hole.

In this case, it will be assumed that the stresses vanish at infinity. It is easily seen that the kernel does not rotate. In general, the kernel renains in its original position.

Hence, one assumes $\Gamma = \Gamma^* = f(t) = 0$ and $k^* = x$.

Gaursat's functions are

$$\phi(\zeta) = \frac{c}{2\pi\text{x}(1+\text{x})}\sum_{j,k=1}^{3}\frac{ch_j}{\zeta - n_j}\left[\frac{\text{x}h_j d_{j,k}(X+iY)}{c(\text{x}^2 - h_j^2 d_{j,k}^2)} - \left\{1 + \frac{h_j^2 d_{j,k}^2}{c(\text{x}^2 - h_j^2 d_{j,k}^2)}\right\}(X - iY)\right] \tag{4.21}$$

$$\Psi(\zeta) = \sum_j \frac{h_j \zeta}{1 - n_j \zeta}\phi_*(n_j^{-1}) - \frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta), \tag{4.22}$$

where

$$\phi_*(\zeta) = \phi'(\zeta) - \frac{X+iY}{2\pi(1+\mu)\zeta}. \tag{4.23}$$

Therefore, we have the solution of the second fundamental problem in the case, when a force (X , Y) acts on the center of the curvilinear kernel.

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