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ROTATION NUMBER OF A CIRCLE HOMEOMORPHISM

BONG JO KIM

ABSTRACT. We knew attractive qualities of a rotation number for an analytic diffeomorphism, for example, Arnold tongues and staircase. Numerical computer simulation let us know easily that the pattern of rotation number of a non analytic homeomorphism is similar to its of an analytic homeomorphism for some cases. Also we define new rotation function and investigate it's properties.

1. INTRODUCTION

A circle homeomorphism and a rotation number were considered first by Poincare (1885), who introduced the important dynamical invariant called the rotation number. He proved that the rotation number of a circle homeomorphism is irrational if and only if it has no periodic points. In this paper, we introduce many computer simulations. These make us understand this thesis easily.

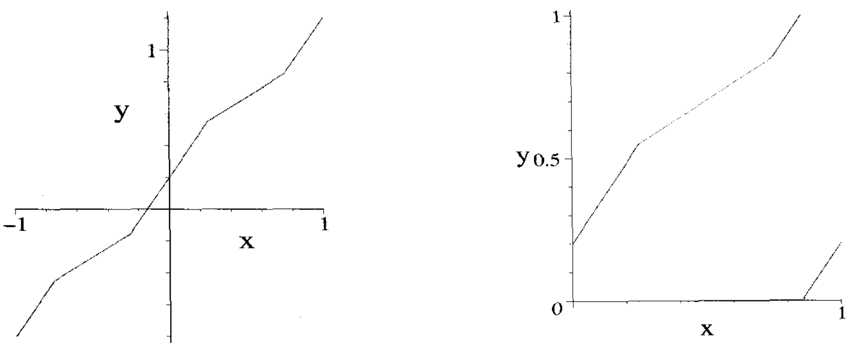

FIGURE 1. Graphs of $y = f(x)$(left) and $y = Tx$(right) represented by $f(x)$

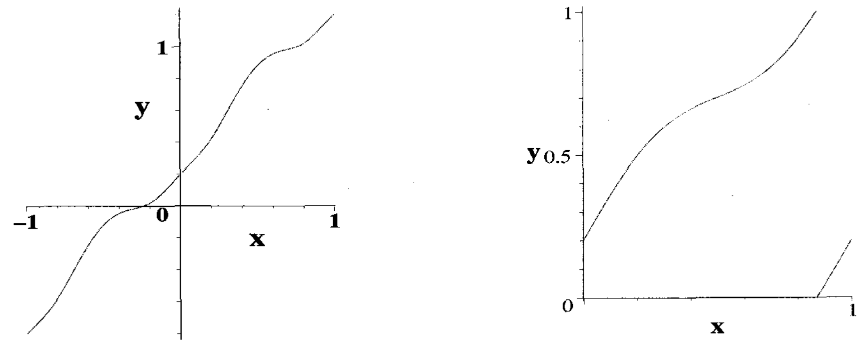

FIGURE 2. Graphs of $y = f(x)$(left) and $y = Tx$(right) represented by $f(x+n) = x + n + ae^{\frac{-1}{x}} e^{\frac{1}{x-1}} + \alpha$ for $0 \leq x \leq 1$, $n \in \mathbb{Z}$

Let T be an orientation preserving circle homeomorphism. For the sake of notational simplicity, the unit circle is identified with the interval $[0, 1)$. Every such homeomorphism may obviously be given in the form $Tx = f(x) \pmod 1$, $0 \leq x < 1$ where $f(x)$ is a continuous monotonically increasing function defined for all $x \in \mathbb{R}^1$, and satisfies the condition

$$f(x+1) = f(x) + 1$$

We say that the function f represents the homeomorphism T. If T is a diffeomorphism, then the function f is smooth. We shall say that T is a diffeomorphism of class C^k, if $f \in C^k(\mathbb{R}^1)$.

Clearly if f_1, f_2 represent the same orientation preserving circle homeomorphism then

$$f_1(x) = f_2(x) + k$$

for all $x \in \mathbb{R}^1$, where k is an integer.

Fact 1. *For any orientation preserving circle homeomorphism T and any function $f(x)$ representing T, the limit*

$$\lim_{n\to\infty} \frac{f^n(x)}{n} = \alpha$$

exists and does not depend on the choice of the point $x \in \mathbb{R}^1$.

Let T be an orientation preserving circle homeomorphism represented by f. We can define the rotation number ρ of T by

$$\rho = \rho(T, f) = \lim_{n\to\infty} \frac{f^n(x)}{n} \pmod 1$$

since this rotation number is independent on $x \in S^1$ by the previous fact.

We does not consider an orientation reversing circle homeomorphism T since if T is not orientation preserving, then the rotation number of T is not defined. We regard a circle homeomorphism T as an orientation preserving circle homeomorphism T in this paper.

Remark 2. Let T be a circle homeomorphism with a rational rotation number. Then there exists f such that f represents T where $0 \leq f(0) < 1$. Then we know that the rational rotation number of T is $\frac{p}{q}$ (mod 1) if there exists $x_0 \in S^1$ such that $f^{(q)}(x_0) = x_0 + p$. So if the image of T intersects the image of $y = x$, then the rotation number of T is 0 obviously. But if the image of T intersects the image of $y = x+1$, then we take the rotation number of T to be 1 to guarantee the continuity of the rotation function ρ_1 in Sec. 2.

Because succeeding fact, a circle homeomorphism T with an irrational rotation number does not have a periodic point. Thus we classify circle homeomorphisms by checking wether they have periodic points.

Fact 3. *Let T be a circle homeomorphism represented by f where $f : \mathbb{R} \to \mathbb{R}$. Then $\rho(T, f)$ is rational if and only if T has a periodic point for some x.*

2. Rotation number

Let $T : S^1 \to S^1$ be a circle homeomorphism represented by f. We define the rotation function, $\rho_1 : \alpha \longmapsto \rho(R_\alpha \circ T)$ where $R_\alpha(x) = x + \alpha$, $\alpha \in S^1$. We know that the function $\rho_1 : S^1 \to S^1$ is continuous and assumes every irrational value in S^1 exactly once. It is known that if f satisfies additional conditions that are f is analytic and there does not exist n such that $f^n =$ identity, then for each rational number $\frac{p}{q}$, $\rho_1^{-1}(\frac{p}{q})$ has a non-empty interior. Consequently, for such maps T, the function $\rho_1 : \alpha \longmapsto \rho(R_\alpha \circ T)$ is locally constant at each rational value. This phenomenon is called phase locking. For $0 < |a| < \frac{1}{2\pi}$, the map T: $S^1 \to S^1$ defined by $T(x) = x + a\sin(2\pi x)$ is analytic diffeomorphism satisfying this additional condition. For convenience of notation let $T_\alpha = R_\alpha \circ T$ and $f_\alpha = R_\alpha \circ f$. The mapping $T_\alpha = x + a\sin(2\pi x) + \alpha$ is called a standard map. In this section and next section we see that there is homeomorphism T that does not satisfy additional condition such that similar to standard map for rotation number by computer simulation.

Fact 4. *Let T: $S^1 \to S^1$ be a circle homeomorphism without a periodic point, i.e., the rotation number of T is an irrational number. Then $\rho(T_\alpha) > \rho(T)$ if $\alpha > 0$.*

In particular, for each a, the rotation function $\rho_1 : \alpha \longmapsto \rho(T_\alpha)$ is monotone, locally constant at each α for which $\rho(T_\alpha)$ is rational, and non-constant at each irrational value if T is standard map. A function with these properties is often called a staircase. It is known that $\rho_1 : \alpha \longmapsto \rho(T_\alpha)$ is increasing and strictly increasing if $\rho(T_\alpha)$ is irrational, see Figure 3 (left). However, for T represented

$$f(x+n) = x + n + ae^{\frac{-1}{x}}e^{\frac{1}{x-1}} \text{ where } 0 \leq x \leq 1, n \in \mathbb{Z}$$

which is not analytic, we see that it's ρ_1 is similar to ρ_1 of standard map, see Figure 3.

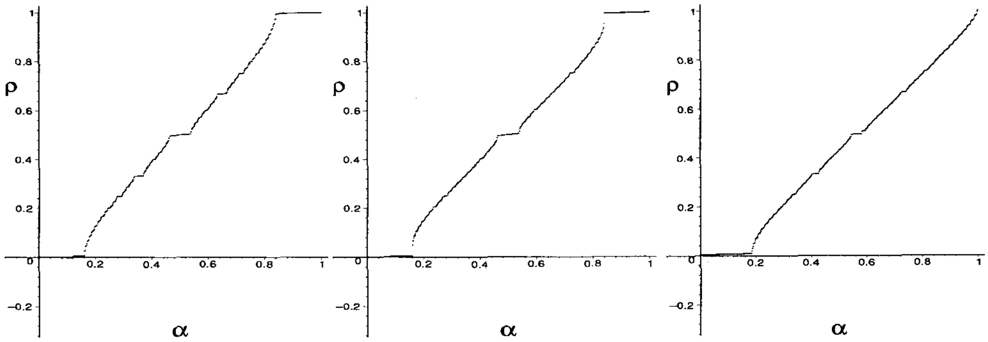

FIGURE 3. The function $\rho_1 : \alpha \longmapsto \rho(T_\alpha)$ for T represented $f(x)=x + a\sin(2\pi x)$ (left) and T represented by f is left function in Figure 1 (center), T represented $f(x+n)=x+n+ae^{\frac{-1}{x}}e^{\frac{1}{x-1}}$ for $0 \le x \le 1$, $n \in \mathbb{Z}$ (right)

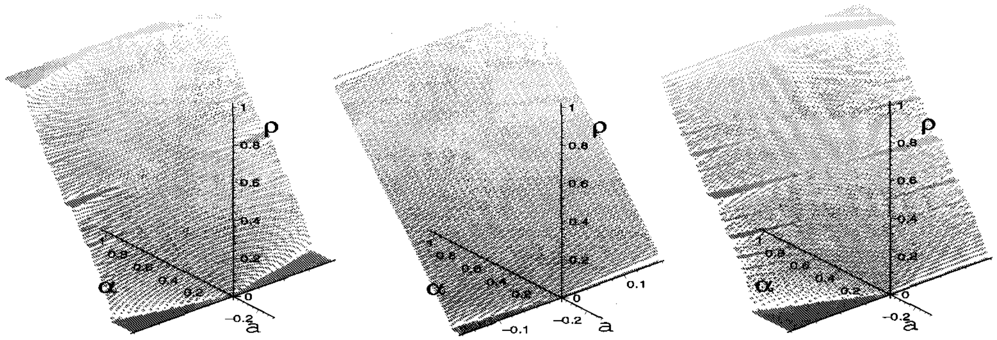

FIGURE 4. Graph of rotation numbers of T represented by f for $f(x)=x+a\sin(2\pi x)+\alpha$ (left) and $f(x)$ is defined in Figure 1(center), T represented by f where $f(x+n) = x+n+ae^{\frac{-1}{x}}e^{\frac{1}{x-1}}+\alpha$ for $0 \le x \le 1$, $n \in \mathbb{Z}$ (right)

Fact 5. *Assume that $T_\alpha^m \neq id$ for all $\alpha \in S^1$ and all $m \in \mathbb{N}$. Then $\rho_1^{-1}(\frac{r}{s})$ has a non-empty interior for each rational number $\frac{r}{s}$. In particular, the set $\{\alpha\ ;\ \rho_1(\alpha)$ is irrational $\}$ is nowhere dense in S^1.*

It is known that ρ_1 is Cantor variation. It is known that the image of the rotation function ρ_1 is the Cantor set with fractal dimension $\cong 0.75$ if T is standard map.

Furthermore, consider the two parameter family (α, a) such that $T_\alpha = x+a\sin(2\pi x)+\alpha$. For each irrational number ρ, it is known that the set $\{(\alpha, a) : \rho(T_\alpha) = \rho, T = x + a\sin(2\pi x) \pmod 1\}$ is continuous curve. For rational ρ, this set has a non-empty

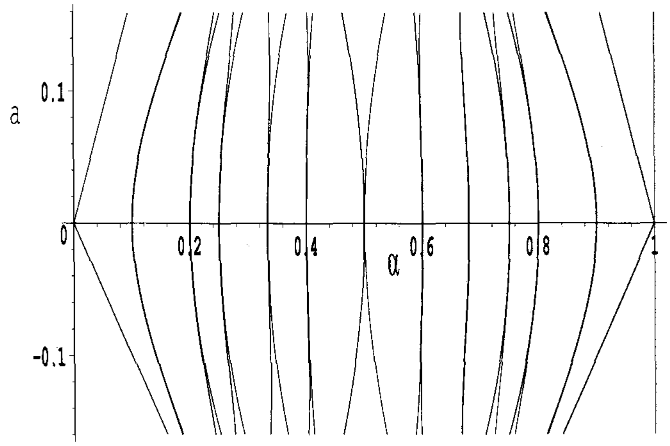

FIGURE 5. Arnold tongues. The boundary set $\{(\alpha, a) | \rho(T_\alpha) =$ constant, $T : x \longmapsto x + a\sin(2\pi x) \pmod 1\}$. In vertical direction the parameter a is drawn. The tongues corresponding to rotation numbers $\frac{1}{10}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{3}{5}, \frac{17}{25}, \frac{3}{4}, \frac{4}{5}, \frac{9}{10}$, and 1

interior and is bounded by two continuous curves from Fact 5. The wedges between these two curves are commonly referred to as Arnold tongues, see Figure 5. Although $\{\alpha\ ;\ \rho(T_\alpha)$ is irrational$\}$(where a is fixed as before) is nowhere dense in S^1, this set has positive Lebesgue measure. This follows from the result of M. Herman. For the details about Arnold tongues, refer to [8]. It is known that Arnold tongues and staircase are taken by most of circle homeomorphisms.

By computer simulation, we gain that the set $\{(\alpha, a) : \rho(T_\alpha) = \rho,\ T$ represented $f(x+n){=}x+n+ae^{\frac{-1}{x}}e^{\frac{1}{x-1}}$ for $0 \le x \le 1,\ n \in \mathbb{Z}\}$ is continuous curve and similar to it of standard map, see Figure 6.

3. ROTATION FUNCTION

In this section, we will define another rotation function and see the behavior of the rotation function. Also we will investigate interesting patterns of rotation number of specified circle homeomorphisms. We will see a rotation number of a circle homeomorphism T which has coefficient a and α, for example, $T_\alpha : x \longmapsto x + a\sin(2\pi x) + \alpha \pmod 1$, $T_\alpha : x \longmapsto x + ae^{\frac{1}{x(x-1)}} + \alpha \pmod 1$. The rotation number of these circle homeomorphisms is regarded as two parameter (α, a) function. So we write $T_{a,\alpha} = T_\alpha$ and $f_{a,\alpha} = f_\alpha$. We define a new rotation function ρ_2 by $\rho_2 : a \longmapsto \rho(T_{a,\alpha})$ where α is fixed.

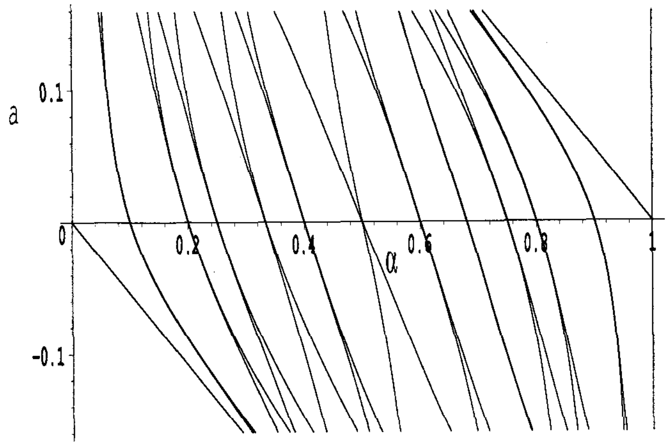

FIGURE 6. The boundary set $\{(\alpha, a) | \rho(T_\alpha) = \text{constant},\ T : x \longmapsto x + ae^{\frac{1}{x(x-1)}} \pmod 1\}$. In vertical direction the parameter a is drawn. The tongues corresponding to rotation numbers $\frac{1}{10}, \frac{1}{5}, \frac{1}{4}, \frac{33}{100}, \frac{2}{5}, \frac{3}{5}, \frac{17}{25}, \frac{3}{4}, \frac{4}{5}, \frac{9}{10}$, and 1

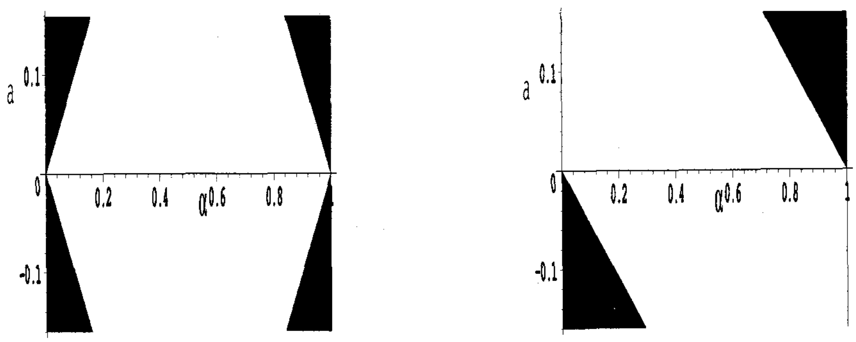

FIGURE 7. The set $\{(\alpha, a) |\ \rho(T_{a,\alpha}) = 0 \text{ or } 1\}$ for $T_{a,\alpha} : x \longmapsto x + a\sin(2\pi x) + \alpha \pmod 1$ (left) and $T_{a,\alpha} : x \longmapsto x + n + ae^{\frac{-1}{x}} e^{\frac{1}{x-1}} + \alpha \pmod 1$ for $0 \le x \le 1$, $n \in \mathbb{Z}$ (right)

We investigate the rotation number of standard map $T_{a,\alpha} : x \longmapsto x + a\sin(2\pi x) + \alpha \pmod 1$ where $0 \le \alpha < 1$, $|a| < \frac{1}{2\pi}$. If $a = 0$, then the rotation number of $T_{0,\alpha}$ is equal to α since $T_{0,\alpha} : x \longmapsto x + \alpha \pmod 1$. We know that the rotation number of $T_{a,\frac{1}{2}}$ is $\frac{1}{2}$ by induction.

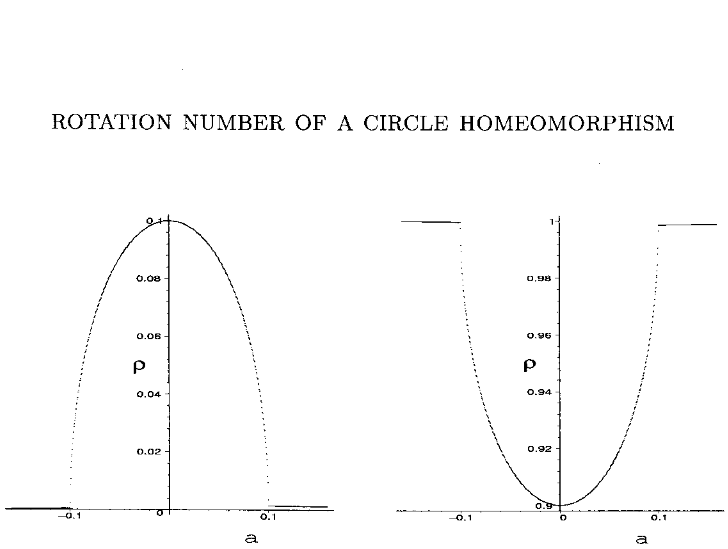

FIGURE 8. The function $\rho_2 : a \longmapsto \rho(T_{a,\alpha})$ where $T_{a,\alpha}$ represented by f such that $f(x){=}x + a\sin(2\pi x) + \alpha$ for $\alpha = 0.1(\text{left}),\ 0.9(\text{right})$

It is obvious that

$$\lim_{n\to\infty} \frac{f^n_{a,\alpha}(x)}{n} = - \lim_{n\to\infty} \frac{f^n_{a,-\alpha}(x)}{n}.$$

So we know $1 - \rho(T_{a,\alpha}) = \rho(T_{a,1-\alpha})$ by definition of a rotation number. The rotation number of T_{a_1,α_1} is 0 if and only if $\{(\alpha_1, a_1)|\ |a_1| \geq \alpha_1\}$. The rotation number of T_{a_2,α_2} is 1 if and only if $\{(\alpha_2, a_2)|\ |a_2| \geq 1 - \alpha_2\}$. The reason of these phenomena is following.

Remark 6. **Case I.** $\rho(T_{a,\alpha})$ =0 if and only if $(\alpha, a) \in \{(\alpha, a)|\ |a| \geq \alpha\}$
($\Rightarrow$) If $|a| \geq \alpha$, then $\{(x, y)|y = x\}$ and the graph of $T_{a,\alpha}$ have intersection point. Then the rotation number of $T_{a,\alpha}$ is 0 by Remark 2.
($\Leftarrow$) If $|a| < \alpha$, then $T_{a,\alpha} > x + t$ where $t = a\sin(2\pi x) + \alpha > 0$ since $|a| < \alpha$. So $\rho(T_{a,\alpha}) \geq \rho(x + t) = t > 0$.
Case II. $\rho(T_{a,\alpha})$ =1 if and only if $(\alpha, a) \in \{(\alpha, a)|\ |a| \geq 1 - \alpha\}$
Since $\rho(T_{a,1-\alpha}) = 1 - \rho(T_{a,\alpha})$, we know the graph $\{(\alpha, a)|\rho(T_{a,\alpha}) = 1\}$.

We know the rotation number of $T_{a,\alpha}$ and the rotation number of $T_{-a,\alpha}$ are same in Figure 2 and Figure 8. Hence we know that the graph of the rotation number of $T_{a,\alpha}$ is symmetric by a. For the details, let $f_{a,\alpha}$ represent $T_{a,\alpha}$ and we take any $x_0 \in S^1$ then we can choose x_1 for such that the increment of $f_{a,\alpha}(x_0)$ under iteration of $f_{a,\alpha}$ equals to that of $f_{-a,\alpha}(x_1)$ under iteration of $f_{-a,\alpha}$ since sine curve is periodic function. Hence $f^n_{a,\alpha}(x_0){=}f^n_{-a,\alpha}(x_1) + \frac{1}{2}$. Since rotation number is independent on $x \in S^1$ by Fact 3, $\rho(T_{a,\alpha}) = \rho(T_{-a,\alpha})$.

The symmetric properties of the graph of the rotation number and the shape of graph of rotation number 0, 1 of standard map are adopted in $Tx = f(x) \pmod 1$ where f is defined in Figure 1. The reason of these facts is same with that of standard map.

Remark 7. We investigate $T_{a,\alpha}{=}\ f \pmod 1$ where $f(x + n){=}x + n + ae^{\frac{-1}{x}}e^{\frac{1}{x-1}} + \alpha$ for $0 \leq x \leq 1$, $n \in \mathbb{Z}$. See Figure 2. We know that the rotation function $\rho_2 : a \longmapsto \rho(T_{a,\alpha})$

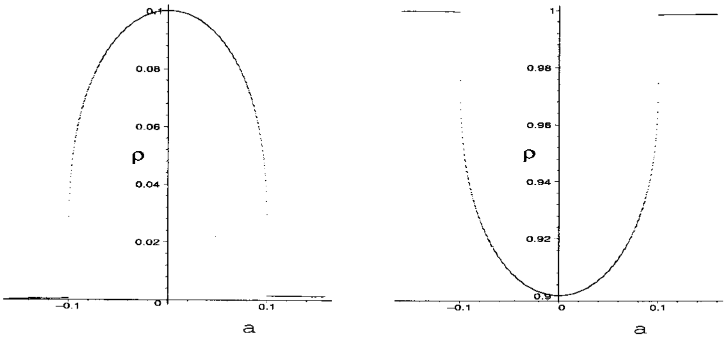

FIGURE 9. The function $\rho_2 : a \longmapsto \rho(T)$ where T represented by f such that f is left function in Figure 1 for $\alpha = 0.1$(left), 0.9(right)

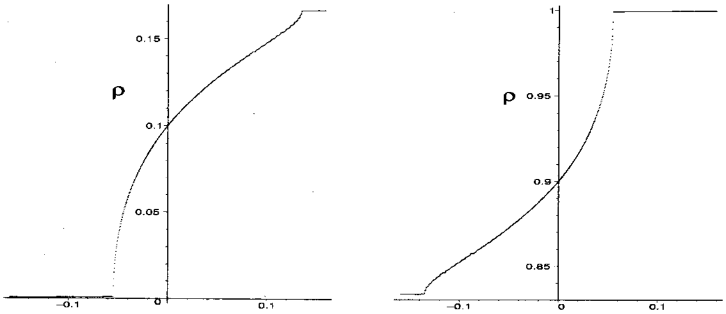

FIGURE 10. The function $\rho_2 : a \longmapsto \rho(T_{a,\alpha})$ where $T_{a,\alpha}$ represented by f such that $f(x+n){=}x+n+ae^{-1/x}e^{1/(1-x)}+\alpha$ for $0 \le x \le 1$, $n \in \mathbb{Z}$ for $\alpha = 0.1$(left), 0.9(right)

is increasing. The reason of this fact is following. $f_{a_1,\alpha}(x) \ge f_{a_2,\alpha}(x)$ for every $x \in S^1$ if $a_1 \ge a_2$. Then $\rho(T_{a_1,\alpha}) \ge \rho(T_{a_2,\alpha})$ by definition of a rotation number.

It is obvious that

$$\lim_{n\to\infty} \frac{f^n_{a,\alpha}(x)}{n} = -\lim_{n\to\infty} \frac{f^n_{-a,-\alpha}(x)}{n}.$$

So we know that $1-\rho(T_{-a,1-\alpha}) = \rho(T_{a,\alpha})$ by definition of a rotation number. See Figure 10. The rotation number of T_{a_1,α_1} is 0 if and only if $(\alpha_1, a_1) \in \{(\alpha, a) |\ a \le -\alpha e^4\}$. The rotation number of T_{a_2,α_2} is 1 if and only if $(\alpha_2, a_2) \in \{(\alpha, a) |\ a \ge (1-\alpha)e^4\}$. See Figure 7. The reason of these facts is same with that of standard map.

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Department of Mathematics
KAIST
Daejeon, 305-701 KOREA
E-mail: bongjo@math.kaist.ac.kr