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BOUNDARY CONTROLLABILITY OF ABSTRACT INTEGRODIFFERENTIAL SYSTEMS

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ABSTRACT. In this paper we establish a set of sufficient conditions for the boundary controllability of nonlinear integrodifferential systems and Sobolev type integrodifferential systems in Banach spaces by using fixed point theorems.

1. INTRODUCTION

The controllability of nonlinear systems represented by ordinary differential equation in finite and infinite dimensional spaces has been extensively studied by means of fixed-point principles [1,12]. Controllability of Sobolev-type nonlinear integrodifferential systems in Banach spaces has been discussed by Balachandran and Dauer [2] with the help of the Schauder fixed point theorem. In [6], Balachandran and Sakthivel studied the controllability of Sobolev-type semilinear functional integrodifferential systems in Banach spaces by using the Schaefer fixed point theorem.

Several abstract settings have been developed to describe the distributed control systems on a domain in which the control is acted through the boundary. Balakrishnan [8] showed that the solution of a parabolic boundary control equation with L^2 controls can be expressed as a mild solution to an operator equation using semigroup theory. Fattorini [10] developed a semigroup approach for boundary control systems. In [9] Barbu discussed the general theory of boundary control systems and the existence of solutions for boundary control systems governed by parabolic equations with nonlinear boundary conditions.

The formulation of boundary control problems in terms of semigroup theory offers the following advantage over a variational approach. The semigroup approach can treat a problem where the spatial domain does not have C^∞ boundary, such as for an n-dimensional parallelepiped. Han and Park [11] studied the boundary controllability of semilinear systems with nonlocal condition. Recently the problem of boundary controllability of semilinear systems and delay integrodifferential systems in Banach spaces

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has been investigated by Balachandran and Anandhi [3,4,5] and Balachandran et al [7]. Here we study the boundary controllability of nonlinear integrodifferential systems in Banach space by using the Schauder fixed point theorem and Sobolev-type integrodifferential systems by using the Banach contraction principle.

2. INTEGRODIFFERENTIAL SYSTEMS

Let E and U be a pair of real Banach spaces with norm $\|.\|$ and $|.|$, respectively.Let σ be a linear closed and densely defined operator with $D(\sigma) \subseteq E$ and let $\tau \subseteq X$ be a linear operator with $D(\sigma)$ and $R(\tau) \subseteq X$, a Banach space.

Consider the boundary control nonlinear integrodifferential system of the form

$$
\begin{aligned}
(1) \qquad \dot{x}(t) &= \sigma x(t) + f(t, x(t)) + \int_0^t g(t, s, x(s))ds, \quad t \in J = [0, b], \\
\tau x(t) &= B_1 u(t), \\
x(0) &= x_0,
\end{aligned}
$$

where $B_1 : U \to X$ is a linear continuous operator , the control function $u \in L^1(J, U)$, a Banach space of admissible control functions.The nonlinear operators $f : J \times E \to E$ and $g : \Delta \times E \to E$ are given and $\Delta : (t, s); 0 \leq s \leq t \leq b$. Let $A : E \to E$ be the linear operator defined by

$$D(A) = \{x \in D(\sigma); \tau x = 0\}, Ax = \sigma x, \quad \text{for} \quad x \in D(A).$$

We shall make the following hypotheses:

: (H_1) $D(\sigma) \subset D(\tau)$ and the restriction of τ to $D(\sigma)$ is continuous relative to the graph norm of $D(\sigma)$.

: (H_2) The operator A is the infinitesimal generator of a compact semigroup $T(t)$ and there exists a constant $M_1 > 0$ such that $\|T(t)\| \leq M_1$.

: (H_3) There exist a linear continuous operator $B : U \to E$ such that

$$\sigma B \in L(U, E), \tau(Bu) = B_1 u,$$

for all $u \in U$. Also $Bu(t)$ is continuously differentiable and $\|Bu\| \leq C\|B_1 u\|$ for all $u \in U$, where C is a constant.

: (H_4) For all $t \in (0, b]$ and $u \in U, T(t)Bu \in D(A)$.Moreover, there exists a positive constant $K_1 > 0$ such that $\|AT(t)\| \leq K_1$.

: (H_5) The nonlinear operators $f(t, x(t))$ and $g(t, s, x(s))$, for $t, s \in J$, satisfy

$$\|f(t, x(t))\| \leq L_1 \quad \|g(t, s, x(s))\| \leq L_2,$$

where $L_1 > 0$ and $L_2 > 0$.

: (H_6) The linear operator W from $L^2(J, U)$ into E defined by

$$Wu = \int_0^b [T(b-s)\sigma - AT(b-s)]Bu(s)ds$$

induces an invertible operator $\tilde{W}^{-1}$ defined on $L^2(J,U)/KerW$ and there exists a positive constant $M_2 > 0$ and $M_3 > 0$ such that $\|B\| \leq M_2$ and $\|\tilde{W}^{-1}\| \leq M_3$.

Let $x(t)$ be the solution of (1). Then we define a function $z(t) = x(t) - Bu(t)$ and from our assumption we see that $z(t) \in D(A)$. Hence (1) can be written in terms of A and B as

$$
\begin{aligned}
\dot{x}(t) &= Az(t) + \sigma Bu(t) + f(t, x(t)) + \int_0^t g(t, s, x(s))ds, \quad t \in J, \\
x(t) &= z(t) + Bu(t), \\
x(0) &= x_0.
\end{aligned} \tag{2}
$$

If u is continuously differentiable on [0,b], then z can be defined as a mild solution to the Cauchy problem

$$
\begin{aligned}
\dot{z}(t) &= Az(t) + \sigma Bu(t) - B\dot{u}(t) + f(t, x(t)) + \int_0^t g(t, s, x(s))ds, \\
z(0) &= x_0 - Bu(0)
\end{aligned}
$$

and the solution of (1) is given by

$$
\begin{aligned}
x(t) &= T(t)[x_0 - Bu(0)] + Bu(t) + \int_0^t T(t-s)f(s, x(s))ds \\
&\quad + \int_0^t T(t-s)[\sigma Bu(s) - B\dot{u}(s)]ds + \int_0^t T(t-s)[\int_0^s g(s, \tau, x(\tau))d\tau]ds
\end{aligned} \tag{3}
$$

Since the differentiability of the control u represents an unrealistic and severe requirement, it is necessary to extend the concept of the solution for the general inputs $u \in L^1(J, U)$. Integrating (3) by parts, we get

$$
\begin{aligned}
x(t) &= T(t)x_0 + \int_0^t [T(t-s)\sigma - AT(t-s)]Bu(s)ds \\
&\quad + \int_0^t T(t-s)f(s, x(s))ds + \int_0^t T(t-s)[\int_0^s g(s, \tau, x(\tau))d\tau]ds.
\end{aligned} \tag{4}
$$

Thus (4) is well defined and it is called a mild solution of the system (1).

Definition: The system (1) is said to be controllable on the interval J if for every $x_0, x_1 \in E$, there exists a control $u \in L^2(J, U)$ such that $x(.)$ of (1) satisfies $x(b) = x_1$.

Theorem.1 If the hypotheses $(H_1) - (H_6)$ are satisfied, then the boundary control integrodifferential system (1) is controllable on J.

Proof: Using the hypotheses (H_6), for an arbitrary function $x(\cdot)$ define the control

$$
u(t) = \tilde{W}^{-1}\{x_1 - T(b)x_0 - \int_0^b T(b-s)[f(s, x(s)) + \int_0^s g(s, \tau, x(\tau))d\tau]ds\}(t)
$$

We shall now show that, when using this control, the operator defined by

$$\begin{aligned}(\Phi x)(t) &= T(t)x_0 + \int_0^t [T(t-s)\sigma - AT(t-s)]Bu(s)ds \\ &+ \int_0^t T(t-s)f(s,x(s))ds + \int_0^t T(t-s)[\int_0^s g(s,\tau,x(\tau))d\tau]ds\end{aligned}$$

has a fixed point. This fixed point is then a solution of (1).

Clearly, $(\Phi x)(b) = x_1$, which means that the control u steers the nonlinear integrodifferential system from the initial state x_0 to x_1 in time T, provided we can obtain a fixed point of the nonlinear operator Φ.

Let $Y = C(J, X)$ and $Y_0 = \{x \in Y : \|x(t)\| \le r, \text{ for } t \in J\}$, where the positive constant r is given by

$$\begin{aligned} r &= M_1\|x_0\| + b[M_1\|\sigma\| + K_1]M_2M_3[\|x_1\| + M_1\|x_0\| + M_1L_1b \\ &+M_1L_2b^2] + M_1L_1b + M_1L_2b^2\end{aligned}$$

Then, Y_0 is clearly a bounded, closed, convex subset of Y. We define a mapping $\Phi : Y \to Y_0$ by

$$\begin{aligned}(\Phi x)(t) &= T(t)x_0 + \int_0^t [T(t-s)\sigma - AT(t-s)]B\tilde{W}^{-1}\{x_1 - T(b)x_0 \\ &- \int_0^b T(b-s)[f(s,x(s)) + \int_0^s g(s,\tau,x(\tau))d\tau]ds\}(s)ds \\ &+ \int_0^t T(t-s)f(s,x(s))ds + \int_0^t T(t-s)[\int_0^s g(s,\theta,x(\theta))d\theta]ds\end{aligned}$$

Consider

$$\begin{aligned}\|(\Phi x)(t)\| &\le \|T(t)x_0\| + \int_0^t \|[T(t-s)\sigma - AT(t-s)]\|\|B\|\|\tilde{W}^{-1}\|\{\|x_1\| - \|T(b)x_0\| \\ &- \int_0^b \|T(b-s)\|\|[f(s,x(s)) + \int_0^s g(s,\tau,x(\tau))d\tau]\|ds\}(s)ds \\ &+ \int_0^t \|T(t-s)\|\|f(s,x(s))\|ds + \int_0^t \|T(t-s)\| \int_0^s \|g(s,\theta,x(\theta))d\theta\|ds \\ &\le M_1\|x_0\| + b[M_1\|\sigma\| + K_1]M_2M_3[\|x_1\| + M_1\|x_0\| + M_1L_1b \\ &+M_1L_2b^2] + M_1L_1b + M_1L_2b^2 \\ &\le r\end{aligned}$$

Since f and g are continuous and $\|(\Phi x)(t)\| \le r$, it follows that Φ is also continuous and maps Y_0 into itself. Moreover, Φ maps Y_0 into a precompact subset of Y_0. To prove this, we first show that, for every fixed $t \in J$, the set

$$Y_0(t) = \{(\Phi x)(t) : x \in Y_0\}$$

is precompact in X. This is clear for $t = 0$, since $Y_0(0) = \{x_0\}$. Let $t > 0$ be fixed and for $0 < \epsilon < t$ define

$$\begin{aligned}(\Phi_\epsilon x)(t) &= T(t)x_0 + \int_0^{t-\epsilon} [T(t-s)\sigma - AT(t-s)]B\tilde{W}^{-1}\{x_1 - T(b)x_0 \\ &\quad - \int_0^b T(b-s)[f(s,x(s)) + \int_0^s g(s,\tau,x(\tau))d\tau]ds\}(s)ds \\ &\quad + \int_0^{t-\epsilon} T(t-s)f(s,x(s))ds + \int_0^{t-\epsilon} T(t-s)[\int_0^s g(s,\theta,x(\theta))d\theta]ds\end{aligned}$$

Since $T(t)$ is compact for every $t > 0$, the set

$$Y_\epsilon(t) = \{(\Phi_\epsilon x)(t) : x \in Y_0\}$$

is precompact in X for every $\epsilon, 0 < \epsilon < t$. Furthermore, for $x \in Y_0$, we have

$$\begin{aligned}&\|(\Phi x)(t) - (\Phi_\epsilon x)(t)\| \\ &\leq \|\int_{t-\epsilon}^t [T(t-s)\sigma - AT(t-s)]B\tilde{W}^{-1}\{x_1 - T(b)x_0 \\ &\quad - \int_0^b T(b-s)[f(s,x(s)) + \int_0^s g(s,\tau,x(\tau))d\tau]ds\}(s)ds\| \\ &\quad + \|\int_{t-\epsilon}^t T(t-s)f(s,x(s))ds\| + \|\int_{t-\epsilon}^t T(t-s)[\int_0^s g(s,\theta,x(\theta))d\theta]ds\| \\ &\leq \epsilon[M_1\|\sigma\|M_2M_3 + K_1M_2M_3][\|x_1\| + M_1\|x_0\| + bM_1L_1 + b^2M_1L_2] \\ &\quad + \epsilon M_1L_1 + \epsilon bM_1L_2\end{aligned}$$

which implies that $Y_0(t)$ is totally bounded, that is , precompact in X. We want to show that

$$\Phi(Y_0) = \{\Phi x : x \in Y_0\}$$

is an equicontinuous family of functions. For that , let $t_2 > t_1 > 0$.Then we have,

$$\begin{aligned}
&\|(\Phi x)(t_1) - (\Phi x)(t_2)\| \\
\leq\ & \| \int_0^{t_1} [T(t_1-s)\sigma - AT(t_1-s) - T(t_2-s)\sigma + AT(t_2-s)] \times \\
& B\tilde{W}^{-1}\{x_1 - T(b)x_0 - \int_0^b T(b-s)[f(s,x(s)) + \int_0^s g(s,\tau,x(\tau))d\tau]ds\}(s)ds\| \\
& +\| \int_{t_1}^{t_2} [T(t_2-s)\sigma - AT(t_2-s)]B\tilde{W}^{-1}\{x_1 - T(b)x_0 \\
& - \int_0^b T(b-s)[f(s,x(s)) + \int_0^s g(s,\tau,x(\tau))d\tau]ds\}(s)ds\| \\
& +\| \int_0^{t_1} [T(t_1-s) - T(t_2-s)]f(s,x(s))ds\| \\
& +\| \int_0^{t_1} [T(t_1-s) - T(t_2-s)][\int_0^s g(s,\theta,x(\theta))d\theta]ds\| \\
& +\| \int_{t_1}^{t_2} T(t_2-s)f(s,x(s))ds\| + \| \int_{t_1}^{t_2} T(t_2-s)[\int_0^s g(s,\theta,x(\theta))d\theta]ds\| \\
\leq\ & \int_0^{t_1} \|[T(t_1-s)\sigma - AT(t_1-s) - T(t_2-s)\sigma \\
& +AT(t_2-s)]\|[M_2M_3\{\|x_1\| + M_1\|x_0\| + M_1(L_1b + L_2b^2)\}]ds \\
& + \int_{t_1}^{t_2} \|[T(t_2-s)\sigma - AT(t_2-s)]\|[M_2M_3\{\|x_1\| + M_1\|x_0\| \\
& +M_1(L_1b + L_2b^2)\}]ds + \int_0^{t_1} \|[T(t_1-s) - T(t_2-s)]\|[L_1 + L_2b]ds \\
& + \int_{t_1}^{t_2} \|T(t_2-s)\|[L_1 + L_2b]ds \qquad (5)
\end{aligned}$$

The compactness of $T(t)$, $t > 0$, implies that $T(t)$ is continuous in the uniform operator topology for $t > 0$. Thus, the right-hand side of (5), which is independent of $x \in Y_0$, tends to zero as $t_2 - t_1 \to 0$. So, $\Phi(Y_0)$ is an equicontinuous family of functions.

Also, $\Phi(Y_0)$ is bounded in Y, and so by the Arzela-Ascoli theorem, $\Phi(Y_0)$ is precompact. Hence, from the Schauder fixed point theorem, Φ has a fixed point in Y_0. Any fixed point of Φ is a mild solution of (1) on J satisfying

$$(\Phi x)(t) = x(t) \in X.$$

Thus, the system (1) is controllable on J.

3. SOBOLEV-TYPE INTEGRODIFFERENTIAL SYSTEMS

Let Y and Z be Banach spaces with norms $|.|$ and $\|.\|$ respectively. Let σ be a linear, closed and densely defined operator with domain $D(\sigma) \subseteq Y$ and range $R(\sigma) \subseteq Z$. Let θ be a linear operator with $D(\theta) \subseteq Y$ and $R(\theta) \subseteq X$, a Banach space.

Consider the boundary control nonlinear system

$$\begin{aligned} (6)\ (Ex(t))' &= \sigma x(t) + f(t, x(t)) + \int_0^t k\left(t, s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau\right) ds, \quad t \in J \\ \theta x(t) &= B_1 u(t) \\ x(0) &= x_0 \end{aligned}$$

where $E : D(E) \subset Y \to R(E) \subset Z$ is a linear operator, the control function $u \in L^1(J, U)$, a Banach space of admissible control functions with U as a Banach space , $B_1 : U \to X$ is a linear continuous operator, and the nonlinear operators $f : J \times Y \to Z$, $g : \Delta \times Y \to Y$, $k : \Delta \times Y \times Y \to Z$ are given and $\Delta = \{(t, s); 0 \leq s \leq t \leq b\}$.

Let $y(t) = Ex(t)$ for $x \in Y$, then (6) can be written as

$$\begin{aligned} (7)\qquad y'(t) &= \sigma E^{-1} y(t) + f(t, E^{-1} y(t)) \\ &\quad + \int_0^t k\left(t, s, E^{-1} y(s), \int_0^s g(s, \tau, E^{-1} y(\tau))d\tau\right) ds, \quad t \in J = [0, b] \\ \tilde{\theta} y(t) &= B_1 u(t) \\ y(0) &= y_0 \end{aligned}$$

where $\tilde{\theta} = \theta E^{-1} : Z \to X$ is a linear operator. Let $A : Y \to Z$ be a linear operator defined by $D(AE^{-1}) = \{w \in D(\sigma E^{-1}) : \tilde{\theta} = 0\}$, $AE^{-1} w = \sigma E^{-1} w$, for $w \in D(AE^{-1})$

The operators $A : D(A) \subset Y \to Z$ and $E : D(E) \subset Y \to Z$ satisfy the following hypotheses.

: (i) A and E are closed linear operators.
: (ii) $D(E) \subset D(A)$ and E is bijective.
: (iii) $E^{-1} : Z \to D(E)$ is continuous.
: (iv) The resolvent $R(\lambda, AE^{-1})$ is a compact operator for some $\lambda \in \rho(AE^{-1})$, the resolvent set of AE^{-1}.

The hypotheses (i), (ii) and the closed graph theorem imply the boundedness of the linear operator $AE^{-1} : Z \to Z$.

Let $B_r = \{y \in Y : |y| \leq r\}$, for some $r > 0$. We shall make the following hypotheses:

: (C_1) $D(\sigma E^{-1}) \subset D(\tilde{\theta})$ and the restriction of $\tilde{\theta}$ to $D(\sigma E^{-1})$ is continuous relative to the graph norm of $D(\sigma E^{-1})$.

: (C_2) The operator AE^{-1} is the infinitesimal generator of a C_0-semigroup $T(t)$ on Z and there exists a constant $M > 0$ such that $\|T(t)\| \leq M$.

: (C_3) There exists a linear continuous operator $B : U \to Z$ such that $\sigma E^{-1}B \in L(U, Z), \tilde{\theta}(Bu) = B_1 u$, for all $u \in U$. Also, $Bu(t)$ is continuously differentiable and $\|Bu\| \leq C\|B_1 u\|$ for all $u \in U$, where C is a constant.

: (C_4) For all $t \in (0, b]$ and $u \in U, T(t)Bu \in D(AE^{-1})$. Moreover, there exists a positive function $\nu \in L^1(0, b)$ such that $\|AE^{-1}T(t)B\| \leq \nu(t)$, a.e. $t \in (0, b)$.

: (C_5) $f : J \times Y \to Z$ is continuous and there exist constants $M_1, M_2 > 0$ such that for $t \in \Delta$ and $v_1, v_2 \in B_r$ we have

$$\|f(t, v_1) - f(t, v_2)\| \leq M_1\|v_1 - v_2\|$$

and

$$M_2 = \max_{t \in J} \|f(t, 0)\|.$$

: (C_6) $k : \Delta \times Y \times Y \to Z$ is continuous and there exist constants $N_1, N_2 > 0$ such that for $(t, s) \in \Delta$, $x_1, x_2 \in B_r$ and $y_1, y_2 \in Y$ we have

$$\|k(t, s, x_1, y_1) - k(t, s, x_2, y_2)\| \leq N_1[\|x_1 - x_2\| + \|y_1 - y_2\|]$$

and

$$N_2 = \max_{(t,s) \in \Delta} \|k(t, s, 0, 0)\|$$

: (C_7) $g : \Delta \times Y \to Y$ is continuous and there exist constants $L_1, L_2 > 0$ such that for $(t, s) \in \Delta$, and $x_1, x_2 \in B_r$ we have

$$\|g(t, s, x_1) - g(t, s, x_2)\| \leq L_1\|x_1 - x_2\|$$

and

$$L_2 = \max_{(t,s) \in \Delta} \|g(t, s, 0)\|$$

Let $y(t)$ be the solution of (7). Then define the function $z(t) = y(t) - Bu(t)$ and from the assumption it follows that $z(t) \in D(AE^{-1})$. Hence (7) can be written in terms of A and B as

$$\begin{aligned} y'(t) &= AE^{-1}z(t) + \sigma E^{-1}Bu(t) + f(t, E^{-1}y(t)) \\ &\quad + \int_0^t k\left(t, s, E^{-1}y(s), \int_0^s g(s, \tau, E^{-1}y(\tau))d\tau\right) ds, \quad t \in J \\ y(t) &= z(t) + Bu(t) \\ y(0) &= y_0 \end{aligned}$$

If u is continuously differentiable on $[0, b]$ then z can be defined as a mild solution to the Cauchy problem

$$
\begin{aligned}
z'(t) &= AE^{-1}z(t) + \sigma E^{-1}Bu(t) - Bu'(t) + f(t, E^{-1}y(t)) \\
&\quad + \int_0^t k\left(t, s, E^{-1}y(s), \int_0^s g(s, \tau, E^{-1}y(\tau))d\tau\right) ds, \\
z(0) &= y(0) - Bu(0)
\end{aligned}
$$

and the solution of (7) is given by

$$
\begin{aligned}
y(t) &= T(t)[y(0) - Bu(0)] + Bu(t) \\
&\quad + \int_0^t T(t-s)\left[\sigma E^{-1}Bu(s) - Bu'(s) + f(s, E^{-1}y(s))\right] ds \\
&\quad + \int_0^t T(t-s)\left[\int_0^s k\left(s, \tau, E^{-1}y(\tau), \int_0^\tau g(\tau, \eta, E^{-1}y(\eta))d\eta\right) d\tau\right] ds
\end{aligned}
$$

Since the differentiability of the control u represents an unrealistic and severe requirement, it is necessary to extend the concept of a solution for general inputs $u \in L^1(J, U)$. Integrating the above equation by parts, we get

$$
\begin{aligned}
y(t) &= T(t)y(0) + \int_0^t \left[T(t-s)\sigma E^{-1}B - AE^{-1}T(t-s)B\right] u(s)ds \\
&\quad + \int_0^t T(t-s)f(s, E^{-1}y(s))ds \\
&\quad + \int_0^t T(t-s)\left[\int_0^s k\left(s, \tau, E^{-1}y(\tau), \int_0^\tau g(\tau, \eta, E^{-1}y(\eta))d\eta\right) d\tau\right] ds
\end{aligned}
$$

which is well defined. Hence the mild solution of (6) is given by

$$
\begin{aligned}
x(t) &= E^{-1}T(t)Ex(0) + \int_0^t E^{-1}\left[T(t-s)\sigma E^{-1}B - AE^{-1}T(t-s)B\right] u(s)ds \\
&\quad + \int_0^t E^{-1}T(t-s)f(s, x(s))ds \\
&\quad + \int_0^t E^{-1}T(t-s)\left[\int_0^s k\left(s, \tau, x(\tau), \int_0^\tau g(\tau, \eta, x(\eta))d\eta\right) d\tau\right] ds \qquad (8)
\end{aligned}
$$

Further, assume the following conditions.

: (C_8) There exist constants $N, K > 0$ such that $\int_0^b \nu(t) \leq K$ and $|E^{-1}| \leq N$.

: (C_9) The linear operator W from $L^2(J,U)$ into Y defined by

$$Wu = \int_0^b E^{-1}\left[T(b-s)\sigma E^{-1}B - AE^{-1}T(b-s)B\right]u(s)ds$$

induces an invertible operator $\tilde{W}^{-1}$ defined on $L^2(J,U)/KerW$, there exists a constant $K_1 > 0$ such that $\|\tilde{W}^{-1}\| \leq K_1$.

: (C_{10}) $NM\|Ex_0\| + N\left[bM\|\sigma E^{-1}B\| + K\right]K_1[|x_1| + NM\|Ex_0\| + L] + L \leq r$, where $L = bNM[M_1 r + M_2] + bNM(N_1 br + bN_1 L_1 r + bN_1 L_2 + N_2)$

: (C_{11}) Let $q = bNMM_1 + b^2 NMM_1 + b^3 NML_1 N_1(1 + K_1 bNM\|\sigma E^{-1}B\| + NKK_1)$ be such that $0 \leq q \leq 1$.

Theorem.2 If the hypotheses $(C_1)-(C_{11})$ are satisfied , then the boundary control nonlinear system (6) is controllable on J.

Proof: Using the hypotheses (C_9), for an arbitrary function $x(\cdot)$, define the control

$$\begin{aligned}
u(t) &= \tilde{W}^{-1}\{x_1 - E^{-1}T(b)Ex_0 - \int_0^b E^{-1}T(b-s)f(s,x(s))ds \\
&\quad - \int_0^b E^{-1}T(b-s)\left[\int_0^s k\left(s,\tau,x(\tau),\int_0^\tau g(\tau,\eta,x(\eta))d\eta\right)d\tau\right]ds\}(t)
\end{aligned}$$

Let $V = C(J,B_r)$. Using this control, it will be shown that the operator Φ defined by

$$\begin{aligned}
\Phi x(t) &= E^{-1}T(t)Ex_0 + \int_0^t E^{-1}\left[T(t-s)\sigma E^{-1}B - AE^{-1}T(t-s)B\right] \\
&\quad \tilde{W}^{-1}\{x_1 - E^{-1}T(b)Ex_0 - \int_0^b E^{-1}T(b-\tau)f(\tau,x(\tau))d\tau \\
&\quad - \int_0^b E^{-1}T(b-s)\left[\int_0^s k\left(s,\tau,x(\tau),\int_0^\tau g(\tau,\eta,x(\eta))d\eta\right)d\tau\right]ds\}(s)ds \\
&\quad + \int_0^t E^{-1}T(t-s)f(s,x(s))ds \\
&\quad + \int_0^t E^{-1}T(t-s)\left[\int_0^s k\left(s,\tau,x(\tau),\int_0^\tau g(\tau,\eta,x(\eta))d\eta\right)d\tau\right]ds
\end{aligned}$$

has a fixed point. This fixed point is then a solution of (6).

Clearly $\Phi x(b) = x_1$, which means that the control u steers the system from the initial state x_0 to x_1 in time b provided the operator Φ has a fixed point.

First to see that Φ maps V into itself. For $x \in V$,

$$
\begin{aligned}
&\|\Phi x(t)\| \\
\leq\ & \|E^{-1}T(t)Ex_0\| + \|\int_0^t E^{-1}\left[T(t-s)\sigma E^{-1}B - AE^{-1}T(t-s)B\right] \\
& \tilde{W}^{-1}\{x_1 - E^{-1}T(b)Ex_0 - \int_0^b E^{-1}T(b-\tau)f(\tau, x(\tau))d\tau \\
& - \int_0^b E^{-1}T(b-s)\left[\int_0^s k\left(s,\tau,x(\tau),\int_0^\tau g(\tau,\eta,x(\eta))d\eta\right)d\tau\right]ds\}(s)ds\| \\
& +\|\int_0^t E^{-1}T(t-s)f(s,x(s))ds\| \\
& +\|\int_0^t E^{-1}T(t-s)\left[\int_0^s k\left(s,\tau,x(\tau),\int_0^\tau g(\tau,\eta,x(\eta))d\eta\right)d\tau\right]ds\| \\
\leq\ & |E^{-1}|\|T(t)Ex_0\| + \int_0^t |E^{-1}|\left[\|T(t-s)\|\|\sigma E^{-1}B\| + \|AE^{-1}T(t-s)B\|\right] \\
& \|\tilde{W}^{-1}\|\{|x_1| + |E^{-1}|\|T(b)Ex_0\| \\
& + \int_0^b |E^{-1}|\|T(b-\tau)\|\left[\|f(\tau,x(\tau)) - f(\tau,0)\| + \|f(\tau,0)\|\right]d\tau + \int_0^b |E^{-1}|\|T(b-s)\| \\
& \int_0^s \left[\|k\left(s,\tau,x(\tau),\int_0^\tau g(\tau,\eta,x(\eta))d\eta\right) - k(s,\tau,0,0)\| + \|k(s,\tau,0,0)\|\right]d\tau ds\}ds \\
& + \int_0^t |E^{-1}|\|T(t-s)\|\left[\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\|\right]ds + \int_0^t |E^{-1}|\|T(t-s)\| \\
& \int_0^s \left[\|k\left(s,\tau,x(\tau),\int_0^\tau g(\tau,\eta,x(\eta))d\eta\right) - k(s,\tau,0,0)\| + \|k(s,\tau,0,0)\|\right]d\tau ds \\
\leq\ & NM\|Ex_0\| + N\left[bM\|\sigma E^{-1}B\| + K\right]K_1\{|x_1| + NM\|Ex_0\| + bNM\{M_1 r + M_2\} \\
& +bNM\left[\int_0^s N_1\|x(\tau)\|d\tau + \int_0^s N_1\|g(\tau,\eta,x(\eta))d\eta\|d\tau + N_2\right]\} \\
& +bNM[M_1 r + M_2] + bNM\{\int_0^s N_1\|x(\tau)\|d\tau + \int_0^s N_1\|g(\tau,\eta,x(\eta))d\eta\|d\tau + N_2\} \\
\leq\ & NM\|Ex_0\| + N\left[bM\|\sigma E^{-1}B\| + K\right]K_1\{|x_1| + NM\|Ex_0\| + L\} + L \\
\leq\ & r
\end{aligned}
$$

Thus Φ maps V into itself. Now, for $x_1, x_2 \in V$ we have

$$\begin{aligned}
&\|\Phi x_1(t) - \Phi x_2(t)\| \\
\leq\ & \int_0^t |E^{-1}| \left[\|T(t-s)\|\|\sigma E^{-1}B\| + \|AE^{-1}T(t-s)B\|\right] \\
& \|\tilde{W}^{-1}\|\{\int_0^b |E^{-1}|\|T(b-\tau)\|\|f(\tau, x_1(\tau)) - f(\tau, x_2(\tau))\|d\tau \\
& + \int_0^b |E^{-1}|\|T(b-s)\|\{\| \int_0^s \{k\left(s, \tau, x_1(\tau), \int_0^\tau g(\tau, \eta, x_1(\eta))d\eta\right) \\
& -k\left(s, \tau, x_2(\tau), \int_0^\tau g(\tau, \eta, x_2(\eta))d\eta\right)\}d\tau\|\}d\eta\}ds \\
& + \int_0^t |E^{-1}|\|T(t-s)\| \left(\|f(s, x_1(s)) - f(s, x_2(s))\|\right) ds \\
& + \int_0^t |E^{-1}|\|T(t-s)\|\{\| \int_0^s \{k\left(s, \tau, x_1(\tau), \int_0^\tau g(\tau, \eta, x_1(\eta))d\eta\right) \\
& -k\left(s, \tau, x_2(\tau), \int_0^\tau g(\tau, \eta, x_2(\eta))d\eta\right)\}d\tau\|\}ds \\
\leq\ & \int_0^t N\left[M\|\sigma E^{-1}B\| + \nu(t)\right] K_1\{bNMM_1\|x_1(\tau - x_2(\tau)\| \\
& + bNMbN_1\{\|x_1(\tau) - x_2(\tau)\| \\
& + \| \int_0^\tau g(\tau, \eta, x_1(\eta))d\eta - \int_0^\tau g(\tau, \eta, x_2(\eta))d\eta\|\}\}ds \\
& + bNMM_1\|x_1(s) - x_2(s)\| + bNMbN_1\|x_1(\tau) - x_2(\tau)\| \\
& + bNMbN_1\| \int_0^\tau g(\tau, \eta, x_1(\eta))d\eta - \int_0^\tau g(\tau, \eta, x_2(\eta))d\eta\| \\
\leq\ & \left([bNM\|\sigma E^{-1}B\| + NK]K_1[bNMM_1 + b^2NMM_1 + b^3NMN_1L_1]\right. \\
& \left. + bNMM_1 + b^2NMM_1 + b^3NMN_1L_1\right) \|x_1(t) - x_2(t)\| \\
\leq\ & q\|x_1(t) - x_2(t)\|
\end{aligned}$$

Therefore, Φ is a contraction mapping.

Hence by the Banach fixed point theorem there exists a unique fixed point $x \in Y$ such that $\Phi x(t) = x(t)$. Any fixed point of Φ is a mild solution of (6) on J satisfying $x(b) = x_1$. Thus system (6) is controllable on J.

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